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On certain sequence of matrices

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ABSTRACT

The sequence of adic automorphisms (and corresponding substitutional dynamical systems) is considered, which was studied by the author earlier. The properties of special automorphisms over such systems are discussed. The sufficient condition of weak mixing is formulated. Some conclusions of preceding work are revised, concerning the concrete dynamical systems. Different possibilities are considered, appearing in more general case.

In [1] the matrices

$$A_n = \begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & \ddots & \mathbb{O} & \\ & \ddots & \ddots & \ddots & \\ & \mathbb{O} & \ddots & 2 & 1 \\ & & & 1 & 2 \end{pmatrix}$$

were considered together with corresponding adic and substitutional dynamical systems. Here, for the case $n \geqslant 4$ we consider more general case (making some corrections, concerning mentioned paper).

Keeping the notations and definitions from [1] at first formulate the condition of weak mixing of special automorphisms over substitutional or adic dynamical systems with function, depending only on zero coordinate.

As in [1] consider the adic transformation T_M, acting in Markov compact X_M, where $\{1,\dots,n\}$-states and M — the primitive transition matrix of adic transformation. Let $f : \{1,\dots,n\} \to N$ — the function and f^* — corresponding map $X_M \to N$, $f^*(\{x_0,\dots\}) = f(x_0)$. Let T_M^f be the special automorphism, built by T_M and function f^*. It is easy to prove its strict ergodicity (in natural sense).

Here we give the definition of good sequences for T_M^f. Let $P_1 = y_0,\dots y_l$; $P_2 = z_0,\dots,z_l$ be two paths of length l with $y_0 = z_0$; $y_l = z_l$; $P_1 < P_2$. Define the sequence of natural numbers $N_m^f(P_1,P_2)$. For any path $x_0,\dots,x_m,y_0,\dots,y_l,x_{m+l+2},\dots \in X_M$ the path $x_0,\dots,x_m$ $z_0,\dots,z_l,x_{m+l+2},\dots$ belongs to X_M too and for some K, not depending on $\{x_i\}$, we have $(T_M^f)^K(\{x_0,\dots,x_m,y_0,\dots,y_l,x_{m+l+2},\dots\},0) = (\{x,\dots,x_m,z_0,\dots,z_l,x_{m+l+2},\dots\},0)$ (these are two points of phase space of special automorphism). Put $N_m^f(P_1,P_2) = K$. If P_1, P_2 are fixed, $N_m^f(P_1,P_2)$ is the recurrent sequence. The characteristic polynomial of the matrix M is the appropriate polynomial. Call good all the sequences $N_m^f(P_1,P_2)$. The same definition is to be used for the generalized adic transformation [2] and corresponding special automorphisms.

For the primitive substitutions with unicity of admissible decoding [2] over alphabet $\{1,\dots,n\}$ the function $f : \{1,\dots,n\} \to N$ defines the special automorphism too. The sequence L_k^f is called good iff there exists admissible sequence of sort $a_1,\dots,a_l$; $a_1 = a_l$, such that if for any k to denote as $b_1^k\dots b_{s(k)}^k$ the block $w_A^k(a_1\dots a_{l-1})$ then $L_k^f = \sum_{i=1}^{s(k)} f(b_i^k)$. Of course $L_k^f = (\{c_1,c_2,\dots,c_n\}, G_A^k U_f)$ where $U_f = \{f(1),\dots,f(n)\}$ and c_i, $1 \leqslant i \leqslant n$ — the number of symbols i among $a_1,\dots,a_{l-1}$.

Recall now the definition of the polynomial $g_v(u)$ for recurrent sequence (the notation from [1]). Let $v = \{v_1,\dots,v_m\}$ be the complex vector, $f(u)$ be the polynomial $a_0u^m + a_1u^{m-1} + \dots + a_m$, having the simple roots $u_1,\dots,u_m$. As follows from the theory, there exists the only polynomial $g_v(u)$ of the degree $< m$, such that the sequence $h_l = \sum_{k=1}^{m} g_v(u_k)(u_k f'(u_k))^{-1} u_k^l$ satisfies the recurrence $a_m h_l + a_{m-1}h_{l+1} + \dots + a_1 h_{l+m+1} + a_o h_{l+m} = 0$, $l > 0$, and $h_l = v_l$ for $1 \leqslant l \leqslant m$. The explicit formula for the polynomial g_v is the following one:

$$g_v(u) = \sum_{i=1}^{m} v_i a_{m-i} + u\sum_{i=1}^{m-1} v_i a_{m-i-1} + u^2 \sum_{i=1}^{m-2} v_i a_{m-i-2} + \dots u^{m-1} \sum_{i=1}^{1} v_i a_{1-i}.$$

The formulation of the condition of the weak mixing from [1] carries without changes (together with the remark). Still list it.

