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MATHEMATICS

On Spectral Asymptotics for a Family of Finite-Dimensional Perturbations of Operators of Trace Class

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Abstract—Spectral asymptotics for a family of finite-dimensional perturbations of operators of trace class are found. The results are used to find exact asymptotics of small ball probabilities in the Hilbert norm for finite-dimensional perturbations of Gaussian functions. As an example, Durbin processes appearing in the study of empirical processes with estimated parameters are considered.

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1. SPECTRAL PROBLEM

Let $\mathbb{O}$ be a bounded domain in $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\overline{\mathbb{O}}$ be the closure of $\mathbb{O}$. Suppose that $\mathbb{G}_0 \in \mathfrak{S}^1$ is a positive integral operator in $L_2(\mathbb{O})$ with a trace (which means that its eigenvalues $\mu_k^0 > 0$, $k \in \mathbb{N}$, have the property $\sum \mu_k^0 < \infty$) and $G_0(s,t)$ is its kernel, so that

$$(\mathbb{G}_0 u)(s) = \int_{\mathbb{O}} G_0(s,t)u(t)dt.$$

Consider a finite-dimensional perturbation of the operator $\mathbb{G}_0$ of rank m. Its kernel can be represented in the form

$$G(s,t) := G_0(s,t) + \boldsymbol{\psi}^{\mathrm{T}}(s)\cdot D\cdot \boldsymbol{\psi}(t), \qquad s,t \in \overline{\mathbb{O}}, \qquad (1)$$

where $\boldsymbol{\psi}(t) = (\psi_1(t), \ldots, \psi_m(t))^{\mathrm{T}}$, $\psi_j(t) \in L_2(\mathbb{O})$, and $D \in M_{m\times m}$ is a square matrix of order m (without loss of generality, we can assume that D is a symmetric matrix). Let μ_k, $k \in \mathbb{N}$, denote the positive eigenvalues of the operator with kernel $G(s,t)$. By the minimax principle (see [1, Subsection 9.2]), $\mu_{k+m}^0 \le \mu_k \le \mu_{k-m}^0$ and, hence, $\sum \mu_k < \infty$. In this paper, we examine the conditions on the perturbation parameters $\psi_j(t)$ and D under which the spectrum μ_k of the perturbed operator is "asymptotically close" to the spectrum μ_k^0 of the original operator (see Theorem 1).

The case of a one-dimensional perturbation ($m = 1$) was considered by Nazarov in [2]. Specifically, it was shown that there exists a critical value of the parameter $D \in \mathbb{R}$ (denoted by D_{cr}) such that

(i) for $D > D_{\mathrm{cr}}$ and $\psi \in \mathrm{Im}(\mathbb{G}_0^{1/2})$, the spectra μ_k^0 and μ_k are asymptotically close, namely, $\prod_{k=1}^{\infty} \frac{\mu_k^0}{\mu_k} < \infty$;

(ii) for $D = D_{\mathrm{cr}}$ and $\psi \in \mathrm{Im}(\mathbb{G}_0)$, the spectrum μ_k is asymptotically close to the shifted spectrum μ_{k+1}^0, namely, $\prod_{k=1}^{\infty} \frac{\mu_{k+1}^0}{\mu_k} < \infty$.

Note that the condition $\psi \in \mathrm{Im}(\mathbb{G}_0^{1/2})$ is equivalent to the fact that the quantity $q := \langle \mathbb{G}_0^{-1}\psi, \psi\rangle$ is well defined. Here, $\langle\cdot,\cdot\rangle$ denotes duality in dual spaces. Moreover, $D_{\mathrm{cr}} = -q^{-1}$, which is equivalent to the condition $1 + qD_{\mathrm{cr}} = 0$.

In the case of a finite-dimensional perturbation, the answer depends on the rank of the matrix $(E_m + QD)$, where E_m is the identity matrix of order m and $Q := \langle \mathbb{G}_0^{-1}\boldsymbol{\psi}, \boldsymbol{\psi}^{\mathrm{T}}\rangle$. The main spectral result can be formulated as follows.

