NON-EXISTENCE OF
A UNIVERSAL ZERO ENTROPY SYSTEM
FOR NON-PERIODIC AMENABLE GROUP ACTIONS*

BY

GEORGII VEPREV

Leonhard Euler International Mathematical Institute in St. Petersburg
14th Line 29B, Vasilyevsky Island, St. Petersburg, 199178, Russia
e-mail: egor.veprev@mail.ru

ABSTRACT

Let G be a non-periodic amenable group. We prove that there does not
exist a topological action of G for which the set of ergodic invariant mea-
sures coincides with the set of all ergodic measure-theoretic G-systems of
entropy zero. Previously J. Serafin, answering a question by B. Weiss,
proved the same for $G = \mathbb{Z}$.

1. Introduction

In this work, we generalize the result of [8] to the case of a non-periodic amenable
group G. Namely, we prove that there does not exist a topological zero entropy
system which is universal for ergodic measure-theoretic G-actions with zero
entropy. The main tool that we implement in the proof is the notion of scaling

entropy which was proposed in works by A. M. Vershik [12, 13]. The theory of scaling entropy was developed in [7, 14, 18, 19]. In this work, we prove the lower bound for the ε-entropy of an averaging of independent metrics (Lemma 5) which allows us to estimate a scaling entropy for a special series of examples of G-actions. The existence of such a series (see Definition 4) implies the absence of a universal zero entropy system.

1.1. CLASSICAL NOTIONS. Let us recall several basic notions of the entropy theory of dynamical systems (see, e.g., [4]). A countable group G is called **amenable** if it satisfies the Følner condition, meaning that there is a sequence $\{F_n\}_{n=1}^{\infty}$ of finite subsets of G such that for any $g \in G$

$$\lim_{n \to \infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0.$$

In this case, the sequence is called a left Følner sequence. A right Følner sequence can be defined in a similar way. We will consider left actions of amenable groups on Lebesgue measure spaces without point masses, i.e., measure spaces that are isomorphic to the unit segment with the Lebesgue measure.

1.1.1. *Amenable Topological Entropy.* Let an amenable group G act by homeomorphisms on a compact metric space (X, d). The amenable topological entropy of this action can be defined in the following way:

$$h_{\text{top}}(X, G) = \sup_{\varepsilon > 0} \limsup_{n \to +\infty} \frac{1}{|F_n|} \log \text{spn}(d, F_n, \varepsilon)$$

$$= \sup_{\varepsilon > 0} \limsup_{n \to +\infty} \frac{1}{|F_n|} \log \text{sep}(d, F_n, \varepsilon),$$

where $\text{spn}(d, F_n, \varepsilon)$ and $\text{sep}(d, F_n, \varepsilon)$ are cardinalities of the minimal ε-net and the maximal ε-separated set respectively for the maximized metric

$$G_{max}^n d(x, y) = \max_{g \in F_n} d(gx, gy), \quad x, y \in X.$$

The value of $h_{\text{top}}(X, G)$ does not depend on the choice of Følner sequence and forms a topological invariant of a dynamical system.

1.1.2. *Amenable Measure Entropy.* Assume that G acts by automorphisms on a standard probability space (X, μ). The **entropy** of a measurable partition ξ is defined as the following non-negative value:

$$H(\xi) = - \int_X \log \mu(\xi(x)) \, \mathrm{d}\mu(x),$$

where $\xi(x)$ stands for the cell of ξ that contains a point $x \in X$. For a partition ξ with finite entropy, define its entropy with respect to the measure-preserving action of G:

$$h(\xi) = \lim_{n \to +\infty} \frac{1}{|F_n|} H\left(\bigvee_{g \in F_n} g^{-1} \xi \right),$$

where $\vee$ is the refinement sign. The **amenable measure entropy** of the action is defined by

$$h(X, \mu, G) = \sup\{h(\xi) \colon H(\xi) < +\infty\}.$$

Amenable measure entropy is independent of the choice of Følner sequence and forms an invariant of a measure-preserving system.

1.1.3. *The Variational Principle.* The variational principle is a well-known relation between topological and measure-theoretic entropies. Let $G \curvearrowright (X, d)$ be a continuous action of G on a compact metric space and $M_G(X)$ be a set of all G-invariant Borel probability measures on X. Then $M_G(X)$ is non-empty and

$$h_{\text{top}}(X, G) = \sup_{\mu \in M_G(X)} h(X, \mu, G).$$

1.2. UNIVERSAL SYSTEMS OF ENTROPY ZERO. Questions about the existence of universal dynamical systems in various senses have been studied, for example, in [1, 8, 9, 16, 17]. We will use the following definition (see, e.g., [1, 8]).

Definition 1: A topological system (X, G) is called **universal** for some class $\mathcal{S}$ of ergodic measure-preserving actions of G if the following two conditions hold:

  (1) For any ergodic $\mu \in M_G(X)$ the system (X, μ, G) belongs to $\mathcal{S}$.
  (2) For any $(Y, \nu, G) \in \mathcal{S}$ there exists an invariant measure μ on (X, G) such that (X, μ, G) is measure-theoretically isomorphic to (Y, ν, G).

In [8], the question about the existence of a universal system for all zero entropy systems appears. This question goes back to B. Weiss. Due to the variational principle and the first condition in Definition 1, such a system must have zero topological entropy.

Question 1: Does there exist a system (X, G) with zero topological entropy which is universal for the class of all ergodic measure-preserving actions of zero entropy?[1]

The work by J. Serafin [8] gives the negative answer to Question 1 for the case of the group $\mathbb{Z}$. However, this question is still open for general amenable groups. The approach of [8] is based on the notions of symbolic and measure-theoretic complexity of a dynamical system (see also [2]) and special constructions of systems with rapidly growing measure-theoretic complexity. The author of that work points out that this approach did not work for the case of amenable groups due to insufficient development of the theory of symbolic extensions. Let us remark that the notion of measure-theoretic complexity is closely related to the notion of scaling entropy that we use. The main result of our work is the following theorem, which gives the negative answer to Question 1 in the case of a non-periodic amenable group G.

THEOREM 1: *Let $G \curvearrowright (X, d)$ be a continuous action of a countable non-periodic amenable group G on a compact metric space (X, d). Suppose that for any ergodic measure-preserving dynamical system (Y, ν, G) with zero entropy there exists an invariant measure μ on X with*

$$(X, \mu, G) \cong (Y, \nu, G).$$

Then the topological entropy of (X, d, G) is positive.

2. Scaling entropy

2.1. EPSILON-ENTROPY AND SCALING ENTROPY SEQUENCE. The main tool in the proof of Theorem 1 is the notion of scaling entropy introduced by A. Vershik in [11, 12, 13]. The closely related notion of measure-theoretic complexity appears in the works by S. Ferenczi [2], A. Katok and J.-P. Thouvenot [3] and uses symbolic encoding and Hamming metrics. Vershik's approach is based on the dynamics of functions of several variables, namely admissible semimetrics[2]

---

[1] Note that the notion of a "universal" system is often used in a slightly different sense. Sometimes it is only required to satisfy the second condition in Definition 1. In this case, the question can be easily solved via the famous Krieger's finite generator theorem (see [5]): every ergodic automorphism T with entropy less than one can be realized in the left shift on $\{0, 1\}^{\mathbb{Z}}$.

[2] Occasionally the term "pseudometric" is used instead of "semimetric".

(see [14] for details). The theory of scaling entropy was developed by A. Vershik, F. Petrov, and P. Zatitskiy in [7, 14, 18, 19]. Let us recall the basic concepts and statements of this theory.

Let $\rho \colon (X^2, \mu^2) \to [0, +\infty)$ be a measurable semimetric on a measure space (X, μ). That is, ρ is a non-negative symmetric function which is measurable with respect to μ^2 and satisfies the triangle inequality. For a positive ε, we define its ε-**entropy** as follows. Let k be the minimal positive integer such that X can be represented as a union of measurable subsets $X_0, X_1, \ldots, X_k$ with $\mu(X_0) < \varepsilon$ and $\mathrm{diam}_\rho(X_i) < \varepsilon$ for all $i > 0$. Put

$$\mathbb{H}_\varepsilon(X, \mu, \rho) = \log_2 k.$$

If there is no such finite k, define

$$\mathbb{H}_\varepsilon(X, \mu, \rho) = +\infty.$$

We call a semimetric **admissible** if it is separable on some subset of full measure. Properties of admissible semimetrics are studied in detail in [6, 14]. In particular, it is proved that a semimetric is admissible if and only if all its ε-entropies are finite for all $\varepsilon > 0$.

Suppose that $G \curvearrowright (X, \mu)$ is a measure-preserving action of a countable group G on a Lebesgue space (X, μ). For an element $g \in G$ denote a shifted semimetric $g^{-1}\rho$:

$$g^{-1}\rho(x, y) = \rho(gx, gy),$$

where $x, y \in X$. Evidently, if ρ is admissible, then $g^{-1}\rho$ is admissible as well.