If $L = \{L_k\}_{k=1}^{\infty}$ the recurrent sequence, the vector $v_L = \{v_1,\dots,v_n\}$ corresponds to it. The polynomial g_{v_L} corresponds in its turn to this vector. Suppose that the roots $t_1,\dots t_n$ of the polynomial $f_T(t)$ are simple.

Theorem. *Let $L^f = \{L^f_k\}$ be a good sequence and there exist no k_0 such that all L_k, $k > k_0$, have the common divisor > 1 and that no polynomial with integer coefficients exist, constant and irrational on the set $\{t_i | g_{v_L}(t_i) \neq 0 \ \& \ |t_i| \geqslant 1\}$. Then the special automorphism T^f_A possesses the weak mixing.*

Analogous is the formulation for adic transformations. The eigenfunctions, if the discrete spectrum exists, are continuous too in the weak topology (natural topology of the phase space of special automorphism). If the recurrent sequence is fixed then it can satisfy many recurrences. But the set $\{t_i | t_i \neq 0 \ \& \ g_{v_L}(t_i) \neq 0\}$ does not depend on the choice of recurrence. We shall use it later.

The case, considered in [1], did correspond to $f \equiv 1$. The investigation of concrete good sequence L in [1] led us to the expression of g_{v_L}. We used the admissible sequence nn. Use it here as well.

Recall that for the polynomial $P_n(t) = \det(A_n - tI)$ the roots are $t^{(n)}_k = 2 + 2\cos\alpha^n_k$, where $\alpha^n_k = \pi\frac{n+1-k}{n+1}$. In fact $P_n(t) = U_n(1 - \frac{t}{2})$ where the U_n's are the Chebyshev polynomials of the second kind ([3], 10.11). Now we have $L_l = \sum_{k=1}^{n} (t^{(n)}_k)^l y^n_k \sum_{j=1}^{n} f(j) \sin j\alpha_k$ (because A_n is symmetric) where y^n_k are the coefficients of the decomposition $e_n = \sum y^n_k X^n_k$, where $e_n = (0, \dots, 0, 1)$ is the vector-column and X^k_n are the eigenvectors of A_n with $x^n_{k,j} = \sin\left(j\frac{\pi(n+1-k)}{n+1}\right), 1 \leqslant k,\, j \leqslant n$. The calculations, almost identical to those of [1], using the same trigonometrical identities and formula for the determinant, lead us to the conclusion that g_{v_L} is up to constant factor is determined by the formula

$$q_{v_L}(t^{(n)}_k) = ct^{(n)}_k \sum_{j=1}^{n} f(j) \frac{\sin j\alpha^n_k}{\sin \alpha^n_k}, \quad \text{where every} \quad \frac{\sin j\alpha^n_k}{\sin\alpha^n_k}$$

is of course the polynomial of $t^{(n)}_k$ (Tchebichef polynomial of the second kind of $(t^{(n)}_k - 2)$).

Contrary to [1] here we shall suppose, that U_f is such that the second condition of our theorem concerning the nonexistence of polynomial, is fulfilled. Maybe it is allways or almost allways so. The arguments of [1] help to check it at least in every concrete case. We shall control the first condition-about the divisibility. For the case $f \equiv 1$ (considered in [1]) it is often violated, contrary to the formulation of [1].

