The Legacy of Vladimir Andreevich Steklov in Mathematical Physics: Work and School

Nikolay Kuznetsov (Russian Academy of Sciences, St. Petersburg)

The 150th anniversary of the birth of the outstanding Russian mathematician Vladimir Andreevich Steklov falls on 9 January 2014. All over the world, active researchers in all areas of mathematics know his name. Indeed, well-known mathematical institutes of the Russian Academy of Sciences in Moscow and St. Petersburg are named after Steklov. This commemorates that he was the founding father of their predecessor: the Physical-Mathematical Institute established in 1921 in starving Petrograd (the Civil War still persisted in some corners of what would become the USSR the next year). Steklov was the first director of the institute, until his untimely death on 30 May 1926.

Meanwhile, Steklov's personality is less known even in present-day Russia. Of course, the biographical sketch [39] by J. J. O'Connor and E. F. Robertson is available online but still the best source of information about Steklov and his work is the very rare book [16]: the proceedings of a session of the Leningrad Physical-Mathematical Society, which took place on the occasion of the first anniversary of Steklov's death. It must be said that the lack of knowledge about his work was the reason for translating into Russian and publishing a collection of Steklov's papers concerning various problems in mechanics [62]. Not too much is also written about his relations with a group of bright students (most of them graduating from the St. Petersburg University in 1910). Together with students of Andrey Andreevich Markov Sr. and Nikolay Maksimovich Günther, they formed the germ which later developed into the Petrograd–Leningrad–Petersburg school, famous for contributions of its scholars to mathematical physics, functional analysis and some other areas of mathematics as well as theoretical physics.

To clarify the word "school", which has various meanings in Russian as well as in English, it is worth quoting A. N. Parshin's recent note [40].

> A school is a community of individuals who work in the same branch of science, who are in close communication with each other, who have a leader, a teacher, amongst whom each generation passes on the torch to the next one, and all this forms one integral organism.

After this "definition", Parshin describes the branching at the Mekh-Mat (the Faculty of Mechanics and Mathematics) of the Moscow University from the original school founded by N. N. Luzin.

Speaking about Steklov's school, founded a little bit earlier than Luzin's, I understand it in the same sense as the latter is treated by Parshin. Of course, after the lapse of 100 years, the school of which Steklov was the founding father has given rise to various schools in the widely understood field of mathematical physics. It is worth mentioning that his school very soon became international. Indeed, J. D. Tamarkin and J. A. Shohat (they were members of Steklov's circle of students) emigrated in the 1920s to the USA, where they had many PhD students. In particular, Tamarkin had 28 students, and so his 1495 descendants (the Mathematics Genealogy Project as of 22 August 2013) are Steklov's "scientific" grandchildren, great-grandchildren and so on. At the same time, the project gives only 888 as his own descendants.

Of course, Steklov's interests in science were much wider than mathematical physics; for example, the above mentioned collection [62] includes 12 of his major contributions to mechanics (440 pages in total). An idea of development of Steklov's work in mechanics during the past decade can be obtained from the papers [5], [66] and references cited therein. Another area which he studied during 26 years is theory of orthogonal polynomials. In the reference book [46], one finds 31 papers by Steklov (the first two published in 1900, whereas the last two dated 1926), and the properties of polynomials investigated in each of them are clearly indicated. Fortunately, this topic of Steklov's research is covered in the survey article [63]; further progress can be found in [43] and [44].

The present paper consists of three sections. In the first one, I briefly describe the non-scientific legacy of Steklov and then turn to quoting his own writings taken from various sources. These excerpts describe his personality and the way he started creating his school. The latter involves his relations with J. D. Tamarkin, A. A. Friedmann, V. I. Smirnov, J. A. Shohat and his other talented students that are shown through the prism of Steklov's diaries and recollections. Then his activity as Vice-President of the Russian Academy of Sciences during the last seven years of his life is presented in the same way. Steklov's role was crucial for Academy's survival during the period of revolutions and the Civil War in Russia. He exemplifies how to withstand governmental attacks on the Academy, something that is particularly important nowadays.

