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On the Attainability of the Best Constant in Fractional Hardy–Sobolev Inequalities Involving the Spectral Dirichlet Laplacian

N. S. Ustinov

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Abstract. We prove the attainability of the best constant in the fractional Hardy–Sobolev inequality with a boundary singularity for the spectral Dirichlet Laplacian. The main assumption is the average concavity of the boundary at the origin. A similar result has been proved earlier for the conventional Hardy–Sobolev inequality.

Key words: fractional Laplacian, attainability of the best constant, Navier Laplacian, spectral Dirichlet Laplacian.

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1. Introduction. In this paper we study the problem of the attainability of the best constant $\mathscr{S}^{Sp}_{s,\sigma}(\Omega)$ in the fractional Hardy–Sobolev inequality with *spectral Dirichlet Laplacian* in a C^1 bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$:

$$\mathscr{S}^{Sp}_{s,\sigma}(\Omega) \cdot \||x|^{\sigma-s}u\|^2_{L_{2^*_\sigma}(\Omega)} \leqslant \langle(-\Delta)^s_\Omega u, u\rangle, \qquad u \in \widetilde{\mathscr{D}}^s(\Omega), \tag{1}$$

where $0 < \sigma < s < 1$ and $2^*_\sigma \equiv 2n/(n-2\sigma)$.

The fractional *spectral Dirichlet Laplacian* $(-\Delta)^s_\Omega$ is the sth power of the Dirichlet Laplacian on Ω in the sense of spectral theory. This is a self-adjoint operator, which can be restored from its quadratic form: in the case $\Omega = \mathbb{R}^n$, this form is

$$\langle(-\Delta)^s_{\mathbb{R}^n} u, u\rangle := \int_{\mathbb{R}^n} |\xi|^{2s}|\mathscr{F}u(\xi)|^2\,d\xi, \qquad \mathscr{F}u(\xi) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}u(x)\,dx, \tag{2}$$

i.e., $(-\Delta)^s_{\mathbb{R}^n}$ coincides with the conventional fractional Laplacian on $\mathbb{R}^n$. In the case of the half-space

$$\mathbb{R}^n_+ := \{x \equiv (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n > 0\},$$

the quadratic form is equal to

$$\langle(-\Delta)^s_{\mathbb{R}^n_+} u, u\rangle := \int_{\mathbb{R}^n_+} |\xi|^{2s}|\widehat{\mathscr{F}}u(\xi)|^2\,d\xi,$$

$$\widehat{\mathscr{F}}u(\xi) := \frac{2}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} u(x)e^{-i\xi'\cdot x'}\sin(x_n\xi_n)\,dx,$$

and for a bounded domain Ω,

$$\langle(-\Delta)^s_\Omega u, u\rangle := \sum_{j=1}^{\infty} \lambda_j^s \langle u, \phi_j\rangle^2; \tag{3}$$

here the λ_j and the ϕ_j are, respectively, the eigenvalues and eigenfunctions (orthonormalized in $L_2(\Omega)$) of the Dirichlet Laplacian on Ω.

Inequality (1) for $s \in (0, n/2)$ follows from a general theorem by Il'in [17, Theorem 1.2, (22)] on estimates of integral operators on weighted Lebesgue spaces. In $\mathbb{R}^n$, in the cases of $\sigma = 0$ and $\sigma = s$, inequality (1) reduces to the fractional Hardy and Sobolev inequalities

$$\langle(-\Delta)^s_{\mathbb{R}^n} u, u\rangle \geqslant \mathscr{S}_{s,0}\||x|^{-s}u\|^2_{L_2(\mathbb{R}^n)} \quad \text{and} \quad \langle(-\Delta)^s_{\mathbb{R}^n} u, u\rangle \geqslant \mathscr{S}_{s,s}\|u\|^2_{L_{2^*_s}(\mathbb{R}^n)}. \tag{4}$$

The explicit values of $\mathscr{S}_{s,0}$ and $\mathscr{S}_{s,s}$ were computed in [6] and [3], respectively.