Let us fix a sequence $\lambda = \{S_n\}_{n=1}^{\infty}$ of non-empty finite subsets of G. Here and in what follows, we call it the **equipment** of the group. A measurable semimetric ρ is called **generating** if all its shifts by elements of $\bigcup_n S_n$ together separate points up to a null set. This means that there exists a subset $X_0 \subset X$ of full measure such that for any pair of distinct points $x, y \in X_0$ there is an element $g \in \bigcup_n S_n$ with $g^{-1}\rho(x, y) > 0$. Note that any actual (measurable) metric is always generating. Next, define a semimetric $G_{av}^n \rho$ averaged by S_n in a natural way:

$$G_{av}^n \rho(x, y) = \frac{1}{|S_n|} \sum_{g \in S_n} \rho(gx, gy), \quad x, y \in X.$$

Sometimes we will emphasize the set over which the averaging is taken. In this case, we will write $G_{av}^{S_n} \rho$ instead of $G_{av}^n \rho$.

Consider then the following function:

$$\Phi(n, \varepsilon) = \mathbb{H}_\varepsilon(X, \mu, G_{av}^n \rho).$$

Actually, the function $\Phi(n, \varepsilon)$ depends on n, ε, and semimetric ρ. However, its asymptotic behaviour is supposed to be independent of ρ and ε in some sense (see [12, 13]). Let us recall a definition proposed in [7, 19].

Definition 2: Let $G \curvearrowright (X, \mu)$ be a measure-preserving action of a group G equipped with λ and ρ be an admissible semimetric on (X, μ). We call a sequence $\{h_n\}_{n=1}^\infty$ a **scaling entropy sequence** of this action of the equipped group G and the semimetric ρ if for all sufficiently small $\varepsilon > 0$ the following asymptotic relation holds:

$$\mathbb{H}_\varepsilon(X, \mu, G_{av}^n \rho) \asymp h_n.$$

Here, for two functions $\phi(n)$ and $\psi(n)$ relation $\phi(n) \asymp \psi(n)$ means that there are two positive constants c and C such that $c\phi(n) \leqslant \psi(n) \leqslant C\phi(n)$. Note that it makes sense to consider the whole class of equivalent scaling entropy sequences. Indeed, it is easy to see from Definition 2 that if $\{h_n\}$ is a scaling sequence of the action and $h'_n \asymp h_n$, then the sequence $\{h'_n\}$ is scaling as well. We need some additional requirements on the equipment λ proposed in [19].

Definition 3: Equipment $\lambda = \{S_n\}$ of a countable group G is called **suitable** if for any $g \in \bigcup S_n$ and $\delta > 0$ there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there are $g_1, \ldots, g_k \in G$ with

$$\left| gS_n \setminus \bigcup_{j=1}^k S_n g_j \right| \leqslant \delta |S_n|.$$

Any left Følner sequence in an amenable group forms suitable equipment. For an action of a suitably equipped group, P. Zatitskiy, proving a conjecture by A. Vershik, showed that if a sequence $\{h_n\}$ is a scaling entropy sequence for some summable generating admissible semimetric ρ, then it is a scaling sequence for any other such semimetric [18, 19]. The summability of ρ means that it has finite integral over X^2, i.e., $\rho \in L^1(X^2, \mu^2)$. In particular, any bounded measurable semimetric is summable. Therefore, the class of scaling entropy sequences does not depend on the choice of semimetric and forms a measure-theoretic invariant of an action of a suitably equipped group. Note that this class can be empty; this case is discussed in Section 2.2.

We should note that a scaling entropy sequence may depend on the choice of equipment. It is shown in [7] that under certain conditions for the equipment if a scaling sequence exists, then one could choose a subadditive increasing function $f \colon \mathbb{N} \to \mathbb{N}$ with $h_n \asymp f(|S_n|)$. However, it is still unknown whether such f can be chosen independently of the equipment. Moreover, it is unclear if the stability (see Section 2.2) depends on the choice of equipment.

It is proved in [7] that in the case of one transformation (i.e., action of $\mathbb{Z}$ with the standard equipment $S_n = \{-n, \dots, n\}$) if the class of scaling entropy sequences is non-empty, then it contains an increasing subadditive function. In [19], the explicit examples of automorphisms with a given increasing subadditive scaling entropy sequence are given. In addition, the actions of $\bigoplus \mathbb{Z}_2$ with a given scaling sequence of an intermediate growth were constructed in [19]. This construction can be easily generalized to the case of the group $\bigoplus_k \mathbb{Z}_{r_k}$ for an arbitrary family of positive integers $\{r_k\}$.

2.2. Stable and unstable systems. Examples of almost complete growth. It was recently shown by the author [10] that a scaling entropy sequence may not exist even for one automorphism. We will call a system **stable** if its class of scaling entropy sequences is not empty. That is, in [10], the examples of unstable ergodic systems were constructed.

However, the definition of scaling entropy (Definition 2) can be improved in such a way that it works for unstable actions and is consistent with the scaling sequence if the action is stable. Let us define an equivalence relation on the set of functions from $\mathbb{N} \times \mathbb{R}_+$ to $\mathbb{R}_+$ that decrease with respect to their second arguments. We say that Ψ and Φ are equivalent if and only if

$$(1) \qquad \forall \varepsilon > 0 \; \exists \delta > 0 \quad \Psi(n, \varepsilon) \precsim \Phi(n, \delta) \text{ and } \Phi(n, \varepsilon) \precsim \Psi(n, \delta).$$

We denote the equivalence class of this relation containing Φ by $[\Phi]$. Here and in what follows, for two sequences $\phi(n)$ and $\psi(n)$, we write $\phi \precsim \psi$ if there is a positive constant C such that

$$\phi(n) \leqslant C\psi(n)$$

for all $n \in \mathbb{N}$. We also write $\phi \prec \psi$ if

$$\phi(n) = o(\psi(n)).$$

Let $G \curvearrowright (X, \mu)$ be a measure-preserving action of a suitably equipped group (G, λ) and ρ be an admissible generating summable semimetric on (X, μ). In [18, 19], it is proved (see Lemma 9 in [18] and similar statements in [19]) that the equivalence class of the function

$$\Phi_\rho(n, \varepsilon) = \mathbb{H}_\varepsilon(X, \mu, G_{av}^n \rho)$$

does not depend on a semimetric and forms an invariant $\mathcal{H}(X, \mu, G, \lambda)$ of the measure-preserving action:

$$(2) \qquad \mathcal{H}(X, \mu, G, \lambda) = [\Phi_\rho(n, \varepsilon)].$$

Note that the system is stable if and only if $\mathcal{H}(X, \mu, G, \lambda)$ contains a function $\Phi(n, \varepsilon) = \phi(n)$ independent of ε.

Scaling entropy may also depend on the choice of equipment. However, if $\lambda = \{S_n\}$ and $\theta = \{W_n\}$ are such that $|S_n \triangle W_n| = o(|S_n|)$, then

$$\mathcal{H}(X, \mu, G, \lambda) = \mathcal{H}(X, \mu, G, \theta)$$

for any measure-preserving system (X, μ, G).

Theorem 2 below states that for any amenable group G equipped with a Følner sequence $\lambda = \{F_n\}$, any $\Phi \in \mathcal{H}(X, \mu, G, \lambda)$, and any positive ε the following asymptotic relation holds:

$$(3) \qquad \Phi(n, \varepsilon) \precsim |F_n|.$$

The equivalence in (3) holds if and only if the amenable measure entropy of (X, μ, G) is positive. In [19], the examples of stable ergodic $\mathbb{Z}$-actions whose scaling entropy sequence (with respect to the standard equipment) h_n grows faster than a given function $\phi(n) = o(n)$ are given. In [3], similar constructions are given for $\mathbb{Z}^2$.

Examples of such actions play the crucial role in further arguments, that is we are looking for G-systems whose scaling entropy grows faster than a given function $\phi(n) = o(|F_n|)$ at least along a subsequence, called actions of almost complete growth (see Definition 4). The main theorem of this work (Theorem 4) states that such ergodic actions exist for any countable non-periodic amenable group with arbitrary Følner equipment. The non-existence of a universal zero entropy system for such groups is proved in Theorem 1 by means of constructed actions of almost complete growth.

3. Scaling entropy and amenable measure entropy

In this section, we study the relation between the notions of scaling entropy and amenable measure entropy.

Let us state several technical lemmas that we use in the proof of Theorem 2 below. Note that for any measurable partition ξ of a measure space (X, μ) there is a naturally defined **cut semimetric** $\rho_\xi(x, y)$ which equals zero if x and y both lie in the same cell of ξ and one otherwise. If the partition is finite (or countable) up to a null set then the corresponding cut semimetric is admissible. The following lemma (see [18]) links the ε-entropy of ρ_ξ with the Shannon entropy of ξ.

LEMMA 1: *The following relations between ε-entropy and Shannon entropy hold:*

(1) *For any measurable partition ξ of a standard measure space (X, μ) and any $\varepsilon > 0$*

$$\mathbb{H}_\varepsilon(X, \mu, \rho_\xi) \leqslant \frac{H(\xi)}{\varepsilon},$$

where ρ_ξ is a semimetric corresponding to ξ.

(2) *Let $m, k \in \mathbb{N}$ and $\{\xi_i\}_{i=1}^k$ be a family of finite measurable partitions of (X, μ) such that each ξ_i consists of not more than m cells. Let $\xi = \bigvee_{i=1}^k \xi_i$ be the refinement of these partitions and $\rho = \frac{1}{k} \sum_{i=1}^k \rho_{\xi_i}$ be the averaging of corresponding semimetrics. Then for any $\varepsilon \in (0, \frac{1}{2})$ the following estimate holds:*

$$\frac{H(\xi)}{k} \leqslant \frac{\mathbb{H}_\varepsilon(X, \mu, \rho)}{k} + 2\varepsilon \log m - \varepsilon \log \varepsilon - (1 - \varepsilon) \log(1 - \varepsilon) + \frac{1}{k}.$$

The next lemma (see [7, Lemma 1]) gives an upper bound for the ε-entropy of an averaged semimetric.

