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A SUFFICIENT CONDITION FOR WEAK MIXING OF SUBSTITUTIONS AND STATIONARY ADIC TRANSFORMATIONS

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The goal of the present article is the proof of a sufficient condition for weak mixing for a wide class of stationary adic automorphisms introduced in [1] and [2], and substitution automorphisms, the method based on the results of [3]. All spaces of sequences are provided with the weak topology.

We shall define a substitution automorphism following [4, 6, 8].

1. An initial object is an aggregate of words $\mathcal{A} = \{A_i\}_1^n$ in an alphabet $Z = \{1, \ldots, n\}$. A matrix [4-6] $G_{\mathcal{A}} = (g_{ij})_{i,j=1}^n$ is constructed with respect to $\mathcal{A}$ where $g_{ij} \geq 0$ is the number of occurrences of symbol j in word A_i. In the following we assume that $G_{\mathcal{A}}$ is primitive (i.e., positiveness of some degree) [7, p. 378].

We define a transformation $\omega_{\mathcal{A}}$ of the set of all finite words: $\omega_{\mathcal{A}}: \bigcup_{k=1}^{\infty} Z^k \to \bigcup_{k=1}^{\infty} Z^k$: $\omega_{\mathcal{A}}(a_1, \ldots, a_l) = A_{a_1}A_{a_2}\ldots A_{a_l}$ where the right-hand side is a concatenation of words. The restriction $\omega_{\mathcal{A}}|_Z$ is called a substitution.

A generating quadruple shall be defined as a quadruple of natural numbers (i, j, m, n'), $i \leq n'$, $j + 1 \leq |A_i|$, such that there exist symbols a, b, c, $d \in Z$, such that: 1) c and d are respectively the j-th and j + 1-st symbols of word A_i; 2) word $\omega_{\mathcal{A}}^m c$ ends on a and word $\omega_{\mathcal{A}}^m d$ begins on b; 3) $\omega_{\mathcal{A}}^{n'}(a)$ ends on a and $\omega_{\mathcal{A}}^{n'}(b)$ begins on b.

Generating quadruples always exist. We shall choose an arbitrary one, and by using its corresponding a and b, we shall define a sequence of concatenations of words: $\{a\}\{b\}$; $\omega_{\mathcal{A}}^{n'}(\{a\})\omega_{\mathcal{A}}^{n'}(\{b\})$; $\omega_{\mathcal{A}}^{2n'}(\{a\})\omega_{\mathcal{A}}^{2n'}(\{b\})$, etc. In each of the concatenations $\omega_{\mathcal{A}}^{kn'}(\{a\})\omega_{\mathcal{A}}^{kn'}(\{b\})$ we shall index the symbols in such a way that the last symbol $\omega_{\mathcal{A}}^{kn'}(\{a\})$ has index 0. By the definition of a and b, the sequence of these concatenations is a sequence of words that is increasing on both sides, and its union is a sequence of symbols from Z that is infinite on both sides. We denote it by x(i, j, m, n'). We consider the weak closure (in the space of all sequences that are infinite on both sides) of trajectory x(i, j, m, n') with respect to a two-sided shift. We denote it by $X_{\mathcal{A}}$. We shall call the shift $X_{\mathcal{A}} \to X_{\mathcal{A}}$ a substitution automorphism. Under the condition that $G_{\mathcal{A}}$ is primitive, $X_{\mathcal{A}}$ does not depend on the choice of the generating quadruple. It is known that T is strictly ergodic [5].

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The stationary automorphism defined in [1, 2] or the adic transformation, which we shall discuss below, is also a symbolic dynamical system, however, with a different action. Again let $Z = \{1, \ldots, n\}$ be an alphabet, $\Pi = \{\pi_{ij}\}$ be a transition matrix. We shall assume that all π_{ij} equal 0 or 1 and that Π is primitive. An adic automorphism acts on a Markov compactum, i.e., on the space Ω_Π of paths $x_0, x_1, \ldots, x_i \in Z$ $(i = 0, 1, \ldots)$ that are infinite on one side such that $\pi_{x_i x_{i+1}} = 1$. The action is described in [1] and [2].

2. With respect to an adic transformation matrix Π, we can construct an aggregate of words analogous to the aggregate of words that define a substitution: $\mathcal{A}_\Pi = \{A_i\}_1^n$, where A_i' is a word whose symbols are indices of all the nonnull symbols of the i-th column of Π, enumerated in increasing order [3].

