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SCALING ENTROPY OF UNSTABLE SYSTEMS

Abstract. In this paper, we study the slow entropy-type invariant of a dynamic system proposed by A. M. Vershik. We provide an explicit construction of a system that has an empty class of scaling entropy sequences. For this unstable case, we introduce an upgraded notion of the invariant, generalize the subadditivity results, and provide an exhaustive series of examples.

§1. Introduction

The classical notion of Kolmogorov–Sinai entropy is based on the dynamics of measurable partitions of a measure space. For the case of zero entropy systems, A. M. Vershik proposed (see [1, 2]) a new approach based on the dynamics of functions of several variables. In topological dynamics, a metric space is usually fixed, and one considers invariant measures on it. In contrast to that, we will implement Vershik's approach, which is the following. We fix an automorphism of a measure space and vary a measurable metric (semimetric) on the space. For some metric and sufficiently small $\varepsilon > 0$, we consider the sequence of the *ε-entropies* of averaged metrics. This family of sequences increases in ε and often has a limit as ε goes to zero. If this limit exists, we say that the system is *stable*. The limit itself does not depend on the choice of the metric (see [4]) and is proved to be subadditive (see [5]). Note that similar constructions (*measure–theoretic complexity*) were considered in [7] and [8].

In this paper, we show that this limit does not necessarily exist in the classical sense. However, the invariant can be extended to the general case (see Sec. 3). We also prove its subadditivity and construct a complete family of examples.

Now let us proceed to the formal definitions.

Key words and phrases: zero entropy, scaling entropy, subadditivity, nonstable systems.

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Let (X, μ, ρ) be a standard probability measure space endowed with a measurable semimetric ρ on X, meaning that $\rho(x, y)$ is a symmetric non-negative measurable function on (X^2, μ^2) satisfying the triangle inequality. For a given positive ε, we define the ε-*entropy* of (X, μ, ρ) in the following way.

Definition 1. *Let k be the minimum positive integer such that X decomposes into a union of measurable sets $X_0, X_1, \dots, X_k$ with $\mu(X_0) < \varepsilon$ and $\operatorname{diam}_\rho(X_i) < \varepsilon$ for all $i > 0$. Set*

$$\mathbb{H}_\varepsilon(X, \mu, \rho) = \log_2 k.$$

If there is no such k, set $\mathbb{H}_\varepsilon(X, \mu, \rho) = +\infty$.

Assume that for some semimetric ρ its ε-entropies are finite for all positive ε. In [3] it is shown that this property is equivalent to the separability of ρ on a set of full measure. In this case, the semimetric is said to be *admissible*.

Now let T be an invertible measure-preserving transformation of the standard measure space (X, μ). For $n \in \mathbb{N}$, denote by $T^n_{\mathrm{av}}\rho$ the T-averaged semimetric:

$$T^n_{\mathrm{av}}\rho(x, y) = \frac{1}{n}\sum_{k=0}^{n-1} \rho(T^k x, T^k y), \quad x, y \in X.$$

Clearly, if ρ is admissible, then $T^n_{\mathrm{av}}\rho$ is admissible too.

Another condition requires the semimetric to be nontrivial. We say that ρ is *generating* if there exists a set $X' \subset X$ of full measure such that for any $x, y \in X'$ there is $n \in \mathbb{N}$ with $T^n_{\mathrm{av}}\rho(x, y) > 0$. For example, any measurable metric is generating.

Consider the following function:

$$\Phi_\rho(n, \varepsilon) = \mathbb{H}_\varepsilon(X, \mu, T^n_{\mathrm{av}}\rho), \quad n \in \mathbb{N}, \quad \varepsilon > 0.$$

Note that $\Phi_\rho(n, \varepsilon) < +\infty$ for all ε and n provided that ρ is admissible. In general, the function Φ depends on ε, n, and the semimetric ρ. However, its asymptotic behavior in a sense does not depend on ε and ρ, i.e., it is an isomorphism invariant of the dynamical system. The following result is proved in [4].