Since Steklov's major achievements in mathematical physics have been summarized in his book [57] entitled *Fundamental Problems of Mathematical Physics*, its contents and significance are discussed in the second section, whereas advances in the area of the potential-theoretic approach to boundary value problems for the Laplace equation (the topic of the book's volume 2) are described in the brief third section.

It should be mentioned that many of Steklov's papers are now available online as well as the complete list of his publications. For the latter see http://steklov150.mi.ras.ru/steklov_pub.pdf. The journals *Annales fac. sci. Toulouse* and *Annales sci. ENS*, in which many Steklov's articles are published, are available at http://www.numdam.org. Almost twenty of his papers in French and some papers in Russian can be found at http://www.mathnet.ru/php/person.phtml?option_lang=rus&personid=27728.

V. A. Steklov About Himself, His Students, Science and the Academy of Science

The legacy of Steklov is multifaceted; along with his work in mathematics and mechanics (see [68] for a survey) it includes scientific biographies of Lomonosov and Galileo, an essay about the role of mathematics (these three books in Russian were printed in 1923 in Berlin because Russian economics was ruined during World War I and the Civil War), the travelogue of his trip to Canada, where he participated in the Toronto ICM in 1924, his correspondence — published (see [60] and [61]) and unpublished — recollections [59] and still unpublished diaries.

Fortunately, many excerpts from Steklov's diaries are quoted in [65] (some of them appeared in [39] as well). In my opinion, the most expressive is dated 2 September 1914, one month after war with Germany and Austria-Hungary was declared by the Russian government.

> St. Petersburg has been renamed Petrograd by Imperial Order. Such trifles are all our tyrants can do — religious processions and extermination of the Russian people by all possible means. Bastards! Well, just you wait. They will get it hot one day!

What happened in Russia during several years after that confirms clearly how right was Steklov in his assessment of the Tsarist regime. In his recollections [59] written in 1923, he describes vividly and, at the same time, critically "the complete bacchanalia of power" preceding the collapse of "autocracy and [Romanov's] dynasty" in February 1917 (old style), "the shameful transient government headed by Kerensky, the fast end of which can be predicted by every sane person", how "the Bolshevik government [. . .] decided to accomplish the most Utopian socialistic ideas in multi-million Russia"; the list can be easily continued. My aim is to give quotations from [59] and from the unpublished manuscript *Excerpts from my diaries* (*Excerpts* in what follows) widely quoted in [22], that characterize Steklov's personality, his relations with a group of talented students at the St. Petersburg University, his understanding of the role of science for himself and his work as Vice-President of the Academy of Sciences. Translation from the Russian is mine, if not stated otherwise.

[1] There are two words meaning freethinking in Rus-

Years of Education. In the *Excerpts,* Steklov writes about his final years at Alexander Institute (a kind of gymnasium) in Nizhni Novgorod.

> I turned my room into a kind of "physical cabinet" — laboratory equipped with Leyden jars, an electrical machine and home-made Galvanic elements. Various chemical experiments (of course, elementary) were carried out. [...] I reduced my contacts with schoolmates (previously numerous), continuing to keep in touch only with those of them who, like me, were interested in mathematics and physics.
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> ... ...
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> The topic of "test composition" [preceding the certificate exam] was as follows: *The reign of Catherine II was a great period.* In a [satirical] poem [by A. K. Tolstoy published not long before that], there are several lines characterizing her in a way far from being respectful to "Her Majesty". However, they added a specific colouring to my essay. [...] I wrote, without any idea to manifest political freethinking, that Catherine's period only looks great but, in fact, it puts an end to reforms initiated by Peter I. [...] To my great surprise, Shaposhnikov (director of the Institute) came to the classroom after reading our essays [...] and asked me: "Where have you, our best student, got this inclination to freethinking[1] and such an impermissible attitude toward the Great Empress?" [...] For almost an hour, he was explaining my thoughtlessness and my wrong understanding of history etc. [...] After that, he dragged in by the head and shoulders the following point of view: preferring mathematics, physics and chemistry to other disciplines, I follow an objectionable way. He said: "May be, this is the reason that you 'took those liberties in thinking'. This your feature was noted long before but definitely revealed itself in your essay."
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> I repeat that it was a surprise for me, but did not become a stimulus to change my mind. [...] Just the opposite, I said to myself: "Aha! It occurs that I have my own point of view on historical events and it is different from that of my schoolmates and teachers. [...] It was the director himself who proved that I am, in some sense, a self-maintained thinker and critic." This was the initial impact that led to my mental awakening; I realised that I am a human being able to reason and what is important to reason freely. [...] Soon after that, my freethinking encompassed religion as well. [...] Thus, the cornerstone was laid to my future complete lack of faith.