The substitution constructed with respect to $\mathcal{A}_\Pi$, as in Sec. 1, is metrically isomorphic to the same adic transformation in a wide class of cases.

Below we shall describe constructions for a substitution that put it in correspondence to adic or generalized adic transformations that are metrically isomorphic to it.

After fixing a substitution (i.e., Z and $\mathcal{A}$), we shall say that every sequence belonging to space $X_\mathcal{A}$ is admissible (see above); $X_\mathcal{A} \ni x = \{b_k\}_{k=-\infty}^{\infty}$.

We shall say that a substitution satisfies condition UAD, i.e., uniquely admissible decodability, if every admissible sequence from $X_\mathcal{A}$ is uniquely representable in the form of a union $\ldots A_{i_k} A_{i_{k+1}} \ldots$ such that $b_0 \in A_{i_0}$ and $\ldots i_k i_{k+1} \ldots$ is also admissible. The sequence $\ldots i_k i_{k+1} \ldots$ also has only one admissible decoding by the cited definition, etc.

It is entirely likely that every noncyclic [6] substitution is of this type (see also [9, 10, 12]. We shall assume that the UAD condition is satisfied.

With respect to the set of words $\mathcal{A}$ we construct two dynamical systems of the adic transformation type. For each word $A_i \in \mathcal{A}$ and any symbol j coming into it, we index all occurrences of a symbol in the order in which they are encountered in the word $j^{(1)}$, $j^{(2)}$, $\ldots$, $j^{(k)}$. We shall call an indexed symbol a distinguished symbol. We denote by $\Omega_\mathcal{A}^1$ the space of sequences of distingushed symbols (paths) that are infinite on one side of the type $y_0^{(i_0)}$, $y_1^{(i_1)}$, $\ldots$, $y_s \in Z$ such that for all $s > 0$ the distinguished symbol $y_{s-1}^{(i_{s-1})}$ corresponds to the i_{s-1}-st occurrence of symbol y_{s-1} in word A_{y_s}.

We define a transformation $T_\mathcal{A}^1 : \Omega_\mathcal{A}^1 \to \Omega_\mathcal{A}^1$ in the following manner. If $A_{y_1} = a_1, \ldots, a_r$ and the symbol corresponding to $y_0^{(i_0)}$ has index $s < r$ in word A_{y_1}, then $T_\mathcal{A}^1 (y_0^{(i_0)}, y_1^{(i_1)}, \ldots) = a_{s+1}^j, y_1^{(i_1)}, y_2^{(i_2)}, \ldots$, where for symbol a_{s+1} the index of its occurrence j is also distinguished. If $s = r$, let ℓ be the smallest index >0 such that the symbol corresponding to $y_\ell^{(i_\ell)}$ is not last in word $A_{y_{\ell+1}} = b_1, \ldots, b_{r'}$, and has index $s' < r'$ (for all but a finite set of points, $\ell < \infty$). Then

$$T_\mathcal{A}^1 (y_0^{(i_0)}, y_1^{(i_1)}, \ldots) = z_0^{(1)}, z_1^{(1)}, \ldots, z_l^{(j)}, y_{l+1}^{(j_{l+1})}, y_{l+2}^{(j_{l+2})},$$

where $z_\ell = b_{s'+1}$ (j is also the index of occurrence), each of z_i^1, $i < \ell$ is first in word $A_{z_{i+1}}$ ($A_{z_{i+1}}$ does not depend on the index of occurrence of z_{i+1} in the following word).

Thus, in the space of paths we can define a particular ordering analogous to the lexicographic ordering in a Markov compactum. Let $\Gamma_1 > \Gamma_2$ if there exists $m > 0$ such that $\Gamma_1 = T_{\mathscr{A}}^{1m}\Gamma_2$. It is clear that finite paths with identical length and identical ending are comparable in the same manner.

Again let $\mathscr{A}$ determine a substitution, and describe two sets: $X'_{\mathscr{A}} \subset X_{\mathscr{A}}$ $\Omega^{1'}_{\mathscr{A}} \subset \Omega^1_{\mathscr{A}}$ each of which is an extension of a countable invariant set, and describe a one-to-one relation between them: $\varphi: X'_{\mathscr{A}} \to \Omega^{1'}_{\mathscr{A}}; T^1_{\mathscr{A}} \circ \varphi = \varphi \circ T$.

