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ON THE LIMIT DISTRIBUTIONS AND ASYMPTOTICS OF EXTREMAL VALUES FOR CERTAIN SEQUENCES OF RANDOM VARIABLES

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The limit distributions are studied for some sequences of sums of coordinate random variables over the dynamical system, connected with the Rudin–Shapiro substitution. The description of the limit distributions is presented on the basis of the expression for the sums of Rudin–Shapiro coefficients obtained earlier. The expression of the density is given for particular subsequences and the Markov representation for the "stationary" situation. The evalution of the excess is given for a wide class of substitutions. Bibliography: 11 *titles.*

Introduction

In [1], questions were treated connected with the problem of finding limit distributions for some sequences of coordinate random variables over the strictly ergodic system generated by the Rudin–Shapiro substitution. The goal of this paper is to prove certain general facts about the weak convergence of distributions which were described in [1], to consider special cases using a method different from that used in [1], and to continue the investigation of a general problem set up in [1] about the matrices of generating functions of random variables for obtaining a more direct description of limit distributions. In addition, a simple general method is suggested which enables one to obtain estimates of the excess (sometimes sufficient for proving topological mixing) for substitutional dynamical systems on the basis of the results of [2] about asymptotics of the sums for the fixed points of substitutions.

§1. Weak convergence of subsequences of distributions

Consider the class E of sequences of natural numbers

$$E = \big\{ \{n_k\}_{k=0}^{\infty};\ 0 \le n_0 \le 3;\ n_{k+1} = 4n_k + l_{k+1};\ 0 \le l_{k+1} \le 3 \big\}.$$

Similar to [1], let for some $\{n_k\} \in E$ a sequence $\{\mu_k\}$ of measures on the real line be defined, describing the distribution of the values of $\dfrac{S(n+n_k)-S(n)}{2^k}$, $0 \le n < \infty$:

$$\mu_k\{x \le d\} = \lim_{n\to\infty} \frac{1}{n+1} \#\Big\{ l : 0 \le l \le n,\ \frac{S(l+n_k)-S(l)}{2^k} \le d \Big\},$$

where $-\infty < d < \infty$. The existence of μ_k follows from an interpretation (contained, in particular, in [1]) of $S(n)$ in terms of the Rudin–Shapiro substitution. We prove the following proposition:

Proposition 1. *The sequence μ_k weakly converges to a limit measure $\mu = \mu(\{n_k\})$.*

Proof. It is evident that for any natural k and p there exists a natural number $R_{k,p} \le 4^p$ such that $n_{k+p} = R_{k,p} + 4^p n_k$. Using the evaluations of [3] and the recurrency properties of substitutional dynamical systems, it is easy to deduce (similarly to the proof of the linear evaluation of complexity) the existence of a constant $C > 0$ such that for any n, u, and k, $u < 4^{k+1}$, the relation $\left| \dfrac{S(n+u)-S(n)}{2^k} \right| < C$ holds. Now let a continuous function f with modulus of continuity ω_f be defined on the segment $[-C, C]$. Evaluate the difference $\mu_{k+p}(f) - \mu_k(f)$. We have by the ergodic theorem

$$\mu_{k+p}(f) = \lim_{n\to\infty} \frac{1}{n\cdot 4^p} \sum_{i=0}^{n\cdot 4^p - 1} f\Big(\frac{S(i+n_{k+p}) - S(i)}{2^{k+p}} \Big)$$

$$= \lim_{n\to\infty} \frac{1}{n\cdot 4^p} \sum_{j=0}^{n-1} \sum_{q=0}^{4^p-1} f\Big(\frac{S(j\cdot 4^p + R_{k,p} + q + n_k\cdot 4^p) - S(j\cdot 4^p + q)}{2^{k+p}} \Big).$$

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The definition of C implies that the argument in the last expression has the form

$$t = \frac{S(j \cdot 4^p + n_k \cdot 4^p - 1 + 4^p) - S(j \cdot 4^p - 1 + 4^p)}{2^{k+p}} + \beta : \quad |\beta| < \frac{2C}{2^k}.$$

From properties of the function $S(n)$ (Theorem 3 of [3]), it follows now that

$$t = \beta + \frac{S(j + n_k) - S(i)}{2^k}$$

and, consequently, $|\mu_{k+p}(f) - \mu_k(f)| < \omega_f(2C/2^k)$, which implies our assertion.