Theorem 1. *Suppose that the matrix* $Q := \langle \mathbb{G}_0^{-1}\boldsymbol{\psi}, \boldsymbol{\psi}^{\mathrm{T}}\rangle$ *is defined, or, equivalently,* $\psi_j \in \mathrm{Im}(\mathbb{G}_0^{1/2})$, $j = 1, 2, \ldots, m$, *and all eigenvalues* η_k $(k = 1, 2, \ldots, m)$ *of the matrix* $(E_m + QD)$ *are nonnegative.*

1. *If* $\mathrm{rank}(E_m + QD) = m$, *then* $\prod_{k=1}^{\infty} \frac{\mu_k^0}{\mu_k} < +\infty$.

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2. *If* $\mathrm{rank}(E_m + QD) = m - s$ *and* $\psi_j \in \mathrm{Im}(G_0)$ $(s, j = 1, 2, \ldots, m)$, *then* $\prod_{k=1}^{\infty} \frac{\mu_{k+s}^0}{\mu_k} < +\infty$.

Remark 1. The perturbation corresponding to $s = m$ in item 2 of Theorem 1 is called critical (which corresponds to the critical case introduced in [2] for one-dimensional perturbations). In this case, $D = -Q^{-1}$.

The proof of Theorem 1 is based on Bateman's formula (see [3, Chapter II, Section 4; 4]) for the ratio of Fredholm determinants and on the Jensen theorem for the product of zeros of an entire function (see [5, Subsection 3.6]).

Remark 2. If the matrix $(E_m + QD)$ has several negative eigenvalues, there is also a "failure" in indexing, since we consider only positive eigenvalues. For example, if the matrix has r negative and s zero eigenvalues, then $\prod_{k=1}^{\infty} \frac{\mu_{k+r+s}^0}{\mu_k} < +\infty$.

2. APPLICATION TO THE PROBLEM OF SMALL BALL PROBABILITIES

Consider a random Gaussian function $X(t)$, $t \in \overline{\mathbb{O}}$, with zero mean and covariance $G_X(s,t) = \mathbb{E}X(s)X(t)$. The problem of small ball probabilities for $X(t)$ in the L_2 norm consists in finding the asymptotics, as $\varepsilon \to 0$, of the quantity

$$\mathbb{P}(\|X\|_{L_2(\mathbb{O})} < \varepsilon) = \mathbb{P}\left(\sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2\right). \tag{2}$$

This equality holds by virtue of the Karhunen–Loève expansion (see, e.g., [6, Section 12]). Here, ξ_k, $k \in \mathbb{N}$, are independent normal standard random variables and μ_k are the positive eigenvalues of the integral operator with kernel $G_X(s,t)$ (covariance operator).

An important role is played by the Wenbo Li comparison theorem [7]: Let μ_k and $\hat{\mu}_k$ be two summable sequences. If

$$\prod_{k=1}^{\infty} \frac{\hat{\mu}_k}{\mu_k} < \infty, \tag{3}$$

then, as $\varepsilon \to 0$,

$$\mathbb{P}\left(\sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2\right) \sim \mathbb{P}\left(\sum_{k=1}^{\infty} \hat{\mu}_k \xi_k^2 < \varepsilon^2\right) \cdot \left(\prod_{k=1}^{\infty} \frac{\hat{\mu}_k}{\mu_k}\right)^{1/2}. \tag{4}$$

Thus, to find the asymptotics of small ball probabilities for $X(t)$, it is sufficient to construct a function $\hat{X}(t)$ with known asymptotics of small ball probabilities such that the eigenvalues μ_n and $\hat{\mu}_n$ of their covariance operators are asymptotically close (in the sense of condition (3)).

Consider the problem of small ball probabilities for finite-dimensional perturbations of Gaussian functions. Let $X_0(t)$, $t \in \overline{\mathbb{O}}$, be a Gaussian function with zero mean and covariance function $G_0(s, t) = \mathbb{E}X_0(s)X_0(t)$. The corresponding covariance operator is denoted by $\mathbb{G}_0$. Consider $\boldsymbol{\varphi}(t) = (\varphi_1(t), \ldots, \varphi_m(t))^{\mathrm{T}}$, where φ_j are locally summable functions in $\mathbb{O}$, $j = 1, 2, \ldots, m$. Assume that the vector function

$$\boldsymbol{\psi}(t) = (\psi_1(t), \ldots, \psi_m(t))^{\mathrm{T}} = (\mathbb{G}_0\boldsymbol{\varphi})(t) \tag{5}$$

is defined a.e. in $\mathbb{O}$, $\boldsymbol{\psi} \not\equiv 0$, and a matrix $Q = (Q_{ij})_{i,j=1}^m$ is defined:

$$Q_{ij} = \int_{\mathbb{O}} \psi_i(t)\varphi_j(t)dt \Leftrightarrow \psi_j \in \mathrm{Im}(\mathbb{G}_0^{1/2}). \tag{6}$$