Let $U_{\mathscr{A}} \subset X_{\mathscr{A}}$ be the set of all $x(i, j, m, n')$, where (i, j, m, n') is a generating quadruple (see the definition of a substitution). The set $U_{\mathscr{A}}$ is finite, but the set of generating quadruples is countable. We define a set $X_{\mathscr{A}}'$ as $X_{\mathscr{A}} - \bigcup_{i=-\infty}^{\infty} T^i(U_{\mathscr{A}})$ and the set $\Omega_{\mathscr{A}}''$ as the set of paths $y_0^{(i_0)}, \ldots, y_1^{(i_1)}, \ldots, y_k^{(i_k)}, \ldots$, such that for an infinite set of indices k the symbol $y_{k-1}^{(i_{k-1})}$ is not first in word A_{y_k}, and for an infinite set of indices k the symbol $y_{k-1}^{(i_{k-1})}$ is not last in word A_{y_k}.

If $x \in X'_{\mathscr{A}}, x = \ldots b_{-k}, \ldots, b_0, \ldots, b_k, \ldots$ then it is easy to see that there exists a unique (corresponding to successive admissible decodings) sequence of elements of Z: $y_0, y_1, y_2, \ldots$, where $b_0 = y_0$, y_0 is a symbol of word $\omega_{\mathscr{A}}(y_1) = A_{y_1}$, y_k is a symbol of word $\omega_{\mathscr{A}}(y_{k+1}) = A_{y_{k+1}}$, etc.; for an infinite set of indices k, y_k is not the first symbol of $A_{y_{k+1}}$, and for an infinite set of indices k it is not last (otherwise $x \in \bigcup_{-\infty}^{\infty} T^i U_{\mathscr{A}}$). For $\varphi(x)$ we take the path $y_0^{(i_0)}, y_1^{(i_1)}, \ldots$ with fixed indices of occurrences of symbols y_k in word $\omega(y_{k+1})$. It is obvious that φ is bijective and measurable with respect to any invariant Borel probability measure. From the arguments presented, the following assertion follows:

$T_{\mathscr{A}}^1$ has a unique invariant Borel probability measure, and φ is a metric isomorphism between T and $T_{\mathscr{A}}^1$.

Thus, we have given a definition of one-ergodicity for mapping $T_{\mathscr{A}}^1$ that is not everywhere defined (naturally generalizable for arbitrary adic mappings). Specifically, we have given the definition of minimality, i.e., compactness of all elements of the partition is required everywhere on the trajectory, which in our case does take place.

We note that for an adic transformation with matrix Π we have the equation $(\Omega_\Pi, T) = (\Omega^1_{\mathscr{A}_\Pi}, T^1_{\mathscr{A}_\Pi})$ which can be considered simply the definition of an adic transformation. If a substitution with set of words $\mathscr{A}_\Pi$ satisfies the UAD condition, then it is metrically isomorphic to (Ω_Π, T).

Now we define a dynamical system $T^2_{\mathscr{A}}: \Omega^2_{\mathscr{A}} \to \Omega^2_{\mathscr{A}}$ or more precisely, a class of equivalent adic transformations that define it.

Let $N = \sum_1^n |A_i|$. We cite the elements of alphabet $Z' = \{1, \ldots, N\}$ in one-to-one correspondence with a set of pairs of natural numbers of type (f, g) where $1 \le f \le n$, $1 \le g \le |A_f|$: $\ell \to \{f_\ell, g_\ell\}$, $1 \le \ell \le N$. Accordingly, we shall show a unique requirement: for each q: $1 \le q \le n$, the relation on set $\{(q, s); 1 \le s \le |A_q|\}$ must be strictly monotonic in s. We define matrix

$$\Pi = \{\pi_{ij}\}_{i,j=1}^{N} = \begin{cases} \pi_{ij} = 1, & \text{if } f_i \text{ is a symbol with index } g_i \text{ in } A_{f_j}, \\ \pi_{ij} = 0 & \text{otherwise.} \end{cases}$$

As $\Omega_{\mathcal{A}}{}^2$ we take the Markov compactum Ω_Π.