Assigning to any sequence $\{n_k\} \in E$ the number $h(\{n_k\}) = \lim_{k\to\infty} n_k/4^k$, we get a bijective correspondence between E and $[0,4]$, and a parametrization of limit distributions by the numbers of this segment. Let us try to get a direct definition of the limit measure, using this parameter, similarly to the results about the limit distributions from [1], obtained without using the measures μ_k.

In [4], the function $\lambda(x) = \lim_{n\to\infty} \frac{S([4^n x])}{\sqrt{x} \cdot 2^n}$ is introduced, its analytical properties are studied, and some results about the distribution of its values are obtained. Also an expression of the function $\lambda(x)\sqrt{x}$ in terms of the expansion of x in the 4-adic system of numeration is given. The sums and fractal functions connected with the Rudin–Shapiro coefficients were also studied in other works. We shall use a different expression of this function—one in terms of the diadic expansion, obtained on the basis of the formula for $S(n)$ from [1]. The formula is as follows. Consider the function $\sigma(c,b)$, defined for integer $0 \le c \le b$:

$$\sigma(c,b) = \begin{cases} 2^{(b+2)/2}, & c - \text{odd}, \ b - \text{even}, \\ 2^{(b+1)/2}, & c - \text{odd}, \ b - \text{odd}, \\ 2^{(b+2)/2} - 2^{c/2}, & c - \text{even}, \ b - \text{even}, \\ 2^{(b+1)/2} + 2^{c/2}, & c - \text{even}, \ b - \text{odd}. \end{cases} \tag{1}$$

Let the diadic expansion of n have the form

$$u_k \ldots u_0 = 1\ldots10\ldots01\ldots10\ldots01\ldots1\ldots;$$

thus the series of ones (including the series of length one) are $u_{b_i} \ldots u_{c_i}$, $1 \le i \le p$, $c_i \le b_i < c_{i+1} - 1$, $1 \le i \le p-1$. Introduce the numbers $v_i = \sum_{j=i}^{p} (b_j - c_j)$. Then

$$S(n) = (-1)^{v_1} + \sum_{i=1}^{p-1} (-1)^{v_{i+1}} \sigma(c_i, b_i) + \sigma(c_p, b_p). \tag{2}$$

For the convenience of the subsequent calculations we introduce the enumeration of b_i, c_i in the opposite direction. We have $b'_i = b_{p+1-i}$, $c'_i = c_{p+1-i}$, $1 \le i \le p$;

$$v'_i = \sum_{j=1}^{p+1-i} (b'_j - c'_j), \quad \text{and} \quad S(n) = \sigma(c'_1, b'_1) + \sum_{i=2}^{p} (-1)^{v'_{i-1}} \sigma(c'_i, b'_i) + (-1)^{v'_p}. \tag{3}$$

Now extend the function σ, using the same formulas (1), to all the pairs of integers $c \le b$ and suppose that the diadic expansion of an arbitrary positive real number x is of the form

$$\underset{b'_1}{1} \ldots \underset{c'_1}{1}0 \ldots 0\underset{b'_k}{1} \ldots \underset{c'_k}{1}0 \ldots .$$

In this case, all v_i' are defined and finite, and

$$\sqrt{x}\,\lambda(x) = \sigma(c_1', b_1') + \sum_{i=2}^{p} (-1)^{v_{i-1}'}\sigma(c_i', b_i'),$$

where $p \leq \infty$ is the number of series of ones. This is an evident consequence of the definition of $\lambda(x)$. Denote $\sqrt{x}\,\lambda(x)$ by $\varphi(x)$. Suppose that $x = n+\alpha$, $0 \leq \alpha < 1$, n is natural. Find an explicit expression for $\varphi(x) - S(n)$. Let $[c_1', b_1'], \dots, [c_p', b_p']$ be the series for x, $p \leq \infty$, $[c_1'', b_1''], \dots, [c_{p_0}'', b_{p_0}'']$ be the series for n. It is evident that $b_i' = b_i''$, $1 \leq i \leq p_0$; $c_i' = c_i''$, $1 \leq i \leq p_0 - 1$; $c_{p_0}' \leq c_{p_0}''$. We have

$$\varphi(x) - S(n) = \sum_{i=p_0+1}^{p} (-1)^{v_{i-1}'}\sigma(c_i', b_i') - (-1)^{v_{p_0}''} + (-1)^{v_{p_0-1}'}\sigma(c_{p_0}', b_{p_0}') - (-1)^{v_{p_0-1}''}\sigma(c_{p_0}'', b_{p_0}'').$$