LEMMA 2: *Let $\rho_1, \rho_2, \ldots, \rho_k$ be admissible semimetrics on (X, μ) with $\rho_i \leqslant 1$ for all $i = 1, \ldots, k$. Assume that $\varepsilon \in (0, 1)$ satisfies $\mathbb{H}_\varepsilon(X, \mu, \rho_i) > 0$. Then the following inequality holds*

$$\mathbb{H}_{2\sqrt{\varepsilon}}\left(X, \mu, \frac{1}{k} \sum_{i=1}^k \rho_i\right) \leqslant 2 \sum_{i=1}^k \mathbb{H}_\varepsilon(X, \mu, \rho_i).$$

The theorem below relates notions of scaling entropy and classical measure entropy for amenable groups. The similar result for one automorphism was proved in [18].

THEOREM 2: *Let $G \curvearrowright (X, \mu)$ be a measure-preserving action of an amenable group and $\lambda = \{F_n\}$ be a Følner sequence in G. Choose some $\Phi \in \mathcal{H}(X, \mu, G, \lambda)$.*

(1) *Assume that the amenable measure entropy $h(X, \mu, G)$ is positive. Then (X, μ, G) is stable and for all sufficiently small $\varepsilon > 0$*

$$\Phi(n, \varepsilon) \asymp |F_n|.$$

(2) *If $h(X, \mu, G) = 0$, then for all positive ε*

$$\Phi(n, \varepsilon) = o(|F_n|).$$

Proof. Suppose that the classical entropy $h(X, \mu, G)$ is positive. Let ξ be a finite measurable partition and

$$\zeta_n = \bigvee_{g \in F_n} g^{-1}\xi.$$

Let ρ_ξ be the cut semimetric corresponding to ξ. Also, for $g \in F_n$ set $\xi_g = g^{-1}\xi$ and $m = |\xi| = |\xi_g|$. Due to part 2 of Lemma 1, we have

$$(4) \quad \frac{H(\zeta_n)}{|F_n|} \leqslant \frac{\mathbb{H}_\varepsilon(X, \mu, G^n_{av}\rho_\xi)}{|F_n|} + 2\varepsilon \log m - \varepsilon \log \varepsilon - (1 - \varepsilon)\log(1 - \varepsilon) + \frac{1}{|F_n|}.$$

Since $h(X, \mu, G) > 0$, we can choose ξ satisfying

$$h(\xi) = \lim_{n \to +\infty} \frac{1}{|F_n|} H(\zeta_n) > 0.$$

Hence, while ε is small enough, inequality (4) implies

$$(5) \qquad\qquad \mathbb{H}_\varepsilon(X, \mu, G^n_{av}\rho_\xi) \gtrsim |F_n|.$$

Note that ρ_ξ may not be generating (it happens if and only if ξ is not generating). We can easily overcome this by adding some admissible metric ω to ρ_ξ. Evidently, $\omega + \rho_\xi$ is now generating and

$$\Psi(n, \varepsilon) = \mathbb{H}_\varepsilon(X, \mu, G^n_{av}(\rho_\xi + \omega))$$

lies in $\mathcal{H}(X, \mu, G, \lambda)$. At the same time,

$$\Psi(n, \varepsilon) \geqslant \mathbb{H}_\varepsilon(X, \mu, G^n_{av}\rho_\xi) \gtrsim |F_n|.$$

The upper bound for the asymptotics of the scaling entropy instantly follows from Lemma 2. Indeed, let $\rho \leqslant 1$ be an admissible generating semimetric. Using Lemma 2 for semimetrics $g^{-1}\rho$, where $g \in F_n$, we obtain

$$\mathbb{H}_{2\sqrt{\varepsilon}}(X, \mu, G_{av}^n \rho) \leqslant 2|F_n|\mathbb{H}_\varepsilon(X, \mu, \rho).$$

Therefore, for any $\Phi \in \mathcal{H}(X, \mu, G, \lambda)$ and any sufficiently small (and thus, as well, for all) $\varepsilon > 0$,

$$\Phi(n, \varepsilon) \precsim |F_n|.$$

The first part is proved.

Now assume that $h(X, \mu, G) = 0$. Consider a generating partition ξ with finite entropy and the corresponding (generating) semimetric ρ_ξ. Let $\zeta_n = \bigvee_{g \in F_n} g^{-1}\xi$ be as above. The first part of Lemma 1 implies the following inequality:

$$\mathbb{H}_\varepsilon(X, \mu, G_{av}^n \rho_\xi) \leqslant \mathbb{H}_\varepsilon(X, \mu, \rho_{\zeta_n}) \leqslant \frac{H(\zeta_n)}{\varepsilon}.$$

However, $h(X, \mu, G) = 0$ means that

$$H(\zeta_n) = o(|F_n|).$$

Therefore, for any $\Phi \in \mathcal{H}(X, \mu, G, \lambda)$ and any $\varepsilon > 0$

$$\Phi(n, \varepsilon) = o(|F_n|).$$

The second part is proved.   ∎

4. Proof of the non-existence of a universal system

In this section, we prove the non-existence of a universal zero entropy system for actions of a non-periodic amenable group. However, the proof that we provide deals with a wider class which, we believe, coincides with the class of all amenable groups.[3]

Definition 4: We say that a group G equipped with $\lambda = \{F_n\}$ **admits actions of almost complete growth** if for any non-negative function $\phi(n) = o(|F_n|)$ there exists a measure-preserving system (X, μ, G) such that for any $\Phi \in \mathcal{H}(X, \mu, G, \lambda)$ and for any sufficiently small ε the following holds:

$$\Phi(n, \varepsilon) \not\precsim \phi(n) \quad \text{and} \quad \Phi(n, \varepsilon) = o(|F_n|).$$

---

[3] At least, it contains $\bigoplus \mathbb{Z}_2$ which is a torsion group.

By virtue of Theorem 2, the second condition in Definition 4 is equivalent to $h(X, \mu, G) = 0$ for amenable groups. Note that Definition 4 has a deep connection with the notions of slow entropy [3] and measure-theoretical complexity [2].

THEOREM 3: *Let $G \curvearrowright (X, d)$ be a continuous action of an amenable group G on a compact metric space. Suppose that G admits ergodic actions of almost complete growth for some Følner equipment $\theta = \{W_n\}$. Assume that for any ergodic measure-preserving system (Y, ν, G) with zero entropy there exists an invariant measure μ on X with*

$$(X, \mu, G) \cong (Y, \nu, G).$$

Then the topological entropy of (X, d, G) is positive.

Proof. We will prove the theorem by contradiction. Assume that $h_{\text{top}}(X, G) = 0$. Then

$$\sup_{\varepsilon > 0} \limsup_{n \to +\infty} \frac{1}{|W_n|} \log \operatorname{spn}(d, W_n, \varepsilon) = 0.$$

Hence, for all $\varepsilon > 0$

$$\limsup_{n \to +\infty} \frac{1}{|W_n|} \log \operatorname{spn}(d, W_n, \varepsilon) = 0.$$

It is clear that there exists a function $\phi(n)$ such that $\frac{\phi(n)}{|W_n|} \to 0$ and for any $\varepsilon > 0$

$$\phi(n) \gtrsim \log \operatorname{spn}(d, W_n, \varepsilon).$$

By the theorem assumption, G admits ergodic actions of almost complete growth with respect to θ. Therefore, there exists an ergodic measure-preserving action $G \overset{\alpha}{\curvearrowright} (Y, \nu)$ with zero entropy such that for any $\Phi \in \mathcal{H}(Y, \nu, G, \theta)$ and any sufficiently small ε

(6) $\Phi(n, \varepsilon) \not\gtrsim \phi(n).$

Also by the theorem assumption, this action has a representation in the topological system (X, d, G). Hence, there exists an invariant measure μ on X with

$$(X, \mu, G) \cong (Y, \nu, G).$$

In particular, scaling entropy classes $\mathcal{H}(Y, \nu, G, \theta)$ and $\mathcal{H}(X, \mu, G, \theta)$ coincide. Note that the metric d on X is obviously admissible and summable. Therefore,

$$\mathbb{H}_\varepsilon(X, \mu, G^n_{av}d) \in \mathcal{H}(X, \mu, G, \theta).$$

However,

$$\mathbb{H}_\varepsilon(X, \mu, G_{av}^n d) \leqslant \mathbb{H}_\varepsilon(X, \mu, G_{max}^n d) \leqslant \log \operatorname{spn}\left(d, W_n, \frac{\varepsilon}{2}\right) \precsim \phi(n),$$

and we have obtained a contradiction to (6).   ∎

In view of Theorem 3, it only remains to prove that non-periodic amenable groups admit ergodic actions of almost complete growth to achieve our goal, i.e., Theorem 1. The rest of the work is devoted to proving this.

5. Coinduced actions and scaling entropy

In order to construct the actions of almost complete growth, we implement the procedure of coinduction.