Dynamical system $T^2_{\mathcal{A}} : \Omega^2_{\mathcal{A}} \to \Omega^2_{\mathcal{A}}$ is defined as an adic transformation of Markov compactum Ω_Π. This definition is correct since up to a trivial isomorphism which is a homeomorphism, our dynamic system does not depend on the choice of relation that defines Π. However, we may assume that we are dealing not with a class of isomorphic dynamical systems, but with adic transformations, if in the definition of an adic transformation we require an ordering not of the entire state space Z', but of each of the subsets of type $Y_q \subset Z'$: $y_q = \{s \in Z' | \pi_{sq} = 1\}$, $1 \le q \le N$. It is clear as well that all these transformations are equivalent to $T^1_{\mathcal{A}}$: $\Omega^1_{\mathcal{A}} \to \Omega^1_{\mathcal{A}}$.

Let T be a substitution automorphism constructed with respect to the substitution defined by set of words $\mathcal{A}$, or the adic automorphism defined by matrix Π. We denote by $f_T(t) = t^n - \sum_{i=0}^{n-1} \alpha_i t^i$ the characteristic polynomial of matrix $G_{\mathcal{A}}$ (respectively matrix Π). Further arguments shall be cited for the case of the (more general) transformation $T^1_{\mathcal{A}}$: $\Omega^1_{\mathcal{A}} \to \Omega^1_{\mathcal{A}}$ which under fulfillment of UAD is a generalized adic representation of the substitution transformation.

We shall formulate an obvious assertion. Let $P_1 = y_0{}^{(i_0)}, \ldots, y_\ell$ and $P_2 = z_0{}^{(j_0)}, \ldots, z_\ell$ be two paths of length ℓ, where

$$y_0 = z_0, \quad y_l = z_l, \quad P_1 < P_2. \tag{1}$$

We define a sequence of natural numbers $N_m(P_1, P_2)$ in the following manner: for any sequence (path) $x_0{}^{(s_0)}, \ldots, x_m{}^{(s_m)}, y_0{}^{(i_0)}, \ldots, y_0{}^{(i_0)}, \ldots, y_\ell{}^{(i_\ell)}, x_{m+\ell+2}{}^{(s_{m+\ell+2})}, \ldots$ belonging to $\Omega_{\mathcal{A}}{}^1$, we have

$$T^{N_m(P_1,P_2)}(x_0^{(s_0)}, \ldots, x_m^{(s_m)}, y_0^{(i_0)}, \ldots, y_l^{(i_l)}, x_{m+l+2}^{(s_{m+l+2})}, \ldots) = x_0^{(s_0)}, \ldots, x_m^{(s_m)}, z_0^{(j_0)}, \ldots, z_l^{(j_l)}, x_{m+l+2}^{(s_{m+l+2})}, \ldots$$

It is clear that the number $N_m(P_1, P_2)$ does not depend on the choice of x_i. Our assertion is that $N_m(P_1, P_2)$ satisfies recurrence relation

$$N_{m+n}(P_1, P_2) = \sum_{i=0}^{n-1} \alpha_i N_{m+i}(P_1, P_2).$$

We shall formulate the primary theorem of the article.

<u>THEOREM</u>. For substitution automorphism T with the UAD condition (for a stationary adic or generalized adic transformation) let all roots of the characteristic polynomial $f_T(t)$ lie outside the circle $B = \{z | |z| < 1\}$ and let there exist two paths P_1, P_2 of identical length with conditions (1) such that for any index m_0, when $m > m_0$, all $N_m(P_1, P_2)$ in the aggregate do not have a common divisor larger than one. Then the corresponding automorphism has weak mixing.

<u>Proof of the Theorem</u>. The theorem follows from the following propositions and lemmas.

<u>Proposition</u>. If a sequence of whole numbers N_m satisfies recurrence relation $N_{m+n} = \sum_{i=0}^{n-1} \alpha_i N_{m+i}$ where α_i are rational coefficients of polynomial $t^i - \sum_{i=0}^{n-1} \alpha_i t^i$ with roots modulo ≥ 1,

and there does not exist an index m_0 beginning with which all N_m in the aggregate have a common divisor > 1, then there does not exist a nonintegral λ such that

$$e^{2\pi i N_m \lambda} \xrightarrow[m\to\infty]{} 1.$$

In the proof, the case with rational λ is considered with the help of the condition that a common divisor of N_m be lacking, and the case with irrational λ is considered by analogy with [11, p. 166].

That is, $N_m\lambda$ is a sequence satisfying the same recurrence relation; beginning at a certain place, it is satisfied by a sequence $\beta_m \to 0$, where $\beta_m = -R_m + N_m\lambda$ (R_m is the integer nearest to $N_m\lambda$), which contradicts the assumption about the roots of the polynomial.