Investigate more carefully the polynomials $P_n(t)$ and matrices A_n in order to use the properties of them, described below (maybe all of them are known), to our situation. There are two cases: 1) n is even. Consider the new basis $(e_1, \dots, e_{n/2}, e'_1, \dots e'_{n/2})$; $e_j = (0 \dots 0\underset{j}{1}0 \dots 0)$, $e'_j = (0 \dots 0\underset{j}{1}0 \dots 0 \underset{n+1-j}{1} 0 \dots 0)$. The matrix in this basis looks, as it is easy to see, as

$$A' = \left(\begin{array}{cccccc|cccccc}
2 & 1 & & & & & & & & & & \\
1 & 2 & 1 & & \mathbb{O} & & & & & & & \\
 & 1 & \ddots & \ddots & & & & & \mathbb{O} & & & \\
 & & \ddots & \ddots & 1 & & & & & & & \\
 & \mathbb{O} & & 1 & 2 & 1 & & & & & & \\
 & & & & 1 & 1 & & & & & & \\
\hline
 & & & & & & 2 & 1 & & & & \\
 & & & & & & 1 & 2 & 1 & & \mathbb{O} & \\
 & & \mathbb{O} & & & & & 1 & \ddots & \ddots & & \\
 & & & & & & & & \ddots & \ddots & 1 & \\
 & & & & 0 & 0 & & \mathbb{O} & & 1 & 2 & 1 \\
 & & & & 0 & 1 & & & & & 1 & 3
\end{array}\right)
\begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} \frac{n}{2} \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} \frac{n}{2} \end{array}$$

It follows that $P_n(t) = \left(P_{\frac{n}{2}}(t) - P_{\frac{n}{2}-1}(t)\right)\left(P_{\frac{n}{2}}(t) + P_{\frac{n}{2}-1}(t)\right)$. One can still prove this identity trigonometrically as other identities with U_n (what is mentioned in [3]). Denote by A^* the matrix

$$\left(\begin{array}{ccccc|c}
1 & & & & & \\
 & 1 & & \mathbb{O} & & \\
 & & \ddots & & & \mathbb{O} \\
 & \mathbb{O} & & 1 & & \\
 & & & & 1 & \\
\hline
 & & & & & \\
 & & \mathbb{O} & & & \mathbb{O} \\
 & & & & &
\end{array}\right) * A'$$

We have that the characteristical polynomial of A^* is $P_{\frac{n}{2}}(t) - P_{\frac{n}{2}-1}(t) = \cdots - \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2}t + 1$ (what is easy to calculate). So for any f_1, such that U_{f_1} is the linear combination of e_j's with integer coefficients, having g.c.d. 1, by the reasoning, analogous to the reasoning of [1], we get, that for every $j \leqslant \frac{n}{2}$ the sequence of j-th components of $A^{*l}U_{f_1}$ satisfies our condition about divisibility. Now take arbitrary f_2 with $U_{f_2} \in \operatorname{span}(\{e'_j\})$. For function $f = f_1 + f_2$ we obtain that the condition about the n-th component for it is satisfied too. Really, for every m $A'^m U_f - A^{*m} U_{f_1} \in \operatorname{span}(\{e'_j\})$. Taking into account the symmetry of the last difference, we conclude that the components of $A'^m U_f$ can have the common divisor > 1 only if the components of A^{*m} have, what is not the case as we saw. So the g.c.d. is allways 1 what in the same way implies our assertion.

2) n is odd, $n = 2k+1$. Consider the basis $(e_1, \ldots, e_k, e'_1, \ldots, e'_{k+1})$

$$e_j = (0, \ldots, \underset{j}{1}, 0, \ldots, 0); \quad e'_j = \begin{cases} (0, \ldots \underset{j}{1}, 0, \ldots, 0, \underset{n+1-j}{1}, 0, 0) & j < k+1 \\ (0, \ldots 0 \underset{k+1}{1}, 0, \ldots, 0) & j = k+1 \end{cases}$$

The matrix in this basis is

$$A' = \left(\begin{array}{cccccc|cccccc}
2 & 1 & & & & & & & & & & \\
1 & 2 & 1 & & \mathbb{O} & & & & & & & \\
 & 1 & \ddots & \ddots & & & & & \mathbb{O} & & & \\
 & & \ddots & \ddots & 1 & & & & & & & \\
 & \mathbb{O} & & 1 & 2 & 1 & & & & & & \\
 & & & & 1 & 2 & & & & & & \\
\hline
 & & & & & & 2 & 1 & & & & \\
 & & & & & & 1 & 2 & 1 & & \mathbb{O} & \\
 & & \mathbb{O} & & & & & 1 & \ddots & \ddots & & \\
 & & & & & & & & \ddots & 2 & 1 & \\
 & & & & & & & \mathbb{O} & & 1 & 2 & 1 \\
 & & & & & 1 & & & & & 2 & 2
\end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} k \\ \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} k+1 \end{array}$$

It follows that $P_n(t) = P_k(t)(P_{k+1}(t) - P_{k-1}(t))$. Formulas (37), (38) from [3] 10.11 imply this relation too.