5.1. COINDUCED ACTIONS. Let us recall the construction of a coinduced action (see, e.g., [4, p. 157]). Let G be a countable amenable group and $H \leqslant G$ be a subgroup of G. Let $H \overset{\alpha}{\curvearrowright} (X, \mu)$ be a measure-preserving action of H. Consider the decomposition of G into a disjoint union of left cosets of H:

$$G = \bigsqcup_{i=0}^{\infty} g_i H,$$

where g_i are some representatives of the cosets. For further convenience, we choose $g_0 = e$. Consider a measure space

$$(X^{G/H}, \mu^{G/H}) = \prod_{i=0}^{\infty} (X_i, \mu_i),$$

where each (X_i, μ_i) is an isomorphic copy of (X, μ) corresponding to g_i. For $x \in X^{G/H}$ and $i \geq 0$, we denote the i-th coordinate of x by x_i, $x_i \in X_i$. For any $g \in G$ and any $i \geq 0$, there are unique $k(i, g) \in \mathbb{N} \cup \{0\}$ and $h(i, g) \in H$ with $g g_i = g_{k(i,g)} h(i, g)$. Define a coinduced action $\operatorname{CInd}_H^G \alpha$ of the group G on a measure space $X^{G/H}$ in the following way. Let $x \in X^{G/H}$, put

$$g(x)_i = h(i, g^{-1})^{-1}(x_{k(i, g^{-1})}).$$

Once and for all, we fix a system of cosets representatives $\{g_i\}$.

LEMMA 3: *Let H be a subgroup of a countable amenable group G and $\{\tilde{W}_n\}_{n=1}^{\infty}$ be a Følner sequence in G. Then there exists another Følner sequence $\{W_n\}_{n=1}^{\infty}$ in G such that $|W_n \triangle \tilde{W}_n| = o(|W_n|)$ and*

$$W_n = \bigcup S_n^i g_i^{-1},$$

where $S_n^i \subset H$ satisfy the following condition. For any $h \in H$, $\varepsilon > 0$, the inequality

$$|h S_n^i \triangle S_n^i| \leqslant \varepsilon |S_n^i|$$

holds for all sufficiently large n and for any i.

Proof. Consider some $h \in H$ and $n \in \mathbb{N}$. Let us denote by $\varepsilon(n, h)$ the following value:

$$\varepsilon(n, h) = \frac{|h \tilde{W}_n \triangle \tilde{W}_n|}{|\tilde{W}_n|}.$$

Due to the Følner condition, $\varepsilon(n, h)$ goes to zero when h is fixed. Moreover, each $\tilde{W}_n$ uniquely decomposes into a disjoint union $\bigcup_i S_n^i g_i^{-1}$, where S_n^i are some finite subsets of H. Let $\tilde{I}(n, h)$ be a set of integers i such that

$$|h S_n^i \triangle S_n^i| > \varepsilon(n, h)^{\frac{1}{2}} |S_n^i|.$$

Denote

$$E(n, h) = \bigcup_{i \in \tilde{I}(n, h)} S_n^i g_i^{-1}.$$

The left multiplication by h preserves all right cosets $H g_i^{-1}$. Therefore,

$$|E(n, h)| = \sum_{i \in \tilde{I}(n, h)} |S_n^i| < \varepsilon(n, h)^{\frac{1}{2}} |\tilde{W}_n|.$$

Let $\tau \colon H \to \mathbb{N}$ be an arbitrary enumeration of all elements of H. Define

$$W_n = \tilde{W}_n \setminus \bigcup_{h \colon \varepsilon(n, h) < 2^{-\tau(h)}} E(n, h).$$

Clearly, sequence $\{W_n\}$ is the desired one. Indeed, we have

$$|\tilde{W}_n \triangle W_n| \leqslant \sum_{h \colon \varepsilon(n, h) < 2^{-\tau(h)}} |E(n, h)| < \sum_{h \colon \varepsilon(n, h) < 2^{-\tau(h)}} \varepsilon(n, h)^{\frac{1}{2}} |\tilde{W}_n| = o(|\tilde{W}_n|).$$

Since each term $\varepsilon(n, h)$ is bounded by $2^{-\tau(h)}$ and tends to zero with respect to n, the sum $\sum \varepsilon(n, h)^{\frac{1}{2}}$ is dominated by the convergent series $\sum_h 2^{\frac{\tau(h)}{2}}$ and, therefore, goes to zero as n tends to infinity. ∎

Definition 5: A subset S of integer numbers is said to be ε-**invariant** for some positive ε if it satisfies

$$|(S+1)\triangle S| < \varepsilon|S|.$$

Remark: Evidently, if $H = \mathbb{Z}$ in Lemma 3, then there exists a subsequence $\{n_j\}$ such that all $S_{n_j}^i$ are $\frac{1}{j}$-invariant.

5.2. SCALING ENTROPY OF A COINDUCED ACTION. In this section, we estimate the scaling entropy of a coinduced action. We reduce the question about the existence of almost complete growth actions for non-periodic amenable groups to the case of the group $\mathbb{Z}$ which is considered in Section 6. Also, we use an important technical Lemma 5 whose proof is postponed to Section 7.

THEOREM 4: *Let $\lambda = \{F_n\}_{n=1}^\infty$ be a Følner sequence of a countable non-periodic amenable group G. Then the group G admits ergodic actions of almost complete growth with respect to λ.*

Proof. Let $h \in G$ be an element of infinite order in G and $H = \langle h \rangle$ be the subgroup generated by h. Let $\{g_i\}$ be a system of representatives of left cosets with $g_0 = e$. Lemma 3 states that there is a sequence $\theta = \{W_n\}$ of finite subsets of G with

(7) $$|F_n \triangle W_n| = o(|W_n|) = o(|F_n|),$$

and

$$W_n = \bigcup S_n^i g_i^{-1},$$

where $S_n^i \subset H$ are such that for any $\varepsilon > 0$ and for any sufficiently large n

$$|hS_n^i \triangle S_n^i| \leqslant \varepsilon|S_n^i|.$$

Relation (7) implies that any measure-preserving action α of G satisfies

$$\mathcal{H}(\alpha, \lambda) = \mathcal{H}(\alpha, \theta).$$

Thus, it suffices to prove that the group G equipped with θ instead of λ admits ergodic actions of almost complete growth.

Let $\phi(n)$ be an increasing positive function which goes to infinity. We will apply the following lemma proved in Section 6.

LEMMA 4: *Suppose that a sequence* $\{S_n^i\}_{i=1}^{k_n}$ *of finite families of finite subsets of* $\mathbb{Z}$ *is such that every* S_n^i *is* $\frac{1}{n}$-*invariant. Let* $\phi(n)$ *be a sequence of positive numbers with* $\lim_{n\to\infty} \phi(n) = \infty$. *Then there exist an ergodic automorphism* T *of a Lebesgue space* (X, μ) *and a subsequence* $\{n_j\}$ *such that for any generating admissible summable semimetric* ρ *and sufficiently small* $\varepsilon > 0$ *the following relation holds:*

$$(8) \qquad \frac{1}{\phi(n_j)} \prec \min_{i=1,\ldots,k_{n_j}} \frac{\mathbb{H}_\varepsilon(X, \mu, T_{av}^{S_{n_j}^i} \rho)}{|S_{n_j}^i|} \leqslant \max_{i=1,\ldots,k_{n_j}} \frac{\mathbb{H}_\varepsilon(X, \mu, T_{av}^{S_{n_j}^i} \rho)}{|S_{n_j}^i|} \prec 1.$$

Note that there are only finitely many $S_{n_j}^i$ for each j. We apply Lemma 4 for the sequence of non-empty sets S_n^i given by Lemma 3. Let I_n be the set of those indices i for which S_n^i is not empty and $k_n = |I_n|$. Relation (8) implies, in particular, that T has zero entropy and for any ε small enough for sufficiently large j

$$(9) \qquad \frac{|S_{n_j}^i|}{\phi(n_j)} < \mathbb{H}_{4\varepsilon}(X, \mu, T_{av}^{S_{n_j}^i} \rho) < |S_{n_j}^i|, \quad i \in I_{n_j}.$$

Consider an action $\alpha = \text{CInd}_H^G T$ coinduced from H to the whole group G. Let $\tilde{\rho} \leqslant 1$ be an admissible metric on (X, μ). Define a semimetric ρ on $(X, \mu)^{G/H}$ in the following way:

$$\rho(x, y) = \tilde{\rho}(x_0, y_0), \quad x, y \in X^{G/H}.$$

Although G acts transitively on G/H, the semimetric ρ may not be generating.[4] However, it does not really matter because we are only looking for lower bounds for the scaling entropy. Since by the choice of the representatives $g_0 = e$, elements of H act on the first component of $(X, \mu)^{G/H}$ independently of other coordinates. Thus, for all $x, y \in X^{G/H}$

$$\rho(hx, hy) = \tilde{\rho}(hx_0, hy_0).$$

Then

$$H_{av}^{S_{n_j}^i} \rho(x, y) = H_{av}^{S_{n_j}^i} \tilde{\rho}(x_0, y_0).$$

For each coset representative g_i, define a semimetric ρ_i on $(X, \mu)^{G/H}$ by

$$\rho_i = g_i H_{av}^{S_{n_j}^i} \rho.$$

---

[4] For example, if all S_n^0 are empty.