sian. One of them — "vol'nodumstvo" — has a negative nuance and it was used by director.

In another passage from the *Excerpts*, Steklov describes how he failed to pass an examination at Moscow University.

> The last oral exam was in physical geography taught by the stern professor Stoletov. Rather quickly, I have managed to study this easy discipline within the lecture course. My reply to the questions formulated on the card was excellent. Suddenly, Stoletov asks: "What date is the longest day in Moscow?" I was completely taken aback by this question. My silence lasted several minutes. Stoletov was glassy staring at me and, at last, he said sluggishly: "Complete ignorance". He writes *unsatisfactory* in my record-book, and I am ruined because my marks for all other difficult exams were excellent. [...] It seemed that committing suicide was the best decision to suppress the feelings tearing my soul apart at that moment. However, this idea came to my mind only afterwards when I had already calmed down.

About science. The thought about committing suicide came to Steklov once more, when he was a second-year student in Kharkov. It was caused by his rather complicated love affairs. In his recollections [59], he writes in connection to this.

> Soon I came to the conclusion that any reasoning as to whether it is worth to live or not is an inadmissible stupidity and moral cowardice. It is worth to live for the sake of pursuit of knowledge and even my experience — rather small at that time — had already demonstrated that all other kinds of activity occupying people are deceptive and temporal. Research is the only kind of activity that occupies you forever and

> never deceives a person who wants and is ready to devote himself/herself to it. Soon, I immersed myself into studies once and for all. Moreover, the young professor Alexander Mikhailovich Lyapunov (my fellow countryman who, afterwards, became an outstanding mathematician) joined the faculty shortly after that. He was my teacher and the only friend; his guidance of my first steps in science is unforgettable.

About students.[2] It is an amazing and lucky coincidence that the same year (1906) as Steklov got his professorship at the St. Petersburg University a group of very gifted students entered it to study mathematics. In the file of M. F. Petelin (he was one of them), this fact was noted by Steklov as follows.

> I should note that the class of 1910 is exceptional. In the class of 1911 and among the fourth-year students who are about to graduate there is no one equal in knowledge and abilities to Messrs. Tamarkin, Friedmann, Bulygin, Petelin, Smirnov, Shohat and others. There was no such case during the fifteen years of teaching at the Kharkov University either. This favorable situation should be used for the benefit of the University.

This quotation as well as further ones concerning Steklov and his students show how attentive to them he was. Their future fate was very different; two of them (Bulygin and Petelin), unfortunately, died young.

A. A. Friedmann became famous for his discovery in general relativity; his solution of Einstein's equations was the first one that describes the expanding Universe (see [14] and [15]). However, he died aged 37, just 7 months after his appointment as Director of the Main Geophysical Observatory in Leningrad and 2 months after his flight to the record altitude of 7,400 meters. V. I. Smirnov (a corresponding member of the Academy of Sciences of the USSR since 1932, and a full academician since 1943) is known for his results in complex analysis and mathematical physics. He is the author of *A Course in Higher Mathematics* (the first two of its five volumes were written in collaboration with Tamarkin but revised for later editions). From 1922 until his death in 1974, Smirnov's activity was associated with Leningrad University, where he founded the Research Institute for Mathematics and Mechanics in 1931, and afterwards headed several departments at the Faculty of Mathematics and Mechanics (Mat.–Mekh.). In the 1950s and 1960s, the Leningrad school of mathematical physics founded by Steklov flourished under the direction of Smirnov. His effort in restoring the Leningrad Mathematical Society in 1959 was also crucial. (The existence of its predecessor — the Physical-Mathematical Society — lasted from 1921 until 1930, when it disbanded due to political pressure; see [64].)