Theorem (Zatitskiy, 2015)**.** *Let T be an automorphism of a standard measure space (X, μ). Suppose that for some admissible generating semimetric*

ρ on (X, μ) there is a sequence $\{h_n\}$ such that for all sufficiently small $\varepsilon > 0$

$$\Phi_\rho(n, \varepsilon) \asymp h_n.$$

Then for any admissible generating semimetric ω and any ε small enough,

$$\Phi_\omega(n, \varepsilon) \asymp h_n.$$

Here and in what follows, for two sequences $\phi(n)$ and $\psi(n)$, the relation $\phi \asymp \psi$ means that there exist two positive constants c and C such that $c\phi(n) \leqslant \psi(n) \leqslant C\phi(n)$ for all $n \in \mathbb{N}$.

Definition 2. *In this case, we call the sequence h_n* a scaling entropy sequence *of (X, μ, T), and the system itself is said to be* stable.

Some important properties of the system can be described in terms of its scaling sequence. For example, if the Kolmogorov–Sinai entropy is positive, then one can choose $h_n = n$ as a scaling entropy sequence of the system. In [3] it is shown that the scaling sequence is bounded if and only if the automorphism has a pure point spectrum. In [5] F. V. Petrov and P. B. Zatitskiy proved that if a scaling sequence exists, then it can be chosen to be increasing and subadditive. Conversely, there exists a stable ergodic system with a given increasing subadditive scaling entropy sequence, i. e., the complete classification of possible scaling sequences was obtained in the *stable case*.

Until now, it was unclear whether or not unstable systems exist. In this paper, we give a positive answer to this question and generalize the scaling entropy sequence invariant to the general case. Also, we construct an exhaustive family of examples.

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§2. Construction of an unstable system

Theorem 1. *There exists an ergodic system (X, μ, T) and an admissible semimetric ρ on X such that the asymptotic behavior of $\mathbb{H}_\varepsilon(X, \mu, T^n_{\mathrm{av}}\rho)$ essentially depends on ε, meaning that for any $\varepsilon > 0$ there exists δ such that $\varepsilon > \delta > 0$ and*

$$\mathbb{H}_\varepsilon(X, \mu, T^n_{\mathrm{av}}\rho) \lnsim \mathbb{H}_\delta(X, \mu, T^n_{\mathrm{av}}\rho).$$

Here $\phi \lesssim \psi$ means that there is a positive constant C such that $\phi(n) < C\psi(n)$ for all $n \in \mathbb{N}$. We write $\phi \lnsim \psi$ if $\phi \lesssim \psi$ but not $\phi \asymp \psi$.

Proof. Let $\mathcal{A} = \{\phi_m\}_{m=1}^{\infty}$ be an infinite family of subadditive functions on $\mathbb{N}$ such that $\phi_m \lnsim \phi_{m+1}$. One could choose $\phi_m(n) = \log^m(n)$, for instance. By [6], there is a family of corresponding ergodic systems $S_m = (X_m, \mu_m, T_m)$ such that for all m, any admissible generating semimetric ρ_m on (X_m, μ_m), and all positive ε small enough,

$$\mathbb{H}_\varepsilon(X_m, \mu_m, T_{\mathrm{av}}^n \rho_m) \asymp \phi_m(n).$$

For each $m \in \mathbb{N}$, we fix an admissible generating semimetric $\rho_m \leqslant 1$ on (X_m, μ_m). Then we define a system $\widetilde{U}_{\mathcal{A}}$ as the product of S_m:

$$\widetilde{U}_{\mathcal{A}} = \left(\prod_{m=1}^{\infty} X_m, \prod_{m=1}^{\infty} \mu_m, \prod_{m=1}^{\infty} T_m\right),$$

where the automorhism $\prod\limits_{m=1}^{\infty} T_m$ acts independently on each factor. Now, by the ergodic decomposition theorem, there exists an ergodic measure μ on $\prod\limits_{m=1}^{\infty} X_m$ such that all its coordinate projections coincide with the initial measures μ_m. Let us change the measure to obtain an ergodic system

$$U_{\mathcal{A}} = \left(\prod_{m=1}^{\infty} X_m, \mu, \prod_{m=1}^{\infty} T_m\right).$$

Define a semimetric ρ on the product space as follows: for all $x, y \in \prod\limits_{m=1}^{\infty} X_m$,

$$\rho(x, y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \rho_m(x_m, y_m).$$

Clearly, ρ is generating, and, as we will show below, its ε-entropies are finite for all ε. Therefore, ρ is admissible and generating.