LEMMA. If a strictly ergodic substitution (stationary adic automorphism) with the UAD condition has eigenvalue $e^{2\pi i\lambda}$, and P_1 and P_2 are two paths of identical length that satisfy (1), then

$$e^{2\pi i N_m(P_1, P_2)\lambda} \to 1, \quad m \to \infty.$$

Proof of Lemma. The assertion follows from the coincidence of the invariant measure with the topological structure of $\Omega^1_{\mathscr{A}}$, i.e., an analogue of the theorem about "density points" in the theory of functions of a real variable.

Let $C \subset C_{\mathscr{A}}{}^1$ be a measurable set, $\mu(C) > 0$. The for almost every point x there exists a sequence of cylindrical sets $V_1{}^x > \ldots > V_j{}^x > \ldots \ni x$, corresponding to paths that begin at time 1 such that

$$\lim_{n\to\infty} \mu(C \cap V_n^x)/\mu V_n^x = 1.$$

From this assertion it follows that if U is an eigenfunction of automorphism T with eigenvalue $e^{2\pi i\lambda}$, then for arbitrary positive sequences $\varepsilon_k \to 0$, $\delta_k \to 0$ it is possible to choose a sequence of sets A_k, $\mu(A_k) > 0$, and cylinders B_k such that the deviation of U on A_k is $\le\varepsilon_k$ and

$$\mu(\tilde{B}_k \cap A_k)/\mu A_k > 1 - \delta_k.$$

Since $G_{\mathscr{A}}$ is primitive, and from the properties of measure μ, we get constants $D > 0$ and $M > 0$ such that if Y_{m,P_i} is a set of those x for which the coordinates from $m + 2$ to $m + \ell + 2$ form path P_i ($i = 1, 2$) then for any n_1, n_2 with $n_2 - n_1 > M$ and any path B of length n_1 we have $\mu(Y_{n_2,P_i} \cap \tilde{B}) \ge D\mu\tilde{B}$, where $\tilde{B}$ is the cylindrical set corresponding to B.

From this and from the definition of $N_m(P_1, P_2)$ it follows that if for cylinder B_k the length of a path generating it is N_k, and $D > 2\delta_k$, then for $m > N_k + M$ we have $|e^{2\pi i N_m(P_1,P_2)\lambda} - 1| \le \varepsilon_k$.

The lemma and thus also the theorem are proved.

As an example we can consider the substitution $0 \to 001$, $1 \to 10110$ from [4], where a somewhat different method of equivalent substitution does not make it possible to establish weak mixing. The theorem of the present article is applied for this along with the method of [4] for other examples from [4]. It is sufficient to take paths $P_1 = 0^1 0$ and $P_2 = 0^2 0$. In this case [4]

$$N_m(P_1, P_2)=\frac{1}{3}(2\cdot 4^n+1) \text{ (roots } f_T(t) \text{ are 1 and } 4).$$

The results on the discrete spectrum of substitutions (obtained by a somewhat different method), which imply the analogous theorem on weak mixing, are contained in [12], which was published as the present article went to press. By the method described in the present article, we can prove also the assertion from [12] (obtained independently by the author) on the fact that every measurable eigenfunction almost everywhere coincides with a continuous function. A continuous eigenfunction can be constructed as well by the method of extension from an everywhere dense trajectory, by considering that reduction to the above-proved lemma is exponential, and by considering the obvious existence of a natural M for the primitive matrix such that for any two infinite paths of type

$$x_1^{(i_1)}, \ldots, x_{n'}^{(i_{n'})}, y_1^{(k_1)}, y_2^{(k_2)}, \ldots,$$

$$x_1^{(i_1)}, \ldots, x_{n'}^{(i_{n'})}, z_1^{(l_1)}, z_2^{(l_2)}, \ldots$$

there exist two sequences of natural numbers N_k, L_k, $N_{k-1} < L_k < N_k < L_{k+1}$, $|N_{k+1} - N_k|$, $|L_{k+1} - L_k| < M$, and the path

$$x_1^{(i_1)}, \ldots, x_{n'}^{(i_{n'})}, t_1^{(r_1)}, t_2^{(r_2)}, \ldots \in \Omega^1_{\mathscr{A}},$$
$$t_{N_k}=y_{N_k}, t_{L_k}=z_{L_k} \quad (k=1, 2, \ldots).$$

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