Each ρ_i depends only on the i-th component:

$$\rho_i(x,y) = (H_{av}^{S_{n_j}^i}\rho)(g_i^{-1}x, g_i^{-1}y) = (H_{av}^{S_{n_j}^i}\tilde{\rho})(x_i, y_i), \quad x,y \in (X,\mu)^{G/H}.$$

Hence, we can consider ρ_i as a semimetric on X_i. The averaging of ρ with respect to W_{n_j} can be expressed in terms of ρ_i as follows:

(10)

$$G_{av}^{W_{n_j}}\rho = \frac{1}{|W_{n_j}|}\sum_{i\in I_{n_j}}\sum_{s\in S_{n_j}^i}g_i s^{-1}\rho = \frac{1}{|W_{n_j}|}\sum_{i\in I_{n_j}}g_i\sum_{s\in S_{n_j}^i}s^{-1}\rho$$

$$= \frac{1}{|W_{n_j}|}\sum_{i\in I_{n_j}}|S_{n_j}^i|g_i H_{av}^{S_{n_j}^i}\rho = \frac{1}{\sum_{i\in I_{n_j}}|S_{n_j}^i|}\sum_{i\in I_{n_j}}|S_{n_j}^i|\rho_i.$$

The next step is to estimate the epsilon-entropy of the semimetric given by (10). The proof of the following lemma will be given in Section 7.

LEMMA 5: *Suppose $\varepsilon > 0$ and $\phi > 1$ are fixed. Consider a finite family of admissible semimetric triples (X_i, μ_i, ρ_i), $i = 1, \ldots, k$. Assume that $\{s_i\}_{i=1}^k$ are such that $\phi^{-1}s_i < \mathbb{H}_{4\varepsilon}(X_i, \mu_i, \rho_i) < s_i$. Define a semimetric ρ on*

$$\prod_{i=1}^k (X_i, \mu_i) = (X, \mu)$$

as a weighted averaging:

$$\rho(x,y) = \frac{1}{\sum_{i=1}^k s_i}\sum_{i=1}^k s_i\rho_i(x_i, y_i),$$

where $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$. Then

(11)
$$\mathbb{H}_{\varepsilon^4}(X, \mu, \rho) \geqslant \frac{1}{\phi}\varepsilon^3\sum_{i=1}^k \mathbb{H}_{4\varepsilon}(X_i, \mu_i, \rho_i) - k - 1.$$

Since for large j inequalities (9) hold, we can apply Lemma 5 to semimetrics ρ_i, weights $s_i = |S_{n_j}^i|$, and $\phi = \phi(n_j)$. We obtain the following estimate:

$$\mathbb{H}_{\varepsilon^4}(X^{G/H}, \mu^{G/H}, G_{av}^{W_{n_j}}\rho) \geqslant \frac{1}{\phi(n_j)}\varepsilon^3\sum_{i\in I_{n_j}}\mathbb{H}_{4\varepsilon}(X, \mu, H_{av}^{S_{n_j}^i}\tilde{\rho}) - k_{n_j} - 1$$

(12)

$$\geqslant \frac{1}{\phi(n_j)^2}\varepsilon^3|W_{n_j}| - 2k_{n_j}.$$

Clearly, for any sequence $\psi(n)$, which goes to infinity, there is an increasing $\phi(n)$ also going to infinity with $\phi^2(n) = o(\psi(n))$. Note that $k_n = o(|W_n|)$, so we can choose such ϕ with $k_n = o(\phi(n)^{-2}|W_n|)$. Therefore, the action α constructed by such a slow growing sequence ϕ satisfies the following condition on scaling entropy with respect to the equipment $\theta = \{W_n\}$: For any $\Phi \in \mathcal{H}(\alpha, \theta)$ and sufficiently small $\varepsilon > 0$,

$$\Phi(n_j, \varepsilon) \succ \frac{|W_{n_j}|}{\psi(n_j)}, \quad \text{that is } \Phi(n, \varepsilon) \not\precsim \frac{|W_n|}{\psi(n)}.$$

Let us show that α has zero entropy. The right inequality in (8) combined with Lemma 2 shows that $\Phi_\rho(n, \varepsilon) = o(|W_n|)$. Although ρ may not be generating with respect to θ, we can choose $\theta' = \{W_n'\}$ with $|W_n' \triangle W_n| = o(|W_n|)$ such that ρ is generating with respect to θ'. Hence, $\Phi_\rho(n, \varepsilon) \in \mathcal{H}(\alpha, \theta') = \mathcal{H}(\alpha, \theta)$ and the amenable measure entropy of α is zero due to Theorem 2.

It only remains to prove that the constructed actions are ergodic. It is true if the index of H is infinite. In this case, the ergodicity follows from the similar argument as in the case of Bernoulli shift. Suppose that there is a non-trivial invariant subset $E \subset \prod_{i=1}^{\infty} X_i$. This subset can be approximated by a cylinder set C with $\mu(E \triangle C) < \varepsilon$, where ε is arbitrarily small. Evidently, for any $g \in G$ the set $g^{-1}C$ is cylinder as well and $\mu(g^{-1}C \triangle C) < 2\varepsilon$. However, for any cylinder C we can find an element $g \in G$ such that the basements of C and $g^{-1}C$ do not intersect and, therefore,

$$\mu(g^{-1}C \cap C) = \mu(C)^2 < \mu(C) - 2\varepsilon.$$

In the case of finite index, the coinduced action $G \overset{\alpha}{\curvearrowright} (X, \mu)^{G\!/\!H}$ itself may not be ergodic. However α still has zero entropy. The automorphism T satisfies inequality (9). Consider the ergodic decomposition of the product measure $\mu^{G\!/\!H}$ into G-ergodic invariant components. Since α has zero entropy, almost all its ergodic components have zero entropy as well.

Since H preserves the coset $g_0 H = H$ and, therefore, acts independently on the 0-coordinate, the standard projection $\pi_0 : (X, \mu)^{G\!/\!H} \to (X_0, \mu_0)$ preserves the H-invariance. Thus, the ergodic decomposition is projected to some decomposition of μ into H-invariant measures. Since μ is H-ergodic, almost all such projections have to coincide with μ. Let ν be one of those ergodic components of $\mu^{G\!/\!H}$ that has entropy zero and $\pi_0(\nu) = \mu$. We are left to prove a lower bound for epsilon-entropy. Lemma 5 is not applicable here, but it can be

replaced with a simpler argument. Let $\tilde{\rho}, \rho$, and ρ_i's be as before and L be the index of H. Since there are no more than L different S_n^i for each n and

$$\sum_i |S_n^i| = |W_n|,$$

there exists some $S_n^{i_0}$ with $|S_n^{i_0}| \geqslant \frac{1}{L}|W_n|$. Applying this for $n = n_j$, obtain

$$
\mathbb{H}_\varepsilon(X^{G/H}, \nu, G_{av}^{W_{n_j}}\rho) \geqslant \mathbb{H}_\varepsilon(X^{G/H}, \nu, L^{-1}\rho_{i_0}) \geqslant \mathbb{H}_{L\varepsilon}(X^{G/H}, \nu, \rho_{i_0})
$$

(13)
$$
= \mathbb{H}_{L\varepsilon}(X^{G/H}, \nu, g_{i_0}H_{av}^{S_{n_j}^{i_0}}\rho) = \mathbb{H}_{L\varepsilon}(X^{G/H}, \nu, H_{av}^{S_{n_j}^{i_0}}\rho)
$$

$$
= \mathbb{H}_{L\varepsilon}(X, \mu, H_{av}^{S_{n_j}^{i_0}}\tilde{\rho}).
$$

The first inequality in (13) holds since

$$
G_{av}^{W_{n_j}}\rho = \frac{1}{|W_{n_j}|}\sum_i |S_{n_j}^i|\rho_i > L^{-1}\rho_{i_0}.
$$

The last two equalities follow from the invariance of ν and the fact that $\pi_0(\nu) = \mu$. Finally, using (9), we obtain

(14) $\mathbb{H}_\varepsilon(X^{G/H}, \nu, G_{av}^{W_{n_j}}\rho) \geqslant \mathbb{H}_{L\varepsilon}(X, \mu, H_{av}^{S_{n_j}^{i_0}}\tilde{\rho}) > \dfrac{1}{\phi(n_j)}|S_{n_j}^{i_0}| \geqslant \dfrac{1}{L\phi(n_j)}|W_{n_j}|,$

since $L\varepsilon$ can be arbitrarily small. Then for a given function $\psi(n)$ that goes to infinity, we take $\phi(n) = o(\psi(n))$. Inequality (14) gives that $\mathbb{H}_\varepsilon(X^{G/H}, \nu, G_{av}^{W_{n_j}}\rho)$ grows faster than $\frac{|W_{n_j}|}{\psi(n_j)}$. The proof is completed.   ∎

6. Adic action on the graph of ordered pairs

6.1. GRAPH OF ORDERED PAIRS. In order to construct ergodic actions of almost complete growth, we use the notion of *the adic transformation* (Vershik's automorphism) on *the graph of ordered pairs*. This graph was studied in detail in [15] and [19].

Consider an infinite graded graph $\Gamma = (V, E)$. The set of vertices V is a disjoint union of the levels $V_n = \{0,1\}^{2^n}$, $n \geq 0$. The set of edges is defined together with the coloring $\mathfrak{c}\colon E \to \{0,1\}$ in the following way. Let $v_n \in V_n$ and $v_{n+1} \in V_{n+1}$. The edge $e = (v_n, v_{n+1})$ belongs to E if and only if v_n is a prefix or a suffix of v_{n+1}. We mark this edge (define $\mathfrak{c}(e)$) with 0 or 1 respectively. If v_n simultaneously forms both a prefix and a suffix of v_{n+1}, then we draw two distinct edges between v_n and v_{n+1} also marked with 0 and 1 respectively. The vertices v_n and v_{n+1} are called the initial and terminal points of e. We denote them by $s(e)$ and $r(e)$ respectively.