Lemma 1. *Let $U_{\mathcal{A}}$ and ρ be the product system and the semimetric described above. Then*

$$\mathbb{H}_\varepsilon(U_{\mathcal{A}}, T_{\mathrm{av}}^n \rho) \leqslant \sum_{m=1}^{R(\varepsilon)} \mathbb{H}_{\frac{\varepsilon}{2R(\varepsilon)}}(X_m, \mu_m, (T_m)_{\mathrm{av}}^n \rho_m), \quad \varepsilon > 0,$$

where $R(\varepsilon) = -\log(\varepsilon)$.

Proof. Let us fix some $n \in \mathbb{N}$ and $\varepsilon > 0$. For each X_m consider its decomposition into subsets $A_0^{(m)}, \ldots, A_{k_m}^{(m)}$ such that $\mu\left(A_0^{(m)}\right) < \frac{\varepsilon}{2R(\varepsilon)}$ and $\mathrm{diam}_{T_{\mathrm{av}}^n \rho_m}\left(A_i^{(m)}\right) < \frac{\varepsilon}{2R(\varepsilon)}$ for $i > 0$. Here

$$\log k_m = \mathbb{H}_{\frac{\varepsilon}{2R(\varepsilon)}}(X_m, \mu_m, (T_m)_{\mathrm{av}}^n \rho_m).$$

Let π_m be the standard projection onto X_m. The construction of μ implies that π_m is measure-preserving. Denote

$$\widehat{A}_i^{(m)} = \pi_m^{-1}\left(A_i^{(m)}\right).$$

Define a new error set $K_0 = \bigcup_{m=1}^{R} \widehat{A}_0^{(m)}$, where $R = R(\varepsilon) = -\log(\varepsilon)$. Clearly,

$$\mu(K_0) \leqslant \sum_{m=1}^{R} \widehat{A}_0^{(m)} \leqslant \frac{\varepsilon}{2}.$$

For every $J = (j_1, \ldots, j_R)$, where j_m lies in $\{1, \ldots, k_m\}$, define

$$K_J = \bigcap_{m=1}^{R} \widehat{A}_{j_m}^{(m)} \setminus K_0.$$

Note that

$$T_{\mathrm{av}}^n \rho(x, y) = \sum_{m=1}^{\infty} \frac{1}{2^m} T_{\mathrm{av}}^n \rho_m(x, y).$$

Therefore,

$$\begin{aligned}
\mathrm{diam}_{T_{\mathrm{av}}^n \rho}(K_J) &\leqslant \sum_{m=1}^{\infty} \frac{1}{2^m} \mathrm{diam}_{T_{\mathrm{av}}^n \rho_m} \pi_m(K_J) \\
&\leqslant \sum_{m=1}^{R} \frac{1}{2^m} \mathrm{diam}_{T_{\mathrm{av}}^n \rho_m} A_{j_m}^{(m)} + \sum_{m=R+1}^{\infty} \frac{1}{2^m} \mathrm{diam}_{T_{\mathrm{av}}^n \rho_m} X_m \\
&\leqslant R \cdot \frac{\varepsilon}{2R} + 2^{-R-1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

So, we have constructed a partition $\mathcal{K} = \{K_J\}_J \cup \{K_0\}$ such that $\mu(K_0) < \varepsilon$ and $\mathrm{diam}_{T_{\mathrm{av}}^n \rho} K_J < \varepsilon$. The cardinality of $\mathcal{K}$ does not exceed $k_1 \cdot k_2 \cdot \ldots \cdot k_R + 1$. Thus,

$$\mathbb{H}_\varepsilon(U_{\mathcal{A}}, T_{\mathrm{av}}^n \rho) \leqslant \log(|\mathcal{K}| - 1) \leqslant \sum_{m=1}^{R(\varepsilon)} k_m = \sum_{m=1}^{R(\varepsilon)} \mathbb{H}_{\frac{\varepsilon}{2R(\varepsilon)}}(X_m, \mu_m, (T_m)_{\mathrm{av}}^n \rho_m),$$

as required. $\square$

Now, assume that $U_{\mathcal{A}}$ is stable, i. e., there exists some $\varepsilon_0 > 0$ such that for any $\varepsilon \leqslant \varepsilon_0$ the following equivalence holds:

$$\mathbb{H}_\varepsilon(X, \mu, T_{\text{av}}^n \rho) \asymp \mathbb{H}_{\varepsilon_0}(X, \mu, T_{\text{av}}^n \rho).$$