J. D. Tamarkin and J. A. Shohat emigrated to the USA in 1925 and 1923, respectively. They were active in research in various areas of analysis (the book [47] is their most cited work) and in supervising PhD students (G. Forsythe — one of Tamarkin's students — was afterwards a prolific PhD adviser himself). In 1927, Tamarkin was called to Brown University, whereas Shohat was at the University of Pennsylvania since 1930. Tamarkin was also involved in editing various journals; in particular, he was one of the editors of the Mathematical Reviews when it started in 1940. As a member of the Organizing Committee for the 1940 ICM, Tamarkin was very efficient. (Unfortunately, the congress was postponed because of World War II and took place after Tamarkin's death.) He was also an influential member of the AMS Council since 1931 and Vice-President of the Society in 1942–1943.

In his recollections [59], Steklov also mentions A. S. Besikovitch who graduated in 1912 and was appointed to a professorship 5 years later at the newly opened Perm University (it was Steklov who recommended him). Besikovitch became famous for his contribution to the theory of almost periodic functions and for his results that form the cornerstones of geometric measure theory. He emigrated from the USSR in 1925, and after staying one year in Copenhagen with Harald Bohr, moved to the UK. There, he became a university lecturer in Cambridge in 1927 and the Rouse Ball Chair of Mathematics in 1950. He had been elected F.R.S. in 1934 and received several academic awards.

In Steklov's diaries, the first mention of students is, chronologically, in the entry for 13 January 1908. What follows is a set of most important entries.

> 13 January, 1908. At 4 o'clock Tamarkin and Friedmann (undergraduate students) turned up and brought the continuation of the lectures in integral calculus

[2] In this section, all quotations are taken from the English version of [65].

they had written. They took the ones I had corrected (i.e. looked through. No possibility of correcting them properly!) They said they would come to my lecture on the 16th. They asked me if it was possible to legalize the mathematical society without a supervisor. I told them to make some suggestions. Let us see!

20 February. Brought my collected works to the University and gave them to Tamarkin for the students' mathematical society. Three memoirs are missing.

21 October. Tamarkin and Friedmann came to see me this evening. They are going to organize a mathematics reading room. Asked me to be their supervisor. Declined, but they deserve help.

22 November. Tamarkin and Friedmann came to see me this evening [...] Kept asking me about their delvings into the theory of orthogonal functions. They are having an article published in Crelle's journal. Sharp fellows! They left at half past twelve, after supper.

18 April, 1909. The students Tamarkin, Friedmann, Petelin came to see me this evening [...] I proposed to Tamarkin that he think about the asymptotic solution of differential equations (i.e. stability, in the sense of Poincaré and Lyapunov) or the problem of equilibrium of a rectangular plate. To Friedmann I suggested he find orthogonal substitutions, when fundamental functions are products of two (see my dissertation). I suggested Petelin read what Jacobi had to say about the principle of the last multiplier. I'll think it over again and will probably find some other topics too.

12 September. This evening Tamarkin, Friedmann and Petelin came to see me. They had worked on the assigned topics. Seem to have done something. Promised to submit their essays in a month. Tamarkin seems to be doing better than the others.

Steklov coauthored only 2 papers and one of them was a joint paper with Tamarkin. It was published in *Rend. Circ. Mat. Palermo* in 1911 and so was written when Tamarkin was still a student.

About administration work in the Academy of Sciences. In Steklov's recollections [59], his comments on this topic are rather brief but they show that he clearly understood his role in the survival of the academy as the leading scientific institution.