Note that $v_{p_0-1}' = v_{p_0-1}''$. If $c_{p_0}' = c_{p_0}''$, then $\sigma(c_{p_0}', b_{p_0}') - \sigma(c_{p_0}'', b_{p_0}'') = 0$. If $c_{p_0}' < c_{p_0}''$, then, evidently, $c_{p_0}'' = 0$, and by the definition of $\sigma(c, b)$ (formula (1)) we get

$$\sigma(c_{p_0}', b_{p_0}') - \sigma(c_{p_0}'', b_{p_0}'') = \sigma(c_{p_0}', b_{p_0}') - \sigma(0, b_{p_0}') = (-1)^{b_{p_0}'}\big(u(c_{p_0}') - u(0)\big),$$

where $u(l)$ is $-2^{l/2}\left(\frac{1+(-1)^l}{2}\right)$. Since $(-1)^{b_{p_0}'}u(0)(-1)^{v_{p_0-1}''} = (-1)^{v_{p_0}''}$, in this case we have

$$\varphi(x) - S(n) = \sum_{i=p_0+1}^{p} (-1)^{v_{i-1}'}\sigma(c_i', b_i') + (-1)^{v_{p_0}''}u(c_{p_0}').$$

Compare this expression with the expression for $\varphi(\alpha)$:

$$\varphi(\alpha) = \sigma(c_{p_0-1}', -1) + \sum_{i=p_0+1}^{p} (-1)^{v_{i-1}' - v_{p_0}'' - 1}\sigma(c_i', b_i').$$

It is clear that

$$\sigma(c_{p_0}', -1) = 1 - u(c_{p_0}'), \quad \varphi(\alpha) = 1 - (-1)^{v_{p_0}''}\big(\varphi(x) - S(n)\big),$$

$$\varphi(x) - S(n) = (-1)^{v_{p_0}''}\big(1 - \varphi(\alpha)\big) = a(n)\big(1 - \varphi(\alpha)\big).$$

As for the first case $(c_{p_0}' = c_{p_0}'')$, we obtained instead, as is easy to understand, $a(n)\big(\varphi(\alpha) - 1\big)$.

Denote the function $\varphi(x+\beta) - \varphi(x)$ by $g_\beta(x)$, $x > 0$. Now we can prove the following statement.

Proposition 2. *If $\{n_k\} \in E$ and $h(\{n_k\}) = \beta$, then for arbitrary w the following equality is true:*

$$\mu(\{n_k\})(\{\, x \mid x \leq w \,\}) = \lim_{T\to\infty} T^{-1}\operatorname{mes}\{\, x \mid g_\beta(x) \leq w,\ 0 < x \leq T \,\}.$$

Proof. For an arbitrary x, according to what has been said above, the following equality takes place:

$$g_\beta(x) = S([x+\beta]) - S([x]) + r_1(x)\big(1 - \varphi(\{x\})\big) + r_2(x)\big(1 - \varphi(\{x+\beta\})\big),$$

where $r_1(x) = \pm a([x])$ and $r_2(x) = \pm a([\beta + x])$. Let $\xi_\beta = \{A_1 \dots A_{k(\beta)}\}$ be the partition of the positive real axis in accordance with the values of the vector $\big(S[x+\beta] - S[x],\ r_1(x),\ r_2(x)\big)$. The enumeration of A_i corresponds to the lexicographic ordering of the possible values of the vector. It is evident that the limits $\lim_{T\to\infty} T^{-1}\operatorname{mes}\big(A_i \cap [0,T]\big)$ exist. Passing to the case of continuous time does not bring any particular specific character. For the influence of the fractional part, its correlation with the number $1/2$ is definitive. It implies the existence of the limit from the formulation of Proposition 2. The sets A_i can be described more precisely. If we consider the closures of all the connected components of all shifts $\overline{A}_i \cap [n, n+1] - n$ of intersections $\overline{A}_i \cap [n, n+1]$, then among them there exist only a finite number of different ones. Order them, for example, in order of appearance in the lists for $n = 0, 1 \ldots$: $I_i^1, \ldots, I_i^{l(i)}$. Then there exist increasing sequences of natural numbers $m_{i,r}^j$, $r = 1, \ldots, l(i)$, $1 \le j < \infty$, such that $\overline{A}_i = \bigcup_{r,j} (I_i^r + m_{i,r}^j)$, and if, for arbitrary n, we denote the number $\max\{ j \mid m_{i,r}^j \le n\}$ by $p_{i,r}(n)$, then there exists a limit $\lim_{n\to\infty} \frac{p_{i,r}(n)}{n}$, and it is sufficient to study the behavior of g_β on the segments only.