By Lemma 1, one has

$$\mathbb{H}_\varepsilon(X, \mu, T_{\text{av}}^n \rho) \lesssim \sum_{k=1}^{R(\varepsilon_0)} \mathbb{H}_{\frac{\varepsilon_0}{2R(\varepsilon_0)}}(X_k, \mu_k, (T_k)_{\text{av}}^n) \lesssim \phi_{R(\varepsilon_0)}(n). \tag{1}$$

Let h_n be a scaling entropy sequence of the system. By the previous formula, h_n does not exceed $\phi_{R(\varepsilon_0)}(n)$ asymptotically. However, in [6] it is proved that a scaling entropy sequence of a system grows not slower than the entropy sequence of a factor system. This implies that for any m

$$h_n \gtrsim \phi_m(n).$$

One can, for instance, choose $m = R(\varepsilon_0) + 1$ and obtain a contradiction. Therefore, our assumption is false, and the system $U_{\mathcal{A}}$ is not stable. $\square$

§3. Invariance

The purpose of this section is to generalize the notion of the scaling entropy sequence invariant to the general (unstable) case. In this case, it will be some equivalence class of functions of two variables. Let us define the equivalence relation.

Definition 3. *Let $\Phi, \Psi \colon \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ be two functions that are decreasing with respect to their second arguments. We will write $\Phi \preceq \Psi$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\Phi(n, \varepsilon) \lesssim \Psi(n, \delta).$$

We will say that Φ and Ψ are equivalent if $\Phi \preceq \Psi$ and $\Psi \preceq \Phi$. In this case, we will write $\Phi \sim \Psi$.

Clearly, the relation $\preceq$ is a partial order on the set of equivalence classes. We will denote by $[\Phi]$ the class of a function Φ.

Assume that a measure-preserving system (X, μ, T) with an admissible semimetric ρ is given. Let us define the *scaling entropy of the system (X, μ, T) with respect to the semimetric ρ* as the following equivalence class:

$$\mathcal{H}(X, \mu, T, \rho) = \Big[\mathbb{H}_\varepsilon(X, \mu, T_{\text{av}}^n \rho)\Big].$$

In fact, this class does not depend on the choice of a generating admissible semimetric. Moreover, in [4] the following theorem is proved.

Theorem (Zatitskiy, 2015)**.** *Let (X,μ,T) be a measure-preserving system. Assume that ρ is an admissible generating semimetric with a finite integral over (X^2,μ^2). Then for any admissible semimetric ω with a finite integral and any $\varepsilon>0$ there exist positive c and δ such that*

$$\mathbb{H}_\varepsilon(X,\mu,T^n_{\mathrm{av}}\omega) \leqslant c\mathbb{H}_\delta(X,\mu,T^n_{\mathrm{av}}\rho).$$

Corollary 1 (invariance)**.** *Let ρ and ω be two admissible generating semimetrics with finite integrals. Then*

$$\mathcal{H}(X,\mu,T,\rho)=\mathcal{H}(X,\mu,T,\omega).$$

This corollary allows us to give the following definition.

Definition 4. *The* scaling entropy of the system (X,μ,T) *is the following class:*

$$\mathcal{H}(X,\mu,T)=\mathcal{H}(X,\mu,T,\rho),$$

where ρ is an arbitrary admissible generating semimetric with a finite integral.

An important example of such a semimetric is the cut semimetric corresponding to a (countable) generating partition. Another useful corollary gives an estimate of the scaling entropy of a factor system.

Corollary 2. *Let $\left(\widehat{X},\widehat{\mu},\widehat{T}\right)$ be a factor of a measure-preserving system (X,μ,T). Then*

$$\mathcal{H}\left(\widehat{X},\widehat{\mu},\widehat{T}\right) \preceq \mathcal{H}(X,\mu,T).$$

§4. Subadditivity

In this section, we prove that the scaling entropy of a system is increasing and subadditive with respect to n. Conversely, every increasing in n subadditive function (decreasing in ε) can be obtained as the scaling entropy of some automorphism.