> In 1919, I was unanimously elected to the post of Vice-President of the Russian Academy of Sciences.[3] At the same time, the [Petrograd] University [...] insisted that I have to head it, but this burden was decidedly rejected by me. Indeed, the state-of-affairs existing at that time would not allow any human to do both jobs properly. [...] It was absolutely clear to me that I could really do a lot for the benefit of the Academy. [...] On the other hand, I saw that the university was on the brink of collapse at that time.
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> ... ...
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> First, it must be said that the Academy is still one of a few institutions that were successfully vindicated from various destructive attacks. Moreover, its reputation was growing gradually in the eyes of ruling circles, and now, the Academy is recognized as the leading scientific institution. At last (in September 1923), I have achieved a success in the matter that I tried to accomplish for a long time, namely, that the Academy must be considered on equal terms with Narkompros [the Ministry of Education]. [...] I can say with satisfaction, without boasting, that my contribution to achieving all these results favorable for the Academy is very considerable.

About Steklov's daily schedule. At the end of his recollections [59], one finds the following.

> In my opinion, it is exclusively due to a particular daily schedule that I manage to separate administration and research so that both of them flow parallel not interrupting each other. I adopted this schedule during my student days.
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> My day is divided into two parts as follows. The time from 10 am to 5 or 6 pm I devote to administration at the Academy. Then I dine and about 7 pm go to bed. I sleep until 9:30 pm (sometimes until 10 pm). After awakening, I

[3] In 1925, it was renamed the Academy of Sciences of the USSR.

> have a cup of tea and then, leaving apart all thoughts about administration and having nothing revulsive, calmly do my research.
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> I work until 4 or 5 am in the morning (sometimes longer). [...] Three hours of sleep after dinner allow me to sleep from 4 or 5 am to 9:30 am, that is, 5 and sometimes 6 hours. This is my daily schedule for more than 40 years and I find it expedient to a great extent.

Of course, it is difficult to stop quoting Steklov's recollections and diaries, but enough is enough.

Steklov's Unfinished Monograph *Fundamental Problems of Mathematical Physics*

Mathematical achievements of the first half of the 20th century are described in the book [41]. Its first section entitled "Guidelines 1900–1950" is compiled by P. Dugac, B. Eckmann, J. Mawhin and J.-P. Pier with the assistance of an international team of almost six dozens prominent mathematicians (V. I. Arnold and S. S. Demidov represent the Moscow school). "Guidelines" is a year by year list of major results and their authors; the most important books published during the period from 1900 to 1950 are also presented in this 34-page list. It includes two items concerning mathematical physics published in 1923: *Lectures on Cauchy's problem in linear partial differential equations* by J. Hadamard and the two-volume book [57] by Steklov. Its second edition [58] appeared 60 years later with a vast number of comments and some necessary corrections made by V. P. Mikhailov and A. K. Gushchin (both from the Steklov Mathematical Institute, Moscow).

Here, my first aim is to explain why the treatise [57] is among the most valuable contributions to mathematical literature of the first half of the 20th century despite the fact that it was not finished by Steklov (see below). Second, further advances will be described in the area of applications of potential theory to boundary value problems for the Laplace equation which is the topic of the second volume of [57].

Prior to that, it is worth mentioning several other contributions to "Guidelines" which came from Steklov's mathematical school. What follows are corrected excerpts from the list in [41] supplied with citations of the corresponding original papers:

1932 A. S. Besikovitch, *Almost Periodic Functions.* Cambridge University Press.

1936 S. G. Mikhlin, Symbol of a singular integral operator; [35], [36], [37] (a comprehensive updated presentation can be found in the monograph [38]).

1936 S. L. Sobolev, First results on distributions; [49].

1937 N. M. Krylov, N. N. Bogolyubov, Averaging method for nonlinear differential systems; [31].

1938 S. L. Sobolev, Sobolev spaces; mollifiers; [50].

1950 S. L. Sobolev, *Applications of Functional Analysis in Mathematical Physics.* (In Russian; English translation was published by the AMS in 1964.) This book presents results obtained in [49] and [50] in a comprehensive form.