The attainability of the best constant $\mathscr{S}_\sigma(\Omega)$ in the local case $s=1$ has been well studied (even for the non-Hilbertian case):

• For $0\in\Omega$, $\sigma\in[0,1]$, and $n\geqslant 3$, the best constant $\mathscr{S}_\sigma(\Omega)$ coincides with $\mathscr{S}_\sigma(\mathbb{R}^n)$ and *is not attained* in the case $\widetilde{\mathscr{D}}^1(\Omega)\neq\mathscr{D}^1(\mathbb{R}^n)$.

• If $0\in\partial\Omega$ and Ω is a cone in $\mathbb{R}^n$, then, for $\sigma\in(0,1)$ and $n\geqslant 2$, the best constant $\mathscr{S}_\sigma(\Omega)$ *is attained* ([4]; [19] in the non-Hilbertian case).

• The case of bounded Ω with $0\in\partial\Omega$ is much more complicated and depends on the geometry of $\partial\Omega$ at the origin. In [16] the attainability of $\mathscr{S}_\sigma(\Omega)$ was proved for all $n\geqslant 2$ in the case where the boundary $\partial\Omega$ is regularly varying and average concave at the origin (see Sec. 4). For $n\geqslant 4$, attainability was proved in [5] under stronger conditions.

The author is aware of only a few works on the attainability of the sharp constant in (1) for $s\notin\mathbb{N}$. In [15] attainability was shown for $\Omega=\mathbb{R}^n$ and $s\in(0,n/2)$. In [9] and [13, Sec. 5] attainability in $\Omega=\mathbb{R}^n_+$ was shown for (1) with restricted Dirichlet or Neumann fractional Laplacians on the right-hand side.

In this paper we prove the following results for inequality (1).

• In the case where $0\in\Omega$ and $\widetilde{\mathscr{D}}^s(\Omega)\neq\mathscr{D}^s(\mathbb{R}^n)$, the best constant *is not attained*.

• In the case $\Omega=\mathbb{R}^n_+$, the best constant *is attained*.

• In the case of bounded Ω and $0\in\partial\Omega$, the best constant *is attained* under some geometric assumptions on $\partial\Omega$ at the origin, analogous to the conditions in [16].

2. Preliminaries. Using the Sobolev inequality (4), we define spaces $\mathscr{D}^s(\mathbb{R}^n)$ and $\widetilde{\mathscr{D}}^s(\Omega)$ as

$$\mathscr{D}^s(\mathbb{R}^n):=\{u\in L_{2^*_s}(\mathbb{R}^n)\mid\langle(-\Delta)^s_{\mathbb{R}^n}u,u\rangle<\infty\},$$
$$\widetilde{\mathscr{D}}^s(\Omega):=\{u\in\mathscr{D}^s(\mathbb{R}^n)\mid u\equiv 0\ \text{ outside }\ \overline{\Omega}\}.$$

Recall that (see, e.g., [10, Lemma 1]) the quadratic form $\langle(-\Delta)^s_\Omega u,u\rangle$ defines an equivalent norm on $\widetilde{\mathscr{D}}^s(\Omega)$.

Let us define the Stinga–Torrea extension for $u(x)\in\widetilde{\mathscr{D}}^s(\Omega)$ ([14]; see also [1] for $\Omega=\mathbb{R}^n$): the Dirichlet problem

$$\operatorname{div}(t^{1-2s}\nabla w(x,t))=0\quad\text{in }\ \Omega\times\mathbb{R}_+,\qquad w|_{t=0}=u,\ w|_{x\in\partial\Omega}=0,\tag{5}$$

has a unique solution w_{sp} with finite energy

$$\mathscr{E}_s[w]:=\int_0^{+\infty}\int_\Omega t^{1-2s}|\nabla w(x,t)|^2\,dx\,dt.\tag{6}$$