A path in Γ is a sequence of edges $\{e_i\}$ such that $s(e_{i+1})=r(e_i)$ and $s(e_i)\in V_i$. On the set X of all infinite paths the cylinder topology is imposed in a natural way. A Borel measure on X is called **central** if all possible beginnings of a path have equal probabilities while the tail is fixed. It means that any two cylinder sets whose corresponding finite paths have the same terminal vertex have the same measure.

Define the **adic transformation** T on the path space X. Let $x = \{e_i\}_{i=0}^{\infty}$ be an infinite path. Find the minimal n with $\mathfrak{c}(e_n) = 0$. Transformation T maps x to another path $T(x) = \{u_i\}$ defined in the following way. Let $u_i = e_i$ for $i \geqslant n + 1$, $\mathfrak{c}(u_n) = 1$, and $\mathfrak{c}(u_i) = 0$ for all $i < n$ (see Figure 1). If μ is a central measure, then this transformation is defined on a subset of full measure and forms an automorphism of the measure space (X, μ).
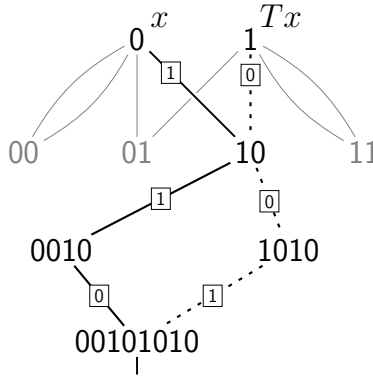

Figure 1. The adic transformation.

Now let $\sigma = \{\sigma_n\}$ be a given sequence of zeroes and ones. Let us construct a special central measure μ^{σ} on X. Note that any Borel measure μ on the path space is uniquely determined by a coherent system $\{\mu_n\}$, where each μ_n is a measure on a space X_n of finite paths of length n. In terms of μ_n, the centrality of μ means that for any n the measure μ_n only depends on the terminal vertex of a path. Define then a measure ν_n on the n-th level V_n as follows:

$$\nu_n(v) = \sum_{\substack{x \in X_n, \\ r(x)=v}} \mu_n(x).$$

The coherent system of measures $\{\nu_n\}$ uniquely determines the central measure μ.

Let us construct a sequence of finite subsets $V_n^\sigma \subset V_n$. Let $V_0^\sigma = V_0$. For $n \geqslant 1$, put $V_n^\sigma = \{ab \colon a, b \in V_{n-1}^\sigma\}$ if $\sigma_n = 1$ and $V_n^\sigma = \{aa \colon a \in V_{n-1}^\sigma\}$ otherwise. Let ν_n^σ be the uniform measure on the finite set $V_n^\sigma \subset V_n$. It is easy to see that $\{\nu_n^\sigma\}$ forms a coherent system. Let μ^σ be the unique central measure on X with the given coherent system $\{\nu_n^\sigma\}$ (see [19] for details).

In [19], it is proved that the system (X, μ^σ, T) is stable with respect to the standard equipment of the group $\mathbb{Z}$. Moreover, it is shown that the sequence $h_n = 2^{s^\sigma(\log n)}$, where $s^\sigma(t) = \sum_{i < t} \sigma_i$, is a scaling entropy sequence of that system. In addition, for every σ with an infinite number of ones, the transformation T is ergodic. The Kolmogorov–Sinai entropy of T is positive if and only if there are only finitely many zeroes in σ. Lemma 4 of this work deals with a more complicated system of sets over which we take an averaging. However, we can restrict ourselves to establishing only lower bounds for epsilon-entropy.

Let $x = \{e_i\} \in X$ be an infinite path. We denote by $\mathfrak{b}_n(x)$ the vertex of the n-th level which lies on x. By $\mathfrak{o}_n(x)$ we denote the value of $\sum_{i=0}^{n-1} \mathfrak{c}(e_i) 2^i$. It is easy to see that if $\mathfrak{o}_n(x) < 2^n - 1$ then

$$\mathfrak{b}_n(Tx) = \mathfrak{b}_n(x) \quad \text{and} \quad \mathfrak{o}_n(Tx) = \mathfrak{o}_n(x) + 1.$$

Thus, we have described the construction of the graph of ordered pairs and the adic transformation on its path space. Now we will use this construction to prove Lemma 4.

6.2. PROOF OF LEMMA 4. In order to prove Lemma 4, we construct special measures μ^σ on the path space X. The adic transformation T on the space (X, μ^σ) produces the desired automorphism. The idea is to choose an appropriate σ in which zeroes occur rarely. We will determine the positions of these zeroes inductively one by one. Note that for any σ with an infinite number of zeroes, the adic transformation has entropy zero. Therefore, the right hand side of inequality (8) holds automatically. Indeed, it follows from Theorem 2 and the fact that the sequence formed by all S_n^i satisfies the Følner condition.

It is sufficient to prove the left part of inequality (8) for an arbitrary admissible summable semimetric (which may not be generating). Indeed, the simple argument, which we have already used, shows that if it holds for some semimetric, then it holds for any generating one as well. Let us once and for all fix the cut semimetric ρ corresponding to a partition that separates paths according

to their first vertices. That is

$$(15) \qquad \rho(x,y) = \begin{cases} 0 & \text{if } \mathfrak{b}_0(x) = \mathfrak{b}_0(y); \\ 1 & \text{if } \mathfrak{b}_0(x) \neq \mathfrak{b}_0(y). \end{cases}$$

Note that by definition $\mathfrak{b}_0(x) = \mathfrak{b}_n(x)_{\mathbf{o}_n(x)}$ for any $n \geqslant 0$.

It is also enough to prove that for some subsequence $\{n_j\}$

$$(16) \qquad \frac{|S_{n_j}^i|}{\phi(n_j)} < \mathbb{H}_\varepsilon(X, \mu, T_{av}^{S_{n_j}^i} \rho), \quad i = 1, \ldots, k_{n_j}.$$

Indeed, applying this inequality to some function $\omega(n) = o(\phi(n))$ instead of ϕ we obtain the desired relation (8).

Let us fix some positive $\varepsilon < \frac{1}{10}$. Suppose that we have already chosen p numbers $q_1, \ldots, q_p$ and another p numbers $n_1, \ldots, n_p$ such that (16) holds for $j = 1, \ldots, p$ for any σ whose first zeroes are exactly $q_1, \ldots, q_p$. Initially, we take $p = 0$.

Without loss of generality, we can assume that all the sets S_n^i consist of positive numbers. For $l > n_p$, we can find $n(l)$ such that all the sets $\{S_l^i\}_{i=1}^{k_l}$ lie in the interval $\{0, \ldots, 2^{n(l)} - 1\}$. Let $N(l) = n(l) + 2$. For a binary word $v \in V_N$, we will denote its k-th entry by v_k. Recall that for the semimetric ρ defined by (15), for $x, y \in X$, the equality $\rho(x, y) = 0$ holds if and only if $\mathfrak{b}_N(x)_{\mathbf{o}_N(x)} = \mathfrak{b}_N(y)_{\mathbf{o}_N(y)}$. Thus,

$$\begin{aligned} T_{av}^{S_l^i} \rho(x,y) &= \frac{1}{|S_l^i|} \sum_{j \in S_l^i} \rho(T^j x, T^j y) \\ &= \frac{1}{|S_l^i|} |\{j \in S_l^i \colon \mathfrak{b}_N(T^j x)_{\mathbf{o}_N(T^j x)} \neq \mathfrak{b}_N(T^j y)_{\mathbf{o}_N(T^j y)}\}|. \end{aligned}$$

Consider the set $A^{S_l^i} = \{0,1\}^{S_l^i}$ and the measure $\mu_{S_l^i}^\sigma$ on it defined as follows:

$$\mu_{S_l^i}^\sigma(w) = \mu^\sigma(x \in X \colon \mathfrak{b}_N(T^j x)_{\mathbf{o}_N(T^j x)} = w_j, \ j \in S_l^i).$$

The mapping

$$\Phi \colon x \mapsto \left(\mathfrak{b}_N(T^j x)_{\mathbf{o}_N(T^j x)} \right)_{j \in S_l^i}$$

produces an isomorphism of semimetric triples $(X, \mu^\sigma, T_{av}^{S_l^i} \rho)$ and $(A^{S_l^i}, \mu_{S_l^i}^\sigma, \rho^H)$, where ρ^H is a Hamming distance on $A^{S_l^i}$. The next step is to construct an appropriate uniform approximation of $\mu_{S_l^i}^\sigma$. Note that for $\mathbf{o}_N(x) < 2^N - 2^n$ we have

$$\Phi(x) = \left(\mathfrak{b}_N(x)_{\mathbf{o}_N(x)+j} \right)_{j \in S_l^i}.$$

Due to centrality of μ^σ, we have for any $N > n$

$$\mu^\sigma(x \in X \colon \mathfrak{o}_N(x) \geqslant 2^N - 2^n) = 2^{n-N}.$$

Therefore, if l is fixed and N is large, then the measure $\mu^\sigma_{S^i_l}$ can be approximated pointwise simultaneously for all i by the following measure:

(17)
$$\mu^\sigma_{S^i_l, N}(w) = \frac{1}{1 - 2^{n-N}} \mu^\sigma(x \colon \Phi(x) = w, \ \mathfrak{o}_N(x) < 2^N - 2^n)$$

$$= \frac{1}{2^N - 2^n} \sum_{k=0}^{2^N - 2^n - 1} \nu^\sigma_N(v \in V_N \colon v_{k+j} = w_j, \ j \in S^i_l).$$

The last equality follows from centrality of μ^σ. Note that for $N > n + 1$, we have the inequality $\mu^\sigma_{S^i_l, N} < 2\mu^\sigma_{S^i_l}$ everywhere on $A^{S^i_l}$. Therefore,

(18)
$$\mathbb{H}_\varepsilon(A^{S^i_l}, \mu^\sigma_{S^i_l}, \rho^H) \geqslant \mathbb{H}_{2\varepsilon}(A^{S^i_l}, \mu^\sigma_{S^i_l, N}, \rho^H).$$

Suppose that q_{p+1}, which is not defined yet, is greater than N. Then any two summands in the last sum in (17) whose difference in indices is a multiple of 2^{q_p} coincide. It follows from the construction of ν^σ_N. Therefore,

$$\mu^\sigma_{S^i_l, N}(w) = \frac{1}{2^{q_p}} \sum_{k=0}^{2^{q_p} - 1} \nu^\sigma_N(v \in V_N \colon v_{k+j} = w_j, \ j \in S^i_l).$$

However, the epsilon-entropy of a semimetric with respect to a convex combination of measures can be estimated from below by the epsilon-entropy of this semimetric with respect to one of these measures. Therefore, it suffices to provide a lower bound for the 2ε-entropy of $(A^{S^i_l}, \mu^\sigma_{S^i_l, N, k}, \rho^H)$ for $k = 0, \ldots, 2^{q_p} - 1$, where

$$\mu^\sigma_{S^i_l, N, k}(w) = \nu^\sigma_N(v \in V_N \colon v_{k+j} = w_j, \ j \in S^i_l).$$

Note that all components of w are divided into groups in such a way that all coordinates in one group are the same, and distinct groups are independent with respect to $\mu^\sigma_{S^i_l, N, k}$. Indeed, that is true for ν_N, and measure $\mu^\sigma_{S^i_l, N, k}$ is a projection of ν_N onto some chosen coordinates.

Now we will use that all S^i_l are $\frac{1}{l}$-invariant sets. Each S^i_l consists of some intervals of integer numbers. The $\frac{1}{l}$-invariance of S^i_l means that the number of these intervals does not exceed $\frac{1}{l}|S^i_l|$. The number of different groups that have points both inside and outside of a given interval does not exceed 2^{q_p+1} because

the length of each group is not greater than 2^{q_p}. Then the total number of points in such groups does not exceed

$$(19) \qquad \frac{2^{q_p+1}}{l}|S_l^i| < l^{-\frac{1}{2}}|S_l^i|,$$

if $l > 2^{2q_p+2}$. We will call a component **proper** if it does not lie in the union of such groups. Let $\tilde{\rho}^H$ be a Hamming metric on the proper coordinates. For $l > 4$, we have

$$\tilde{\rho}^H(x,y) \leqslant \frac{1}{1-l^{-\frac{1}{2}}}\rho^H(x,y) \leqslant 2\rho^H(x,y), \quad x,y \in A^{S_l^i}.$$

Since every set with ρ^H-diameter less than 2ε has $\tilde{\rho}^H$-diameter less than 4ε, the following inequality holds:

$$(20) \qquad \mathbb{H}_{2\varepsilon}(A^{S_l^i}, \mu^\sigma_{S_l^i,N,k}, \rho^H) \geqslant \mathbb{H}_{4\varepsilon}(A^{S_l^i}, \mu^\sigma_{S_l^i,N,k}, \tilde{\rho}^H).$$

The right-hand side of formula (20) is exactly 4ε-entropy of a binary cube whose dimension is at least $|S_l^i|2^{-p-1}$. This value is not less than

$$(21) \qquad c(\varepsilon)\frac{|S_l^i|}{2^{p+1}} > \frac{|S_l^i|}{\phi(l)}$$

for sufficiently large l. It only remains to choose $n_{p+1} = l$ which satisfies condition (21) and the corresponding N. Then put $q_{p+1} = N + 1$.

7. Proof of Lemma 5

Let us proceed to the last step, which is the proof of Lemma 5, that we need in order to complete the proof of the main theorem. First, we construct an appropriate partition of each measure space (X_i, μ_i) with semimetric ρ_i. These partitions provide a useful framework to deal with the epsilon-entropy of a product space. Second, we apply some probabilistic estimates that lead to the desired inequality.

For $i = 1, \ldots, k$, denote by b_i the value of $2^{\mathbb{H}_{4\varepsilon}(X_i,\mu_i,\rho_i)}$. Since all the semimetrics ρ_i are admissible, they have finite ε^2-entropies. Consider the corresponding partition of X_i. Since (X_i, μ_i) is a continuous Lebesgue space, there exists a refinement $Y_0, \ldots, Y_r$ of this partition which satisfies the following: for $j > 0$, we have

$$\text{diam}_\rho(Y_j) < \varepsilon^2, \quad \mu(Y_0) < 2\varepsilon^2,$$

and

$$\mu(Y_{j_1}) = \mu(Y_{j_2})$$

for all $j_1, j_2 > 0$.

Consider the following procedure. Note that for any measurable $Z \subset X_i$ with $\mu_i(Z) < 4\varepsilon$ there exists a 2ε-separated set of size b_i in the difference $X_i \setminus Z$. Put $Z_0 = Y_0$ and choose the corresponding 2ε-separated set $\{p_1, \ldots, p_{b_i}\}$ in $X_i \setminus Z_0$. For each point p_j find a cell Y_j containing it and denote this cell by $a_{1,j}^i$. Thus, we obtain a family $\{a_{1,j}^i\}_{j=1}^{b_i}$ of disjoint subsets. Let us denote the union of these subsets by A_1^i. Note that these sets satisfy the following property. For any $x_i \in a_{1,j_1}^i$, $y_i \in a_{1,j_2}^i$ with $j_1 \neq j_2$ the distance between x_i and y_i is at least $2\varepsilon - 2\varepsilon^2 > \varepsilon$ due to the triangle inequality. If $\mu_i(A_1^i) < \varepsilon$, we can choose $Z_1 = Z_0 \cup A_1^i$ and similarly extract A_2^i from $X_i \setminus Z_1$ that is a union of subsets $a_{2,j}^i$, $j = 1, \ldots, b_i$, satisfying the same property. Thus, we can repeat this procedure until we obtain the following partition of (X_i, μ_i):

$$X_i = \bigcup_{l=0}^{m_i} A_l^i,$$

where $\mu_i(A_0^i) \leqslant 1 - \varepsilon$ and any A_l^i with $l > 0$ admits a decomposition

$$A_l^i = \bigcup_{j=1}^{b_i} a_{l,j}^i$$

such that for any $x \in a_{l,j_1}^i$ and $y \in a_{l,j_2}^i$ the ρ_i-distance between x and y is not less than ε, and all sets $a_{l,j}^i$, $l = 1, \ldots, m_i$, $j = 1, \ldots, b_i$, have the same measure.

Now let us estimate the ε^4-entropy of (X, μ, ρ) from below. Assume that a set $E \subset X$ with measure less than ε^4 is given. We will look for a ε^4-separated set in its complement. For any point $x = (x_1, \ldots, x_k) \in X$, we define a sequence $w = w(x) \in \prod_i \{1, \ldots, m_i\}$ of k non-negative integers $\{w_r\}_{r=1}^k$ in the following way:

$$x_r \in A_{w_r}^r \quad \text{for } r = 1, \ldots, k.$$

Recall that the weights s_i's that appear in the statement of Lemma 5 satisfy $\phi^{-1}s_i < \log b_i < s_i$, where ϕ is a positive parameter. Let us fix an arbitrary w satisfying the inequality

$$(22) \qquad \sum_{w_r \neq 0} s_r \geqslant \varepsilon^2 \sum_{i=1}^{k} s_i.$$

Consider the following set $S_w = \{x \in X : w(x) = w\}$. It is easy to see that S_w can be represented as the following disjoint union:

$$(23) \qquad S_w = \bigcup_{i_r = 1, \ldots, b_r} a^1_{w_1, i_1} \times a^2_{w_2, i_2} \times \cdots \times a^k_{w_k, i_k},$$

where we take the union over those indices i_r for which $w_r \neq 0$, and all the factors corresponding to $w_r = 0$ are equal to A^r_0. We will call the subsets on the right side of formula (23) *the cells* of S_w. Note that all the cells have the same measure and, therefore, the desired inequality (11) reduces to the discrete case as follows. For a point $x_i \in A^r_l$, we denote the set $a^r_{l,j}$ containing x_i by $a^r_l(x_i)$. Let $x, y \in S_w$, then

$$(24) \qquad \begin{aligned} \rho(x, y) &\geqslant \frac{\varepsilon}{\sum\limits_{i=1}^{k} s_i} \sum_{w_r \neq 0} s_r \mathbb{1}\{a^r_{w_r}(x_r) \neq a^r_{w_r}(y_r)\} \\ &\geqslant \frac{\varepsilon^3}{\sum\limits_{w_r \neq 0} s_r} \sum_{w_r \neq 0} s_r \mathbb{1}\{a^r_{w_r}(x_r) \neq a^r_{w_r}(y_r)\}. \end{aligned}$$