Definition 5. *We say that a function $\Phi(n,\varepsilon)$ is* subadditive *if for all $\varepsilon>0$ and for any $k,m\in\mathbb{N}$,*

$$\Phi(k+m,\varepsilon)\leqslant\Phi(k,\varepsilon)+\Phi(m,\varepsilon).$$

Theorem 2. *Let (X, μ, T) be a measure-preserving system. Then there exists a subadditive function $\Phi(n, \varepsilon)$ increasing in n and decreasing in ε such that*

$$\Phi \in \mathcal{H}(X, \mu, T).$$

Proof. We will use the following estimates proved in [5].

Lemma 2 (Petrov, Zatitskiy, 2015)**.** *Let $\rho_1, \ldots, \rho_k$ be admissible semimetrics on a measure space (X, μ) with $\rho_i \leqslant 1$ for all $i \leqslant k$.*

(1) *Suppose that $\varepsilon > 0$ and $\mathbb{H}_\varepsilon(X, \mu, \rho_i) > 0$ for all $i \leqslant k$. Then*

$$\mathbb{H}_{2\sqrt{\varepsilon}}\left(X, \mu, \frac{1}{k}\sum_{i=1}^{k} \rho_i\right) \leqslant 2\sum_{i=1}^{k} \mathbb{H}_\varepsilon(X, \mu, \rho_i).$$

(2) *There exists $m \leqslant k$ such that*

$$\mathbb{H}_{2\sqrt{\varepsilon}}(X, \mu, \rho_m) \leqslant \mathbb{H}_\varepsilon\left(X, \mu, \frac{1}{k}\sum_{i=1}^{k} \rho_i\right).$$

Now, let $\rho \leqslant 1$ be an admissible generating semimetric. Denote

$$\Psi(m, \varepsilon) = \mathbb{H}_\varepsilon(X, \mu, T_{\mathrm{av}}^m \rho).$$

Proposition 1. *The following inequalities hold.*

(1) *For all $k, n \in \mathbb{N}$ and positive ε small enough,*

$$\Psi(kn, \varepsilon) \leqslant 2k\Psi\left(n, \frac{\varepsilon^2}{4}\right).$$

(2) *For all $k, n \in \mathbb{N}$ with $k \leqslant n$,*

$$\Psi(n, \varepsilon) \geqslant \Psi\left(k, 2\sqrt{2\varepsilon}\right).$$

Proof. Let us prove the first inequality. Note that for $\varepsilon < \frac{1}{3}\int_{X^2} \rho$ the ε-entropy of ρ is positive. Also, any averaged semimetric $T_{\mathrm{av}}^m \rho$ has the same integral as the initial one; therefore, its ε-entropy is positive too. Let us fix such $\varepsilon > 0$ and some $k, n \in \mathbb{N}$. For $i \leqslant k$ define

$$\rho_i = T^{(i-1)n}\, T_{\mathrm{av}}^n \rho.$$

Clearly,

$$T_{\mathrm{av}}^{kn} \rho = \frac{1}{k}\sum_{i=1}^{k} \rho_i.$$

Applying Lemma 2, we obtain

$$\mathbb{H}_\varepsilon\big(X,\mu,T_{\mathrm{av}}^{kn}\rho\big) \leqslant 2\sum_{i=1}^{k}\mathbb{H}_{\frac{\varepsilon^2}{4}}(X,\mu,\rho_i) = 2k\mathbb{H}_{\frac{\varepsilon^2}{4}}(X,\mu,T_{\mathrm{av}}^{n}\rho).$$

The first part is proved.

Let us proceed to the second inequality. Let $n = km + r$, where $r < k$. Note that $r \leqslant \frac{n}{2}$. Then

$$T_{\mathrm{av}}^{n}\rho \geqslant \frac{1}{n}\sum_{i=0}^{km}T^i\rho \geqslant \frac{n-r}{n}\frac{1}{km}\sum_{i=0}^{km}T^i\rho \geqslant \frac{1}{2}T_{\mathrm{av}}^{km}\rho.$$

Thus,

$$\mathbb{H}_\varepsilon(X,\mu,T_{\mathrm{av}}^{n}\rho) \geqslant \mathbb{H}_\varepsilon\left(X,\mu,\frac{1}{2}T_{\mathrm{av}}^{km}\rho\right) \geqslant \mathbb{H}_{2\varepsilon}\big(X,\mu,T_{\mathrm{av}}^{km}\rho\big).$$