In addition, the following relation holds for sufficiently smooth u:

$$(-\Delta)^s_\Omega u(x)=C_s\frac{\partial w_{sp}}{\partial\nu_s}(x,0):=-C_s\lim_{t\to 0_+}t^{1-2s}\partial_t w_{sp}(x,t),\qquad C_s:=\frac{4^s\Gamma(1+s)}{2s\cdot\Gamma(1-s)}\,.$$

Moreover, w_{sp} can be derived as a minimizer of (6) over the space

$$\mathfrak{W}_s(\Omega):=\{w(x,t)\mid\mathscr{E}_s[w]<+\infty,\ w|_{t=0}=u,\ w|_{x\in\partial\Omega}=0\},$$

and the quadratic form (3) can be expressed in terms of $\mathscr{E}_s[w_{sp}]$ (see, e.g., [11, (2.6)]) as

$$\langle(-\Delta)^s_\Omega u,u\rangle=C_s\mathscr{E}_s[w_{sp}].\tag{7}$$

We refer to any function $w(x,t)\in\mathfrak{W}_s(\Omega)$ as an *admissible extension* of $u(x)$. Obviously, for any admissible extension w, we have $\mathscr{E}_s[w]\geqslant\mathscr{E}_s[w_{sp}]$.

The attainability of the best constant in (1) is equivalent to the existence of a minimizer for the functional $\mathscr{I}_\sigma$:

$$\mathscr{I}_\sigma[u]:=\frac{\langle(-\Delta)^s_\Omega u,u\rangle}{\||x|^{\sigma-s}u\|^2_{L_{2^*_\sigma}(\Omega)}}\,.\tag{8}$$

A standard variational argument shows that each minimizer of (8) solves the following problem (up to multiplication by a constant):

$$(-\Delta)^s_\Omega u(x) = \frac{u^{2^*_\sigma-1}(x)}{|x|^{(s-\sigma)2^*_\sigma}} \quad \text{in } \Omega, \qquad u \in \widetilde{\mathscr{D}}^s(\Omega). \tag{9}$$

According to [12, Theorem 3], the substitution $u \to |u|$ decreases $\mathscr{I}_{\sigma,\Omega}$. Therefore, if u is a minimizer of (8), then the right-hand side of (9) is nonnegative. Thus, the maximum principle [2, Lemma 2.6] shows that u cannot vanish in Ω and therefore preserves sign.

Theorem 1. 1. *Let* $0 \in \Omega$, *and let* $\widetilde{\mathscr{D}}^s(\Omega) \neq \mathscr{D}^s(\mathbb{R}^n)$. *Then* $\mathscr{S}^{Sp}_{s,\sigma}(\Omega)$ *is equal to* $\mathscr{S}_{s,\sigma}(\mathbb{R}^n)$ *and is not attained.*

2. *If* Ω *is star-shaped about* 0, *then the only nonnegative solution of* (9) *is* $u \equiv 0$.

The proof of the first statement is similar to that in the local case. The second statement follows from a newly invented nonlocal variant of the Pohozhaev identity.

3. Attainability of the sharp constant $\mathscr{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)$.

Theorem 2. *For* $n \geqslant 1$, $n > 2s$, *and* $\sigma \in (0, s)$, *there exists a minimizer of the functional* (8) *in* $\mathbb{R}^n_+$.

Similarly to the local case, the proof is based on the concentration-compactness principle by Lions [7]. However, to justify the passage to the limit, estimates on the Green function of problem (5) in $\mathbb{R}^n_+$ are required.

We denote the obtained minimizer by $\Phi(x)$ and its Stinga–Torrea extension by $\mathscr{W}(x,t)$. Without loss of generality, we can assume that $\||x|^{\sigma-s}\Phi\|_{L_{2^*_\sigma}(\mathbb{R}^n_+)} = 1$; therefore, we have $\mathscr{E}_s[\mathscr{W}] = \mathscr{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)$.

Remark 1. A minimizer of (8) with $\||x|^{\sigma-s}\Phi\|_{L_{2^*_\sigma}(\mathbb{R}^n_+)} = 1$ is not unique. Indeed, the functional (8) is invariant with respect to dilations and multiplication by constants. Compositions of these transformations that preserve the norm $\||x|^{\sigma-s}\Phi\|_{L_{2^*_\sigma}(\mathbb{R}^n_+)} = 1$ give us numerous minimizers.