Assume that the subset E contains less than a half of the cells of S_w entirely. Let us estimate from below the cardinality of the maximal ε^4-separated set in $S_w \setminus E$. Consider the set

$$P = \prod_{w_r \neq 0} \{1, \ldots, b_r\},$$

where each point $u = \{u_r\}_{w_r \neq 0} \in P$ corresponds to the cell of S_w with coordinates $i_r = u_r$, and the semimetric $\tilde\rho$:

$$\tilde\rho(u, v) = \frac{1}{\sum\limits_{w_r \neq 0} s_r} \sum_{w_r \neq 0} s_r \mathbb{1}\{u_r \neq v_r\}.$$

By (24), it is enough to estimate the cardinality of the maximal ε-separated set in a subset of P that contains at least half of its points. Indeed, take a point in each cell that is not entirely covered by E and consider the corresponding points in P. Inequality (24) guarantees that the distances in this subset of X are estimated from below by corresponding distances in the corresponding subset of P multiplied by ε^3. To estimate the cardinality of the maximal ε-separated set, it suffices to establish an upper bound for the measure of an ε-ball on the space P with the uniform measure. Since the mutual distribution of coordinate functions is uniform on $\prod_{w_r \neq 0} \{1, \ldots, b_r\}$, the random variables u_r are mutually

independent, and each u_r is uniformly distributed on the set $\{1, \ldots, b_r\}$. Thus, it is enough to estimate the probability

(25)
$$
\mathbb{P}\left\{\sum_{w_r \neq 0} s_r \mathbb{1}\{u_r \neq 1\} \leqslant \varepsilon \sum_{w_r \neq 0} s_r\right\} = \mathbb{P}\left\{\sum_{w_r \neq 0} s_r \mathbb{1}\{u_r = 1\} \geqslant (1-\varepsilon) \sum_{w_r \neq 0} s_r\right\}
$$
$$
\leqslant \mathbb{P}\left\{\sum_{w_r \neq 0} \log b_r \mathbb{1}\{u_r = 1\} \geqslant \frac{1-\varepsilon}{\phi} \sum_{w_r \neq 0} s_r\right\}
$$
$$
\leqslant 2^{-\frac{(1-\varepsilon) \sum_{w_r \neq 0} s_r}{\phi}} \cdot \mathbb{E}\left(\prod_{w_r \neq 0} 2^{\log b_r \mathbb{1}\{u_r = 1\}}\right).
$$

The first inequality follows from the conditions for the weights. The second inequality holds due to the exponential Chebyshev's inequality. Let us estimate the first factor:

(26)
$$
2^{-\frac{(1-\varepsilon) \sum_{w_r \neq 0} s_r}{\phi}} \leqslant 2^{-\frac{(1-\varepsilon)\varepsilon^2 \sum_{i=1}^{k} s_i}{\phi}} \leqslant 2^{-\frac{(1-\varepsilon)\varepsilon^2 \sum_{i=1}^{k} \log b_i}{\phi}}
$$
$$
= \left(\prod_{i=1}^{k} b_i\right)^{-\frac{(1-\varepsilon)\varepsilon^2}{\phi}} \leqslant \left(\prod_{i=1}^{k} b_i\right)^{-\frac{\varepsilon^3}{\phi}}.
$$

To estimate the second factor we use the independence of u_r:

(27)
$$
\prod_{w_r \neq 0} \mathbb{E}(b_r^{\mathbb{1}\{u_r=1\}}) \leqslant \prod_{i=1}^{k} \left(\frac{b_i}{b_i} + \frac{b_i - 1}{b_i}\right) \leqslant 2^k.
$$

Thus, the desired probability does not exceed $(\prod_{i=1}^{k} b_i)^{-\frac{\varepsilon^3}{\phi}} 2^k$. Therefore, the size of the maximal ε^4-separated set in $S_w \setminus E$ can be estimated from below by the value $\frac{1}{2}(\prod_{i=1}^{k} b_i)^{\frac{\varepsilon^3}{\phi}} 2^{-k}$. Hence,

$$
\mathbb{H}_{\varepsilon^4}(X, \mu, \rho) \geqslant \log\left(\left(\prod_{i=1}^{k} b_i\right)^{\frac{\varepsilon^3}{\phi}} 2^{-k-1}\right) = \frac{1}{\phi}\varepsilon^3 \sum_{i=1}^{k} \mathbb{H}_{4\varepsilon}(X_i, \mu_i, \rho_i) - k - 1.
$$

If the desired inequality (11) does not hold, then E must entirely contain at least half of the cells of S_w for w satisfying condition (22) and, since all the cells have the same measure, $\mu(S_w \cap E) > \frac{1}{2}\mu(S_w)$ for such w. Let us estimate the

measure of those $x \in X$ that do not satisfy condition (22):

$$\mu\left\{x \in X \colon \sum_{i=1}^{k} s_i \mathbb{1}\{x_i \in A_0^i\} \geqslant (1-\varepsilon^2) \sum_{i=1}^{k} s_i\right\}$$

$$\text{(28)} \qquad \leqslant \frac{\mathbb{E}\sum_{i=1}^{k} s_i \mathbb{1}\{x_i \in A_0^i\}}{(1-\varepsilon^2)\sum_{i=1}^{k} s_i} = \frac{\sum_{i=1}^{k} s_i \mu_i(A_0^i)}{(1-\varepsilon^2)\sum_{i=1}^{k} s_i}$$

$$\leqslant \frac{1-\varepsilon}{1-\varepsilon^2} \leqslant 1 - \frac{\varepsilon}{2}.$$

Therefore, the measure of those $x \in X$ that satisfy condition (22) is at least $\frac{\varepsilon}{2}$. Hence, $\mu(E) \geqslant \frac{\varepsilon}{4}$, and we obtain a contradiction to the choice of the exceptional set. The lemma is proved.

References

[1] T. Downarowicz and J. Serafin, *Universal systems for entropy intervals*, Journal of Dynamics and Differential Equations **29** (2017), 1411–1422.

[2] S. Ferenczi, *Measure-theoretic complexity of ergodic systems*, Israel Journal of Mathematics **100** (1997), 187–207.

[3] A. Katok and J.-P. Thouvenot, *Slow entropy type invariants and smooth realization of commuting measure-preserving transformations*, Annales de Ínstitut Henri Poincaré. Probabilités et Statistiques **33** (1997), 323–338.

[4] D. Kerr and H. Li, *Ergodic Theory*, Springer Monographs in Mathematics, Springer, Cham, 2017.

[5] W. Krieger, *On entropy and generators of measure-preserving transformations*, Transactions of the American Mathematical Society **149** (1970), 453–464.

[6] F. V. Petrov and P. B. Zatitskiy, *Correction of metrics*, Zapiski Nauchnykh Seminarov POMI **390** (2011), 201–209; English translation: Journal of Mathematical Sciences **181** (2012), 867–870.

[7] F. V. Petrov and P. B. Zatitskiy, *On the subadditivity of a scaling entropy sequence*, Zapiski Nauchnykh Seminarov POMI **436** (2015), 167–173; English translation: Journal of Mathematical Sciences **215** (2016), 734–737.

[8] J. Serafin, *Non-existence of a universal zero-entropy system*, Israel Journal of Mathematics **194** (2013), 349–358.

[9] O. Shilon and B. Weiss, *Universal minimal topological dynamical systems*, Israel Journal of Mathematics **160** (2007), 119–141.

[10] G. Veprev, *Scaling entropy of unstable systems*, Zapiski Nauchnykh Seminarov POMI **498** (2020), 5–17; English translation: Journal of Mathematical Sciences **255** (2021), 109–118.

[11] A. M. Vershik, *Information, entropy, dynamics*, in *Mathematics of the 20th Century: A View from Petersburg*, Moscow Center for Continuous Mathematical Education, Moscow, 2010, pp. 47–76.

[12] A. M. Vershik, *Dynamics of metrics in measure spaces and their asymptotic invariants*, Markov Processes and Related Fields **16** (2010), 169–185.

[13] A. M. Vershik, *Scaled entropy and automorphisms with pure point spectrum*, St. Petersburg Mathematical Journal **23** (2012) 75–91.

[14] A. M. Vershik, P. B. Zatitskiy and F. V. Petrov, *Geometry and dynamics of admissible metrics in measure spaces*, Central European Journal of Mathematics **11** (2013), 379–400.

[15] A. M. Vershik and P. B. Zatitskiy, *Universal adic approximation, invariant measures and scaled entropy*, Izvestiya Mathematics **81** (2017), 734–770.

[16] A. M. Vershik and P. B Zatitskiy, *Combinatorial invariants of metric filtrations and automorphisms; the universal adic graph*, Functional Analysis and its Applications **52** (2018), 258–269.

[17] B. Weiss, *Countable generators in dynamics-universal minimal models*, in *Measure and measurable dynamics (Rochester, NY, 1987)*, Contemporary Mathematics, Vol. 94, American Mathematical Sociey, Providence, RI, 1989, pp. 321–326.

[18] P. B. Zatitskiy, *Scaling entropy sequence: invariance and examples*, Zapiski Nauchnykh Seminarov POMI **432** (2015), 128–161; English translation: Journal of Mathematical Sciences **209** (2015), 890–909.

[19] P. B. Zatitskiy, *On the possible growth rate of a scaling entropy sequence*, Zapiski Nauchnykh Seminarov POMI **436** (2015), 136–166; English translation: Journal of Mathematical Sciences **215** (2016), 715–733.