Now apply the second part of Lemma 2 for the semimetrics $\rho_i = T^{(i-1)n}T_{\mathrm{av}}^{n}\rho$. We obtain

$$\mathbb{H}_\varepsilon(X,\mu,T_{\mathrm{av}}^{n}\rho) \geqslant \mathbb{H}_{2\varepsilon}\big(X,\mu,T_{\mathrm{av}}^{km}\rho\big) \geqslant \mathbb{H}_{2\sqrt{2\varepsilon}}\big(X,\mu,T_{\mathrm{av}}^{k}\rho\big). \qquad \square$$

Lemma 3. *Let $\eta(n)$, $\phi(n)$, and $\psi(n)$, $n \in \mathbb{N}$, be sequences of nonnegative real numbers. Suppose that*

$$\phi(kn) \leqslant k\psi(n) \quad \textit{for } k,n \in \mathbb{N} \tag{2}$$

and

$$\phi(n) \geqslant \eta(k) \quad \textit{for } k \leqslant n. \tag{3}$$

Then there exists an increasing subadditive function $\theta(n)$ such that

$$\eta(n) \leqslant \theta(n) \leqslant 2\psi(n).$$

Proof. Let

$$\widehat{\phi}(n) = \inf_{m\geqslant n}\phi(m) \leqslant \phi(n).$$

It is clear that $\widehat{\phi}$ is increasing and, by (3), for any $k \leqslant n$

$$\widehat{\phi}(n) \geqslant \eta(k).$$

Also, by (2), for all $k,n \in \mathbb{N}$

$$\widehat{\phi}(kn) \leqslant \phi(kn) \leqslant k\psi(n).$$

Define $\widehat{\theta}$ as follows:

$$\widehat{\theta}(n) = \sup_{k>0}\frac{\widehat{\phi}(kn)}{k} \leqslant \psi(n).$$

Obviously, $\widehat{\theta}(n) \geqslant \widehat{\phi}(n)$. Note that $\widehat{\theta}$ increases, because $\widehat{\phi}$ increases, and

$$\widehat{\theta}(kn) = \sup_{m>0} \frac{\widehat{\phi}(mkn)}{m} = k \sup_{m>0} \frac{\widehat{\phi}(mkn)}{mk} \leqslant k \sup_{l>0} \frac{\widehat{\phi}(ln)}{l} = k\widehat{\theta}(n).$$

Now define θ as

$$\theta(n) = n \sup_{m \geqslant n} \frac{\widehat{\theta}(m)}{m} \geqslant \widehat{\theta}(n) \geqslant \widehat{\phi}(n) \geqslant \eta(n).$$

First, θ is increasing. Indeed,

$$\begin{aligned}\theta(n) &= \max\left(\widehat{\theta}(n), n \sup_{m \geqslant n+1} \frac{\widehat{\theta}(m)}{m}\right)\\ &\leqslant \max\left(\widehat{\theta}(n+1), (n+1) \sup_{m \geqslant n+1} \frac{\widehat{\theta}(m)}{m}\right) = \theta(n+1).\end{aligned}$$

Second, θ is subadditive:

$$\theta(k+n) = (k+n) \sup_{m \geqslant k+n} \frac{\widehat{\theta}(m)}{m} \leqslant k \sup_{m \geqslant k} \frac{\widehat{\theta}(m)}{m} + n \sup_{m \geqslant n} \frac{\widehat{\theta}(m)}{m} = \theta(k) + \theta(n).$$

It only remains to show that $\theta(n) \leqslant 2\psi(n)$. Indeed,

$$\begin{aligned}\theta(n) = n \sup_{m \geqslant n} \frac{\widehat{\theta}(m)}{m} &\leqslant n \sup_{m \geqslant n} \frac{\widehat{\theta}\big(n\left[\frac{m}{n}\right] + n\big)}{m}\\ &\leqslant \sup_{m \geqslant n} \frac{n\left[\frac{m}{n}\right] + n}{m}\, \widehat{\theta}(n) \leqslant 2\psi(n). \qquad \square\end{aligned}$$