Notice that the notion of mollifier proposed by Sobolev is a far-ranging generalisation of the *Steklov mean function*, which is the most simple averaging operator (see [1], § 74, for the definition and properties). It was introduced by Steklov in 1907 for studying the problem of expanding a given function into a series of eigenfunctions defined by a 2nd-order ordinary differential operator; see [54] and [55] for the announcement and full-length paper, respectively. Of course, "Guidelines" contain many other entries due to mathematicians from Russia, beginning with Steklov's teacher A. M. Lyapunov (1901 — Central limit theorem), and ending with several entries for 1950, one of which is M. G. Krein's "Parametric resonance in higher dimensional Hamiltonian systems". The overall best number of entries is 14, unsurprisingly, by A. N. Kolmogorov.

Let us turn back to the monograph [57]; it is based on lectures given to a small group of well prepared audiences in 1918–1920. This is why this book is written in Russian despite the fact that the underlying papers were written in French. Its 1st volume was finished in April 1919 and the 2nd volume was finished in November 1922, respectively). More material was presented in the lecture course than appeared in [57]; Steklov planned to publish the 3rd volume with his results concerning "fundamental" functions (that is, eigenfunctions of various spectral problems for the Laplacian) and some applications of these functions. He describes this objective on p. 257 of the 2nd volume and also mentions it in [59], p. 299. Unfortunately, his administrative duties as Vice-President of the Academy prevented him from realizing this project. However, one gets an idea about the probable contents of the unpublished 3rd volume from the lengthy article [53]. In this paper, which appeared in 1904, Steklov developed his approach to "fundamental" functions (see [26],

§ 5, by A. Kneser for its brief outline). This approach uses two different kinds of Green's function and this allows one to apply theory of integral equations worked out by I. Fredholm [13] and D. Hilbert [21] shortly before that.

In 1923 — the year when [57] had been published — Russian scientists were still cut off from their colleagues in the West after the October Revolution. It is worth emphasizing great efforts of Steklov and his fellow academicians Abram Fedorovich Ioffe (he founded the Physical–Technical Institute in Petrograd in 1918), Alexei Nikolaevich Krylov (naval engineer and applied mathematician known for his work [30] winning a Gold Medal from the Royal Institution of Naval Architects) and Sergey Fedorovich Oldenburg (the Permanent Secretary of the Russian Academy of Sciences) directed towards restoring contacts with colleagues abroad as well as to set up exchanges through scientific publications. Anyway, at that time no attempt was made to translate [57] into French, German or English. However, 11 years later, N. M. Günther (presumably, he attended Steklov's lectures) gave an account of potential theory and its application to the Dirichlet and Neumann problems following the approach proposed by Steklov. First, Günther's book [19] was published in French, then its Russian revised and augmented edition appeared in 1953 and, finally, the English translation [20] of the latter was issued in 1967.

In his book, Steklov considers boundary value problems as mathematical models of physical phenomena and so two essential requirements must be fulfilled for a solution of any such problem: the existence and uniqueness theorems. This is the first important point of vol. 1. Notice that the notion of a well-posed problem was introduced simultaneously by Hadamard in his book mentioned above; he complemented these two requirements with the following one: a solution must depend continuously on the problem's data.

The second important point of vol. 1 is the systematic rigorous justification of the Fourier method for initial-boundary value problems for parabolic and hyperbolic equations with variable coefficients not depending on the time and depending on a single spatial variable. For this purpose the following stages must be accomplished.

- The existence of an infinite sequence of eigenvalues and eigenfunctions must be proved.
- It must be shown that the set of eigenfunctions is "rich enough" for expanding every "sufficiently smooth" function into a Fourier series.
- One has to prove that the obtained Fourier series gives a solution of the problem under consideration.

A detailed study of the Sturm–Liouville problem serves as the basis for the first two of these stages, and 6 of 11 chapters of vol. 1 are devoted to this problem. Steklov had written many papers on this topic (the first of them "On cooling of a heterogeneous bar" was published in Russian and dates back to 1896); the presentation of material in [57] follows his final article [56].