Construct a family of Gaussian functions—a generalization of formula (1.3) from [2]:

$$X_A(t) := X_0(t) - \boldsymbol{\psi}(t)^{\mathrm{T}} \cdot A \cdot \int_{\mathbb{O}} X_0(s) \cdot \boldsymbol{\varphi}(s)ds, \quad t \in \overline{\mathbb{O}}. \tag{7}$$

Here, A is the matrix of perturbation parameters ($A_{ij} \in \mathbb{R}$, $i, j = 1, 2, \ldots, m$). Clearly, $\mathbb{E}X(t) \equiv 0$.

Lemma 1. *The function* $X_A(t)$ *has the covariance*

$$G_{X_A}(s,t) = G_0(s,t) + \boldsymbol{\psi}(s)^{\mathrm{T}} \cdot D \cdot \boldsymbol{\psi}(t), \tag{8}$$

where the matrix D is given by the formula

$$D = -A - A^{\mathrm{T}} + AQA^{\mathrm{T}}. \tag{9}$$

Corollary 1. *The functions* $X_A(t)$ *and* $X_{2Q^{-1}-A}(t)$ *have identical finite-dimensional distributions.*

Corollary 2. *Let* $A = Q^{-1}$. *Then the following assertions hold.*

1. *We have the identity* ($j = 1, 2, \ldots, m$)

$$\int_{\mathbb{O}} X_A(t)\varphi_j(t)dt = 0.$$

2. *The function* $X_A(t)$ *and the random variables* $\int_{\mathbb{O}} X_0(t)\varphi_j(t)dt$, $j = 1, 2, \ldots, m$, *are independent.*

3. *If* $\varphi_j \in L_2(\mathbb{O})$, $j = 1, 2, \ldots, m$, *then the integral operator with kernel* $G_{X_A}(s,t)$ *has a zero eigenvalue of multiplicity m corresponding to the eigenfunctions* φ_j, $j = 1, 2, \ldots, m$.

Theorem 2. 1. *Let the matrix* $(E_m - QA)$ *be nonsingular. Then, as* $\varepsilon \to 0$,

$$\mathbb{P}(\|X_A\|_{L_2(\mathbb{O})} < \varepsilon) \sim \frac{\mathbb{P}(\|X_0\|_{L_2(\mathbb{O})} < \varepsilon)}{\det(E_m - QA)}.$$

2. *Let* $A = Q^{-1}$. *If* $\varphi_j \in L_2(\mathbb{O})$, $j = 1, 2, \dots, m$, *then, as* $r \to 0$,

$$\mathbb{P}(\|X_A\|_{L_2(\mathbb{O})} < \sqrt{r}) \sim \sqrt{\frac{\det(Q)}{\det\left(\int\limits_O \boldsymbol{\varphi}^{\mathrm{T}}(s)\boldsymbol{\varphi}(s)ds\right)}}\left(\sqrt{\frac{2}{\pi}}\right)^m$$

$$\times \int\limits_0^r \int\limits_0^{r_1} \cdots \int\limits_0^{r_{m-1}} \frac{d^m}{dr_m^m}\mathbb{P}(\|X_0\|_{L_2(\mathbb{O})} < \sqrt{r_m})$$

$$\times \frac{dr_m \dots dr_1}{\sqrt{(r - r_1)(r_1 - r_2)\dots(r_{m-1} - r_m)}}.$$

Remark 3. If $0 < \operatorname{rank}(E_m - QA) < m$ and $\varphi_j \in L_2(\mathbb{O})$, $j = 1, 2, \dots, m$, then the small ball probabilities can be calculated by combining cases 1 and 2 from Theorem 2.

The basic motivating example of processes of form (7) is given by Durbin processes naturally arising in statistics. Let us describe them in more detail.