Now let us complete the proof of the theorem. For a fixed $\varepsilon > 0$, we use Lemma 3 with $\eta(n) = \Psi\big(n, 2\sqrt{2\varepsilon}\big)$, $\phi(n) = \Psi(n, \varepsilon)$, and $\psi(n) = 2\Psi\Big(n, \frac{\varepsilon^2}{4}\Big)$; conditions (2) and (3) of Lemma 3 are guaranteed by Proposition 1. We obtain an increasing subadditive function $\Theta(n, \varepsilon)$ such that

$$\Psi\Big(n, 2\sqrt{2\varepsilon}\Big) \leqslant \Theta(n, \varepsilon) \leqslant 4\Psi\left(n, \frac{\varepsilon^2}{4}\right). \tag{4}$$

It only remains to make it decreasing in ε. To do this, let us find a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ such that $\Theta(n, \varepsilon_k) < 4\Theta(n, \varepsilon_{k+1})$ for all $k, n \in \mathbb{N}$ and ε_k tends to 0. This can always be done due to inequality (4). For $\varepsilon > 0$, denote by $\gamma(\varepsilon)$ the smallest k such that $\varepsilon_k < \varepsilon$. Then set

$$\Phi(n, \varepsilon) = 4^k \Theta\big(n, \varepsilon_{\gamma(\varepsilon)}\big).$$

It is easy to see that $\Phi \sim\Theta \sim \Psi$; therefore, $\Phi \in \mathcal{H}(X, \mu, T)$, as required. $\square$

The next theorem gives the opposite result. It completes the description of the possible values of the invariant.

Theorem 3. *Let $\Phi(n, \varepsilon)$ be a subadditive function of two variables increasing in n and decreasing in ε. Then there exists a measure-preserving system (X, μ, T) such that*

$$\Phi \in \mathcal{H}(X, \mu, T).$$

Proof. We will use the construction of an unstable system described in Sec. 2. Let

$$\phi_m(n) = \Phi\left(n, \frac{1}{m}\right).$$

Note that $\phi_m(\cdot)$ is increasing and subadditive. Denote $\mathcal{A} = \{\phi_m\}_{m=1}^{\infty}$. Let us construct the system $U_{\mathcal{A}}$ and show that $\Phi \in \mathcal{H}(U_{\mathcal{A}})$. Lemma 1 gives an upper bound for any fixed $\varepsilon > 0$:

$$\mathbb{H}_\varepsilon(U_{\mathcal{A}}, T^n_{\text{av}}\rho) \leqslant \sum_{k=1}^{R(\varepsilon)} \mathbb{H}_{\frac{\varepsilon}{2R(\varepsilon)}}(X_k, \mu_k, (T_k)^n_{\text{av}}\,\rho_k) \lesssim \phi_{R(\varepsilon)}(n).$$

Therefore,

$$\mathcal{H}(U_{\mathcal{A}}) \preceq \Phi.$$

However, $U_{\mathcal{A}}$ has a stable factor (A_m, μ_m, T_m). Then for all $m \geqslant 1$

$$\mathcal{H}(U_{\mathcal{A}}) \succeq \phi_m(n).$$

Thus, $\mathcal{H}(U_{\mathcal{A}}) \ni \Phi$, as required. $\square$

§5. On the minimality of the relation

It might seem that the equivalence relation we defined is too strong. The question is whether δ in Definition 3 can be improved. Can δ be controlled by some function of ε? Note that the proof of the subadditivity (Theorem 2) requires $\delta = 4\varepsilon^2$ only. If there exists a weaker equivalence relation, then the invariant can separate more systems. However, the following theorem claims that our relation is sharp in this sense.

Theorem 4. *Let $f\colon (0,1) \to (0,1)$ be an increasing function. Then there exists a measure-preserving system (X, μ, T) and two admissible generating semimetrics ρ and ω on X such that for any $\varepsilon_0 > 0$ there is ε such that $0 < \varepsilon < \varepsilon_0$ and*

$$\mathbb{H}_\varepsilon(X, \mu, T^n_{\text{av}}\omega) \gtrsim\!\!\!\!/\; \mathbb{H}_{f(\varepsilon)}(X, \mu, T^n_{\text{av}}\rho). \tag{5}$$

Proof. Let us first prove the following proposition.

Proposition 2. *Let $h\colon (0,1) \to (0,1)$ be some map. Then there exists $\kappa \in (0,1)$ such that there is an increasing sequence in $h^{-1}(\kappa,1)$.*

The proof of the proposition is clear. Indeed, any subset of the real line that contains no infinite increasing sequence is countable.

