FOREWORD

The present edition is the first part of the selected works by Alexandr Danilovich Alexandrov, one of our most brilliant geometricians. His scientific contributions address a wide range of the problems of modern mathematics and its applications.

At the turn of this century, geometry was in a position to deal with the objects "in the large." However, approaches to them were prompted neither by the available methods of differential geometry nor by the methods for studying solvability of the Cauchy problem and boundary-value problems for partial differential equations which were propounded in the 19th century. The efforts of such outstanding mathematicians as H. Minkowski, D. Hilbert, H. Weyl, et al. led only to isolated results. At the same time, their works contained the statements of many important unresolved problems which anticipated the further development of geometry "in the large" in this century. The major discoveries in studying the problems belong to Alexandrov who alone solved many hard and enduring problems of the discipline.

The works by Alexandrov on the theory of generalized Riemannian spaces are closely related to the background ideas of the section of mathematical analysis which studies the concept of weak (generalized) solution to a differential equation (see, for instance, the famous articles by S. L. Sobolev and L. Schwartz).

Alexandrov is also responsible for some important results on existence of weak solutions to some nonlinear partial differential equations (the equations of Monge-Ampère type). In the case considered by Alexandrov, the concept of weak solution is not reducible to that of the theory of partial differential equations.

Alexandrov was the first to apply many of the tools and methods of the theory of real functions and functional analysis which are current today in geometry. One of his first geometric papers using such pioneering methods is the article of Chapter 1 which concerns the infinitesimal bendings of a surface of revolution. Here the concept of infinitesimal bending is proposed and necessary and sufficient conditions are given for a vector field on a surface to be a bending field on an arbitrary surface rectifiable in the sense of Lebesque. Then Alexandrov proves a rigidity theorem for a surface of revolution which asserts absence of infinitesimal bendings other than motions. The subtlety of this result is that in studying the equations of infinitesimal bendings with classical tools there appears the second derivative of the function describing a meridian of the surface under consideration. If the function is an arbitrary convex function then the methods requiring its second derivative cannot be applied directly.

Chapter 2 contains a proof of some uniqueness theorem for convex polyhedra in three-dimensional space. The author derives from it a proof of the Minkowski theorem claiming that two convex polyhedra coincide or are translates of one another provided that their parallel faces have the same area. The proof of the uniqueness theorem is of pure synthetic nature. It is based on the idea underlying the proof given by Cauchy to his remarkable theorem claiming that convex polyhedra composed of congruent similarly-placed faces are congruent.

Chapters 3–6 present parts of a single treatise published in 1937 and 1938 in Matematicheskiĭ Sbornik. Chapter 9 also relates to the treatise. The obtained results are now considered fundamental to the theory of convex bodies and the present collection offers an opportunity to become better acquainted with the theory by consulting its definitive source. A short survey of the contents of the treatise is given in Chapter 3. The main topic of the treatise is the theory of mixed volumes of systems of convex bodies. The concept of mixed volume was suggested by H. Minkowski. In the works by Brunn and Minkowski some inequalities were obtained between various mixed volumes. The purpose of the treatise by Alexandrov which is reprinted in Chapters 3–6 is the establishment of some new inequalities between mixed volumes, derivation of various geometric corollaries of the inequalities and proof of numerous uniqueness theorems of the theory of convex bodies and the theorems on extremal properties of a ball and bodies circumscribed around a ball. (Alexandrov's inequalities were independently obtained by Fenchel and Jessen.) This treatise is characterized by the prolific use of some tools from functional analysis and the theory of real functions. Among the ingenious gadgets of research there are some countably additive set functions on the unit sphere, the area functions, curvature functions and some other functions introduced by Alexandrov. In the series of results of these chapters of the collection we also mention the inequality between mixed discriminants of positive definite quadratic forms which are established in Chapter 6 and are analogous to those for mixed volumes. These results belong principally to algebra. Some of their geometric applications are given in Chapter 6. This chapter also contains the proofs of Alexandrov's main inequalities between mixed volumes which are obtained by means of functional analysis (the Hilbert method, as the author calls it).

In Chapters 7 and 8 one general uniqueness theorem is proven for regular closed convex surfaces in three-dimensional space. In Chapter 7 the validity of the theorem is established under certain hypotheses which are modified in Chapter 8. The uniqueness theorem of Chapter 2 is an analog of the general theorem applied to polyhedra.

Alexandrov indicated a new direction of research in geometry, the theory of nonregular Riemannian spaces. Central to Riemannian geometry is the theory of curvature of a space. Alexandrov constructed a theory of nonregular Riemannian spaces satisfying a specific generalized boundedness condition on curvature. His articles devoted to the topic further develops the geometric concept of space along the lines of the tradition stemming from Lobachevskiĭ, Riemann and E. Cartan. Thereby mathematics was enriched with new beneficial ideas. Chapters 10, 11 and 12 contain the articles by Alexandrov which reflect this side of his activities.

Chapters 10 and 11 deal with the intrinsic geometry of arbitrary convex surfaces in the space $\mathbb{R}^3$. A surface is considered as a metric space whose distance between points is defined as the greatest lower bound of the lengths of curves on the surface which join the points. The challenge here is to find conditions for an arbitrary metric space to be isometric with a convex surface. Alexandrov introduces the concept of a metric of positive curvature. The sought conditions are as follows: the metric space under test is to be a two-dimensional manifold with metric of positive curvature. In particular, it is proven that a two-dimensional manifold with metric of positive curvature and homeomorphic with the sphere is isometric to a closed convex surface. This theorem provides a new solution to the familiar Weyl problem but is stated now in an essentially more general form than that considered by Weyl himself. The solution given by Alexandrov is purely geometric in nature. The crux of the argument is the existence theorem for a closed convex polyhedron with a given development.

Chapter 12 relates to the Alexandrov research into mathematical problems of crystallography. Here he considers the problem of tiling a space with polyhedra among which only finitely many are geometrically distinct.

Chapter 13 is devoted to study of metric spaces of curvature not greater than K, with K a constant. A particular instance of such a space is a Riemannian space whose sectional curvature at every point is at most K. The definition is given of a space with curvature not greater than K and various properties of such spaces are established.

A wide circle of articles by Alexandrov is devoted to the theory of partial differential equations. A starting point for this research is within geometry. The present Part 1 of the collection contains only two of Alexandrov's papers reflecting this side of his activities. In Chapter 14 an existence theorem is proven for a solution to the Dirichlet problem for the equation of the form

$\det \|z_{ij}\| = \varphi(z_1, \dots, z_n, z, x_1, \dots, x_n)$, where z is an unknown function, z_i are its first derivatives and z_{ij} its second derivatives. The concept of solution is treated here in some generalized sense and is determined by geometric means. The proof of the main result is based on making use of the methods developed in the theory of convex bodies. Chapter 15 outlines a method for estimating solutions to second order partial differential equations which is indebted to some geometric considerations connected with using the convex envelope of a solution.

The final Chapter 16 gives an overview of Alexandrov's research which relates to mathematical foundations of relativity theory. The question addressed and solved here is to find minimal conditions characterizing Lorentz transformations in relativity theory.

Alexandrov has many students and followers. Virtually all of his articles, including those reproduced here, are continued and developed in the research of other scientists. Analysis and comparison would require so ample room that we were impelled to decline the challenge (with a natural relief). We hope that the reader will find the book of relevance, the paucity of comments notwithstanding.

Yu. G. Reshetnyak
S. S. Kutateladze