Further, if β is fixed, then for any $\varepsilon > 0$ there exists δ such that if $|\beta' - \beta| < \delta$, $\xi_{\beta'} = \{A'_1 \ldots A'_{k(\beta')}\}$ is the corresponding partition, and $\{I'^{1}_{i'} \ldots I'^{l'(i')}_{i'}\}$ is the corresponding systems of segments, then numbers $1 \le i_1 < \ldots < i_s \le k(\beta)$ and $1 \le i'_1 < \ldots < i'_s \le k(\beta')$, and for any p, $1 \le p \le s$, numbers $1 \le j_p^1 < \ldots$ $\ldots < j_p^{v(p)} \le l(i_p)$ and $1 \le j'^1_p < \ldots < j'^{v(p)}_p \le l'(i'_p)$ can be selected such that the sequences $m^j_{i_p, j_p^u}$ and $m'^j_{i'_p, j'^u_p}$ coincide with one another for all p, u such that $1 \le p \le s$, $1 \le u < v(p)$, and, moreover,

$$M + \sum_{\substack{1\le p\le s\\ 1\le u\le v(p)}} \operatorname{mes}\left(I_{i_p}^{j_p^u} \Delta I'^{j'^u_p}_{i'_p}\right) < \varepsilon,$$

where M is the sum of the measures of all nonselected segments of both collections of systems. Incidentally, this evidently implies that if $\beta' \to \beta$, then $\mu_{\beta'}$ weakly converges to μ_β. The assertion we are proving now appears to be, in fact, of the same kind. Denote the number $n_k/4^k$ by β_k. Then

$$\frac{S(n+n_k) - S(n)}{2^k} = \varphi\Big(\frac{n}{4^k} + \beta_k\Big) - \varphi\Big(\frac{n}{4^k}\Big) + 2^{-k}\big(a(n+n_k) - a(n)\big).$$

We consider the problem for β_k with the discrete parameter, but one can construct for β_k partitions and systems of segments by the scheme considered above as well. Directing ε to 0, for large k one can again single out segments in the way described above. Since the partition for β is fixed, one can (for k large enough) assume that the collection of segments to be selected includes all the segments connected with β. Since g_{β_k} converges uniformly to g_β on each of them, the weak convergence of the corresponding discrete distributions to the limit one is evident. The proof is complete.

§2. Consideration of the cases $n_r = 2 \cdot 4^r$ and $n_r = 4^r$

First, join both sequences into one and consider the general question on the distribution of values of $u_n^l = S(n+2^l) - S(n)$, $l > 1$. It is evident that the sequential differences $\Delta_n^l = u_n^l - u_{n-1}^l$ can be simply expressed in terms of $a(i)$, namely, $\Delta_n^l = a(n+2^l) - a(n)$. We are solving the problem about the distribution of sums

$$S(2^l-1),\quad S(2^l-1) + \Delta_0^l,\quad S(2^l-1) + \Delta_0^l + \Delta_1^l, \ldots .$$

Let n be $k \cdot 2^l + c$, where k is an integer, and $0 \le c < 2^l$. The definition of the Rudin–Shapiro coefficients $a(n)$ implies that if $2^{l-1} \le c$, then $a(n) = a(c) \times a(2k+1)$ (this can be checked by considering the series of those ones in which the most significant digit of c is situated). Therefore,

$$\Delta_n^l = \begin{cases} a(c)\big(a(k+1) - a(k)\big), & 0 \le c < 2^{l-1}, \\ a(c)\big(a(2k+3) - a(2k+1)\big), & 2^{l-1} \le c < 2^l. \end{cases} \tag{4}$$