Now let $\{\psi_\alpha(n)\},\ \alpha \in (0,1)$, be a family of increasing subadditive functions such that $\psi_{\alpha_1}(n) \lnsim \psi_{\alpha_2}(n)$ for any $\alpha_1 < \alpha_2$. For example, one can use $\psi_\alpha(n) = \log^{1+\alpha}(n)$. Consider the corresponding stable measure-preserving systems $(Y_\alpha, \nu_\alpha, R_\alpha)$ and admissible generating semimetrics $\tau_\alpha \leqslant 1$ such that for any positive $\varepsilon < h(\alpha)$

$$\mathbb{H}_\varepsilon(Y_\alpha, \nu_\alpha, (R_\alpha)^n_{\text{av}}\tau_\alpha) \asymp \psi_\alpha(n).$$

By Proposition 2, we obtain some $\kappa > 0$ and an increasing sequence $\{\alpha_m\}_{m=1}^\infty$ such that $h(\alpha_m) > \kappa$ for all $m > 0$. Let us define $\phi_m = \psi_{\alpha_m}$ and let $\mathcal{A}$ be the family of ϕ_m. Now let us construct $U_\mathcal{A}$ as in Sec. 2 using $(Y_{\alpha_m}, \nu_{\alpha_m}, R_{\alpha_m}) = (X_m, \mu_m, T_m)$ as the coordinate factors.

Define ρ as the standard semimetric on $U_\mathcal{A}$:

$$\rho(x,y) = \sum_{m=1}^\infty \frac{1}{2^m}\rho_m(x_m, y_m),$$

where $\rho_m = \tau_{\alpha_m}$. We will look for ω in a similar linear form:

$$\omega(x,y) = \sum_{m=1}^\infty C_m\rho_m(x_m, y_m),$$

where $1 > C_m > 0$ and $\sum_{m=1}^\infty C_m$ is finite. Note that any semimetric of this type is always generating and admissible.

For all $m,\ n \geqslant 1$,

$$\mathbb{H}_\varepsilon(U_\mathcal{A}, T^n_{\text{av}}\omega) \geqslant \mathbb{H}_\varepsilon(U_\mathcal{A}, C_m T^n_{\text{av}}\rho_m) = \mathbb{H}_\varepsilon(X_m, \mu_m, C_m (T_m)^n_{\text{av}}).$$

Using the fact that $C_m \leqslant 1$, we obtain

$$\mathbb{H}_\varepsilon(X_m, C_m T^n_{\text{av}}\rho_m) \geqslant \mathbb{H}_{\frac{\varepsilon}{C_m}}(X_m, T^n_{\text{av}}\rho_m) \asymp \phi_m(n),$$

while $\frac{\varepsilon}{C_m} \leqslant \kappa$. Therefore,

$$\mathbb{H}_\varepsilon(U_\mathcal{A}, T^n_{\text{av}}\omega) \gtrsim \phi_m(n)$$

for all m such that $C_m \geqslant \kappa^{-1}\varepsilon$. Let $K(\varepsilon)$ be the largest such m.

However, Lemma 1 gives an upper bound on the ε-entropy of ρ:

$$\mathbb{H}_\varepsilon(U_\mathcal{A}, T^n_{\text{av}}\rho) \lesssim \phi_{R(\varepsilon)}(n),$$

where $R(\varepsilon) = -\log \varepsilon$. Hence,

$$\mathbb{H}_{f(\varepsilon)}(U_{\mathcal{A}}, T^n_{\mathrm{av}}\rho) \lesssim \phi_{R(f(\varepsilon))}(n).$$

Inequality (5) holds for every ε such that

$$K(\varepsilon) > R(f(\varepsilon)).$$

And this is true when

$$C_{R(f(\varepsilon))+1} > \kappa^{-1}\varepsilon.$$

Now let $\varepsilon_p = \kappa 2^{-p}$ for $p \in \mathbb{N}$ and set $C_{R(f(\varepsilon_p))+1} = 2\kappa^{-1}\varepsilon_p \leqslant 1$. If some C_m is not defined yet, set $C_m = 2^{-m}$. Clearly, the obtained semimetric ω is as required. □

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