Lemma 1. *Any minimizer* $\Phi(x)$ *and its Stinga–Torrea extension* $\mathscr{W}(x,t)$ *satisfy the following estimates*:

$$\begin{aligned} \Phi(x) &\leqslant \frac{Cx_n}{1+|x|^{n-2s+2}}, \qquad x \in \mathbb{R}^n_+, \\ \mathscr{W}(x,t) &\leqslant \frac{Cx_n}{1+|(x,t)|^{n-2s+2}}, \qquad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_+, \end{aligned} \tag{10}$$

$$\mathscr{V}(x) := \int_0^{+\infty} t^{1-2s} |\nabla \mathscr{W}(x,t)|^2 dt \leqslant \frac{C}{1+|x|^{2n-2s+2}}, \qquad x \in \mathbb{R}^n_+, \tag{11}$$

where the constants C *depend on* n, s, *and* σ *and on the choice of the minimizer* Φ.

To obtain these estimates, we first prove the boundedness of the function Φ by using a modification of De Giorgi's technique; see [18, Ch. II, Sec. 5]. More precise estimates follow from estimates on the Green function of problem (5) in $\mathbb{R}^n_+$. Estimates of the behavior of Φ and $\mathscr{W}$ at infinity are obtained from the estimates at the origin via the s-Kelvin transform.

4. Attainability of the sharp constant $\mathscr{S}^{Sp}_{s,\sigma}(\Omega)$. We assume that in a neighborhood $\{x\colon |x| < r_0\}$ of the origin the surface $\partial\Omega$ is given by the equation $x_n = F(x')$, where $F \in C^1$, $F(0) = 0$, and $\nabla_{x'} F(0) = 0$. Following [16], we assume that $\partial\Omega$ is *average concave at the origin*, i.e., for small $\tau > 0$, we have

$$f(\tau) := \frac{1}{|\mathbb{S}^{n-2}_\tau|} \int_{\mathbb{S}^{n-2}_\tau} F(x')\, dx' < 0. \tag{12}$$

Obviously, $f \in C^1$ for small τ. We also assume that f is *regularly varying* at the origin with exponent $\alpha \in [1, n-2s+3)$, i.e., for any $a > 0$,

$$\lim_{\tau \to 0} \frac{f(a\tau)}{f(\tau)} = a^\alpha. \tag{13}$$

Finally, we assume that the following technical assumption holds:

$$\lim_{\tau\to 0}\frac{\tau}{|\mathbb{S}^{n-2}_\tau|\cdot f(\tau)}\int_{\mathbb{S}^{n-2}_\tau}|\nabla_{x'}F(x')|^2\,dx'=0. \tag{14}$$

Remark 2. In the case where $\partial\Omega$ is C^2 and has negative mean curvature at the origin, our assumptions (12)–(14) are fulfilled with $\alpha=2$ (see [16, Remark 1]). We also emphasize that these assumptions admit the absence of mean curvature ($\alpha<2$) or its vanishing ($\alpha>2$).

Theorem 3. *Let $\partial\Omega$ satisfy* (12)–(14). *Then the minimizer of* (8) *exists, i.e., problem* (9) *has a positive solution in* Ω.

Sketch of proof. We consider a minimizing sequence $u_k(x)$ for (8). Using the Lions principle, we obtain two alternatives: either u_k is relatively compact in $\widetilde{\mathscr{D}}^s(\Omega)$, or $||x|^{\sigma-s}u_k|^{2^*_\sigma}\rightharpoonup\delta_0(x)$. We prove that, in the second case,

$$\mathscr{S}^{Sp}_{s,\sigma}(\Omega)=\mathscr{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+). \tag{15}$$