A great part of the contents of vol. 2 is based on the major Steklov's original contribution to the theory of boundary value problems for the Laplace equation: the two-part article [51], [52] published in 1902, the second of which is the most cited of Steklov's work. Confusingly, his intitials are given in a wrong way in almost all its citations. Indeed, R. Weinstock mistook the abbreviation "M. W." ("M." stands for "Monsieur" in French) for Steklov's initials and this was afterwards reproduced elsewhere. Nevertheless, we must be grateful to Weinstock for introducing the term "the Steklov problem" in his paper [69] published in 1954, in which he initiated studies of the following problem:

$$\nabla^2 u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} = \lambda \varphi u \text{ on } \partial D. \quad (1)$$

In fact, Steklov proposed this problem in his talk at a session of the Kharkov Mathematical Society in December 1895; nowadays, it is mainly referred to as the Steklov problem but, sometimes, it is also called the *Stekloff problem* as in [69].

In problem (1), D is, generally speaking, a bounded Lipschits domain in $\mathbb{R}^m$, n is the exterior unit normal existing almost everywhere on ∂D and λ denotes the spectral parameter. This problem is similar to the spectral problem for the Neumann Laplacian in the following sense. The latter problem describes the vibration of a homogeneous free membrane, while the Steklov problem models the vibration of a free membrane with all its inhomogeneous mass $\varphi \geqslant 0$, $\varphi \not\equiv 0$ concentrated along the boundary (see [3], p. 95).

In [69], an isoperimetric inequality is proved for the smallest positive eigenvalue of (1) under the following assumptions: $m = 2$, whereas $\varphi \equiv 1$ on ∂D which is an analytic curve. Further progress achieved about inequalities for eigenvalues of problem (1) and other related problems can be found in the book [3] by C. Bandle's, in § 8 of the survey article [2] by M. S. Ashbaugh and R. D. Benguria, and also in the recent papers [4], [17] and [18] by I. Polterovich and his coauthors.

In the opinion of Steklov's contemporaries (see [26], § 6, and two papers by Günther in [16]), which is shared by the compilers of "Guidelines", his results presented in vol. 2 are of paramount interest. They deal with the Dirichlet and Neumann problems in interior and exterior domains separated by a closed surface in $\mathbb{R}^3$. Steklov was the first who proved the existence of solutions to these problems by means of potential theory in the case of an *arbitrary* (that is, without any shape restriction) $C^{1,\alpha}$-surface, $\alpha \in (0, 1]$. (These surfaces are also referred to as Lyapunov's because they were introduced by him in [33].) In order to prove the existence of solutions Steklov used iterative procedures aimed to finding the densities of the double and single layer potentials which solve the Dirichlet and Neumann problem, respectively. Thus, a definitive solution had been given to a long-standing question concerning these problems. However, unlike many other definitive solutions, Steklov's did not kill the field and further developments are outlined in the next section.

Potential-Theoretic Approach to Boundary Value Problems for the Laplace Equation

The method which is standard in textbooks nowadays is as follows. Potential theory is applied for reducing the Dirichlet and Neumann problems to integral equations which then are investigated with the help of Fredholm's theorems. It seems that it was O. D. Kellogg who first realized this approach in detail in his comprehensive monograph [23]. However, his assumption, that a surface dividing $\mathbb{R}^3$ into two domains belongs to the class C^2, is superfluous. V. I. Smirnov applied the same approach in the case of Lyapunov's surfaces in his classical textbook [48] (its 1st edition was published in 1941). This assumption is sufficient to guarantee that the kernels of arising integral operators have a weak (polar) singularity.

As early as 1916, T. Carleman [9] initiated studies of boundary value problems for the Laplace equation in domains with non-smooth boundaries. In particular, he developed a potential-theoretic approach to the Dirichlet and Neumann problems in the case when a surface dividing $\mathbb{R}^3$ into two domains consists of several pieces each belonging to the C^2 class and overlapping pairwise along edges which are C^2-curves. Moreover, half-planes tangent to two different pieces must not coincide at every edge-point. The method used by Carleman in three dimensions is a straightforward generalization of his technique applied for two-dimensional domains with a finite number of corner points on the boundary (see also [32], § 2.1.3, where this technique is outlined).