Let $x_1, \dots, x_n \in \mathbb{R}$ be a sample with a population distribution function $F(x, \theta)$, $f(x, \theta)$ be the distribution density, and $\theta = (\theta_1, \dots, \theta_s)$, $s \in \mathbb{N}$, be a vector of parameters. Consider an empirical distribution function for fixed parameter values $\theta^0 = (\theta_1^0, \dots, \theta_s^0)$:

$$F_n^0(t) = \frac{\#\{x_i\colon F(x_i, \theta^0) \le t,\ i = 1, 2, \dots, n\}}{n}, \quad t \in [0, 1].$$

It is well known (see [8, Chapter 3]) that the process $n^{1/2}[F_n^0(t) - t]$ weakly converges to the Brownian bridge $B(t)$ in $D[0, 1]$. Here, $D[0, 1]$ is the space of functions on [0, 1] that are right continuous and have only jump discontinuities.

Assume that some of the parameters of the distribution are unknown (without loss of generality, we may assume that these are the first m parameters). The unknown parameters are estimated using the sample (e.g., by applying the maximum likelihood method), and the new parameter vector is denoted by $\hat{\theta} := (\hat{\theta}_1, \dots, \hat{\theta}_m, \theta_{m+1}^0, \dots, \theta_s^0)$. Then the empirical distribution function becomes

$$\hat{F}_n(t) = \frac{\#\{x_i\colon F(x_i, \hat{\theta}) \le t,\ i = 1, 2, \dots, n\}}{n}, \quad t \in [0, 1].$$

It was shown in [9] that the process $n^{1/2}[\hat{F}_n(t) - t]$ converges weakly in $D[0, 1]$ to a finite-dimensional perturbation of the Brownian bridge, namely, to a Gaussian process with zero mean and the covariance function

$$G(s,t) = G_B(s,t) - \boldsymbol{\psi}^{\mathrm{T}}(s)S^{-1}\boldsymbol{\psi}(t), \qquad s, t \in [0, 1];$$

here, $G_B(s,t) = \min(s,t) - st$ is the covariance of the Brownian bridge; S is the Fisher information matrix with elements $S_{ij} = \mathbb{E}\left(\frac{\partial}{\partial\theta_i}\ln(f(x,\theta))\frac{\partial}{\partial\theta_j}\ln(f(x,\theta))\right)\Big|_{\theta=\theta^0}$, where $i, j = 1, 2, \dots, m$ and $\theta^0 = (\theta_1^0, \dots, \theta_s^0)$ is a fixed parameter vector; x and t are related by $t = F(x, \theta)$; and the functions $\psi_j(t)$ are given by the formula $\psi_j(t) = \frac{\partial F(x,\theta)}{\partial\theta_j}\Big|_{\theta=\theta^0}$, $j = 1, 2, \dots, m$.

Theorem 3. *The Durbin processes are critical* (*in the sense of Remark* 1 *and Corollary* 2).

Small ball probabilities for some Durbin processes were considered in [10, 11].

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REFERENCES

1. M. Sh. Birman and M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space* (Reidel, Dordrecht, 1987; Lan', St. Petersburg, 2010).
2. A. I. Nazarov, Theory Probab. Appl. **54** (2), 203–216 (2010).
3. L. V. Kantorovich and V. I. Krylov, *Approximate Methods of Higher Analysis* (Gostekhizdat, Moscow, 1950; Wiley, New York, 1964).
4. S. Sukhatme, Ann. Math. Stat. **43**, 1914–1926 (1972).
5. E. Titchmarsh, *The Theory of Functions*, 2nd ed. (Oxford Univ. Press, London, 1958).
6. M. A. Lifshits, *Lectures on Gaussian Processes* (Lan', St. Petersburg, 2016) [in Russian].
7. W. Li, J. Theor. Probab. **5** (1), 1–31 (1992).
8. P. Billingsli, Billingsley, *Convergence of Probability Measures* (Wiley, New York, 1968).
9. J. Durbin, Ann. Stat. **1** (2), 279–290 (1973).
10. A. I. Nazarov and Yu. P. Petrova, Theory Probab. Appl. **60** (3), 460–480 (2016).
11. Yu. P. Petrova, Zap. Nauchn. Semin. POMI **466**, 211–233 (2017).

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