It remains to show that (15) cannot hold under assumptions (12)–(14). To prove this, we construct a function $\Phi_\varepsilon(x)$ such that $\mathscr{I}_\sigma[\Phi_\varepsilon(x)]<\mathscr{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)$. Let $\Theta(x)=x-F(x')e_n$ be a coordinate transformation that flattens the boundary $\partial\Omega$. For $\delta\in(0,r_0)$, we define $\widetilde{\varphi}(x):=\varphi_\delta(\Theta(x))$, where $\varphi_\delta(x)$ is a cut-off function supported in the δ-neighborhood of the origin. We put

$$\Phi_\varepsilon(x):=\varepsilon^{-(n-2s)/2}\Phi(\varepsilon^{-1}\Theta(x))\widetilde{\varphi}(x),$$
$$w_\varepsilon(x,t):=\varepsilon^{-(n-2s)/2}\mathscr{W}(\varepsilon^{-1}\Theta(x),\varepsilon^{-1}t)\widetilde{\varphi}(x).$$

Obviously, $w_\varepsilon(x,t)$ is an admissible extension of $\Phi_\varepsilon(x)$; therefore,

$$\mathscr{I}_\sigma[\Phi_\varepsilon(x)]=\frac{\langle(-\Delta)^s_\Omega\Phi_\varepsilon,\Phi_\varepsilon\rangle}{\||x|^{\sigma-s}\Phi_\varepsilon(x)\|^2_{L_{2^*_\sigma}(\Omega)}}\leqslant\frac{\int_0^{+\infty}\int_\Omega t^{1-2s}|\nabla w_\varepsilon(x,t)|^2\,dx\,dt}{\||x|^{\sigma-s}\Phi_\varepsilon(x)\|^2_{L_{2^*_\sigma}(\Omega)}}. \tag{16}$$

From (10) and (11) we derive the following estimates for the numerator and denominator on the right-hand side of (16) (the second estimate essentially uses the fact that $\alpha\in[1,n-2s+3)$):

$$\int_\Omega\frac{|\Phi_\varepsilon(x)|^{2^*_\sigma}}{|x|^{(s-\sigma)2^*_\sigma}}\,dx=1-\mathscr{A}_1(\varepsilon)\cdot(1+o_\varepsilon(1)+o_\delta(1)), \tag{17}$$

$$\mathscr{E}_s[w_\varepsilon]=\mathscr{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)+\mathscr{A}_2(\varepsilon)\cdot(1+o_\varepsilon(1)+o_\delta(1))-\frac{2\mathscr{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)}{2^*_\sigma}\mathscr{A}_1(\varepsilon)\cdot(1+o_\varepsilon(1)), \tag{18}$$

where $\mathscr{A}_1(\varepsilon),\mathscr{A}_2(\varepsilon)<0$ and, for fixed δ and $\varepsilon\to 0$,

$$\mathscr{A}_1(\varepsilon)\sim C_1\varepsilon^{-1}f(\varepsilon),\quad \mathscr{A}_2(\varepsilon)\sim C_2\varepsilon^{-1}f(\varepsilon),\qquad C_1,C_2>0.$$

Therefore, for sufficiently small δ and ε, we have

$$\mathscr{I}_\sigma[\Phi_\varepsilon(x)]\leqslant\frac{\mathscr{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)+\mathscr{A}_2(\varepsilon)\cdot(1+o_\varepsilon(1)+o_\delta(1))-\frac{2\mathscr{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)}{2^*_\sigma}\mathscr{A}_1(\varepsilon)\cdot(1+o_\varepsilon(1))}{(1-\mathscr{A}_1(\varepsilon)\cdot(1+o_\varepsilon(1)+o_\delta(1)))^{2/2^*_\sigma}}$$
$$=\mathscr{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)+\mathscr{A}_2(\varepsilon)\cdot(1+o_\varepsilon(1)+o_\delta(1))<\mathscr{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+).$$

Thus, (15) is not fulfilled and a minimizer exists, which proves Theorem 3.

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St.-Petersburg State University, St.-Petersburg, Russia
e-mail: ustinns@yandex.ru

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