In 1919, J. Radon [42] made the next step in developing the potential-theoretic approach for irregular domains in two dimensions. He considered the corresponding integral operators for contours having "bounded rotation" without cusps (see also the survey paper [34], ch. 4, § 1, for the exact definition). It took more than 40 years to generalize Radon's result to boundary value problems in irregular higher-dimensional domains. As often happens, this was accomplished simultaneously in two different places: in Leningrad and in Prague (see [7] published by mathematicians from Steklov's school and [27], respectively). One can find further details in [6] and [28] (see also [34], ch. 4, § 2). In particular, it was shown that the square of C. Neumann's operator (the latter is also referred to as the direct value of the double layer potential) is a contraction operator on the boundary of a convex domain (see the paper [29] by J. Král and I. Netuka, and also Král's lecture notes [28], § 3). Moreover, converging iterative procedures were developed in this case for the integral equations of interior and exterior Dirichlet and Neumann problems. These procedures involve Neumann's operator (and its dual, respectively) and are similar to those proposed by Steklov in the case when an arbitrary Lyapunov surface divides $\mathbb{R}^3$ into two domains.

During the last quarter of the 20th century, the potential-theoretic approach to the Dirichlet and Neumann problems for the Laplace equation was devised for C^1 and Lipschitz domains. The beginning to this development was laid by A. P. Calderon's note [8], in which boundedness of the Cauchy singular integral was proved in L^p over a Lipschitz curve provided the Lipschitz constant is sufficiently small; this resriction was later removed for L^2 (see [10]). These results allowed the investigation of solubility of boundary integral equations for problems with L^p boundary data (see [12] and [67] for the case of C^1 and Lipschitz domains, respectively). A brief review of these results is given in the survey article [34], ch. 4, § 3, whereas the book [24] by C. E. Kenig contains their systematic exposition, some generalizations and an extensive list of references. Besides this, a simple treatment of boundary integral operators on Lipschitz domains was proposed by M. Costabel [11].

Furthermore, B.-W. Schulze and G. Wildenhain [45] presented results concerning potential theory for higher order elliptic equations and covered the usual topics as in the classical case; general boundary value problems, strongly elliptic systems and problems in Beppo Levi spaces are also considered.

In conclusion of this section, one more development of Steklov's approach to iterative solutions of boundary integral equations should be mentioned. In the case of the exterior Dirichlet problem for the Laplace equation, he used iterations which give the problem's solution although the corresponding homogeneous integral equation has a non-trivial solution. Besides, if one applies potentials involving the standard fundamental solution of the Helmholtz equation for solving exterior boundary value problems then the corresponding homogeneous integral equations usually have non-trivial solutions for several values of the frequency problem's parameter. (The values for which the method fails are referred to as *irregular* frequencies.) Nevertheless, it is possible to modify a boundary integral equation so that a properly transformed iteration method gives its solution for all frequencies. It was shown by R. E. Kleinman and G. F. Roach [25] that one has to use modified Green's function and to adapt iteration procedure for this purpose. Modified Green's function is equal to the sum of the standard fundamental solution and a series of some given solutions of the Helmholtz equation with properly chosen coefficients. This technique was introduced in the 1970s and further developed in the 1980s (see references cited in [25]); it allows one to obtain integral equations without irregular frequencies at the expense of loosing self-adjontness of the integral operator. Furthermore, for an integral equation with modified Green's function there exists the iterative solution converging to the exact one as a geometric progression.

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N. Kuznetsov heads the Laboratory for Mathematical Modelling of Wave Phenomena at the Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, St. Petersburg

Institute for Problems in Mechanical Engineering
V.O., Bol'shoy pr. 61
199178, St. Petersburg
Russian Federation
e-mail: nikolay.g.kuznetsov@gmail.com