The matrix A^* here is analogous to A_k. So some vectors from $\operatorname{span}(\{e_j\})$ can satisfy our condition if to consider the matrix A_k. Just like in first case we see that if it is so for some function f_1 with $U_{f_1} \in \operatorname{span}(\{e_i\})$ and f_2 is such that $U_{f_2} \in \operatorname{span}(\{e'_j\})$ then it is so for $f_1 + f_2$. Consider now in both cases the "bad" possibilities $U_f \in \operatorname{span}(\{e'_j\})$ that was the case in [1]. The calculation of $P_k(t) + P_{k-1}(t)$ and $P_{k+1}(t) - P_{k-1}(t)$ will show that rather often the $(n+1)$ adic discrete component of spectrum in the first case and 2-adic component in the second one is guaranted. It is evident that the d-adic component is guaranted if our polynomial is $\pm t^n + dg(t)$, where $g(t)$ has integer coefficients. From the other hand the following trivial lemma (analog of lemma of [1] and other similar assertions) shows which are the obstacles to occuring of such component.

Lemma. *Let $f(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + a_n$ be the polynomial with the integer coefficients, such that for some integer matrix A $f(A) = 0$. Let k, r be the natural numbers, such that $K \nmid a_{n-r-1}$ and $K | a_{n-i}$, $0 \leqslant i < r+1$. If for some integer vector a $K | A^s a$, s-natural, then $K | A^{r+1} a$.*

For calculation of coefficients use the Chebyshev polynomials of the first kind $T_n(x)$.
One can easily check (because of the recurrences), that

$$P_k(t) = \frac{T_{k+1}(1-\frac{t}{2}) \times (1-\frac{t}{2}) - T_k(1-\frac{t}{2})}{t(\frac{t}{4}-1)}. \tag{1}$$

(1) and recurrences imply that $P_{k-1}(t) = \frac{T_{k+1}(1-\frac{t}{2}) - T_k(1-\frac{t}{2})(1-\frac{t}{2})}{t(\frac{t}{4}-1)}$. So $P_k + P_{k-1} = -\frac{2}{t}(T_{k+1}(1-\frac{t}{2}) - T_k(1-\frac{t}{2}))$ There is a representation of T_m by means of hypergeometric function $T_m(x) = F(m, -m, \frac{1}{2}, \frac{1-x}{2})$. So

$$T_m\Big(1-\frac{t}{2}\Big) = F\Big(m, -m, \frac{1}{2}, \frac{t}{4}\Big) = 1 + \frac{m}{2}\sum_{s=1}^{m} \frac{1}{s} C_{m+s-1}^{2s-1}(-t)^s$$

$$P_k(t) + P_{k-1}(t) = t^k(-1)^k + \sum_{s=1}^{k}(-t)^{s-1}\Big(\frac{k+1}{s}C_{k+s}^{2s-1} - \frac{k}{s}C_{k+s-1}^{2s-1}\Big).$$

For many (not for all) values of k all differences are divisible by $2k+1$. It is not the case, for example, for $k = 10$, $s = 2$. For $P_{k+1}(t) - P_{k-1}(t)$ we get analogously

$$P_{k+1}(t) - P_{k-1}(t) = 2T_{k+1}\Big(1-\frac{t}{2}\Big) = 2 + (1+k)\sum_{s=1}^{k+1}\frac{1}{s}C_{k+s}^{2s-1}(-t)^s.$$

Parity of every coefficient is the same as that of C_{k+s}^{2s-1}. In spite of great variety of situations we see that for many values of $n \geqslant 4$ a lot of weak mixing dynamical systems exists. By the simple induction one can see that it is the case for all $n \neq 2^l - 1$, $l > 2$. To prove that only for such n it is the case one must prove that if $k+1 = 2^{l-1}$ then all $\frac{1+k}{s}C_{k+1}^{2s-1}$, $s \leqslant k$ are even. It is evident.

References

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