We also have

$$\begin{aligned} u^l_{k\cdot 2^l - 1} &= \sum_{i=2^l k}^{2^l(k+1)-1} a(i) = a(k) \sum_{i=0}^{2^{l-1}-1} a(i) + a(2k+1) \sum_{i=2^{l-1}}^{2^l-1} a(i) \\ &= R_1^l\, a(k) + R_2^l\, a(2k+1) = a(k)\big(R_1^l + (-1)^k R_2^l\big), \end{aligned} \tag{5}$$

$$u^l_{k\cdot 2^l+2^{l-1}-1} = R^l_2(-1)^k a(k) + R^l_1 a(k+1),$$

because, as is known, $a(2k+1) = (-1)^k a(k)$.

Thus, the value of $a(k)$, $a(k+1)$ and the parity of k determine the values of u^l_i for $k\cdot 2^l - 1 \le i < (k+1)\cdot 2^l - 1$. Since the second of the expressions (4) is $a(c)\big((-1)^{k+1}a(k+1) - (-1)^k a(k)\big)$, it is clear that for any k one of these expressions is identically equal to 0.

Consider the following two measures:

$$\mu^l_1 = \sum_{i=0}^{2^{l-1}-1} \delta_{\rho(i)} \quad \text{and} \quad \mu^l_2 = \sum_{i=2^{l-1}}^{2^l-1} \delta_{S(i)-S(2^{l-1}-1)}.$$

Further, for an arbitrary measure μ and numbers α, β, denote $\mu_{\alpha,\beta}(A) = \mu(A/\alpha - \beta)$, where A is an arbitrary measurable set (we assume that $\mu_{0,\beta} = \delta(0)$ for any β). It is easy to see that all the measures of the form $\sum_{i=k\cdot 2^l-1}^{(k+1)\cdot 2^l-2} \delta_{u^l_i}$ are in fact $2^{l-1}\delta(x) + \mu'_k$, where μ'_k are $\mu^l_{1\,\alpha,\beta}$ or $\mu^l_{2\,\alpha,\beta}$, $\alpha = \pm 2$, $\beta = \pm R^l_1 \pm R^l_2$.

Form for the μ'_k the corresponding table on the basis of correlations (4) and (5) (see Table 1).

TABLE 1

	k even	k odd
$a(k) = -1$ & $a(k+1) = -1$	$\mu^l_{2\;2,-R^l_1-R^l_2}$	$\mu^l_{2\;-2,-R^l_1+R^l_2}$
$a(k) = -1$ & $a(k+1) = 1$	$\mu^l_{1\;2,-R^l_1-R^l_2}$	$\mu^l_{1\;2,-R^l_1+R^l_2}$
$a(k) = 1$ & $a(k+1) = -1$	$\mu^l_{1\;-2,\;R^l_1+R^l_2}$	$\mu^l_{1\;-2,\;R^l_1-R^l_2}$
$a(k) = 1$ & $a(k+1) = 1$	$\mu^l_{2\;-2,\;R^l_1+R^l_2}$	$\mu^l_{2\;2,\;R^l_1-R^l_2}$

Studying the bivariate distributions of coordinates for the dynamical system in the space of sequences generated by the Rudin–Shapiro substitution (cf. (1)) shows that the probability of each of the 8 situations presented in Table 1 is $1/8$. Denote by μ^l the sum of measures from the table. Now our task is to study the measures $\frac{1}{2^{l+3}}\mu^l_{2^{-r},0}$, where $r = l/2$ for l even, and $r = (l-1)/2$ for l odd (continuous measures are being accumulated "half of the time"). One can find all the necessary information about R^l_i in [3], and about μ^l_1 and μ^l_2 below. (Corresponding facts are obtained in [1] on the basis of the results of [3].)

Now we consider the cases of even and odd l separately.

I. l is even; $R^l_1 = 2^r$, $R^l_2 = 0$. The simple application of the argument of the proof of Theorem 22 in [3] gives in this case

$$\begin{aligned}
\#\{n : 0 \le n \le 2^{l-1} - 1 \mid S(n) = v\} &= \begin{cases} v, & 0 < v < 2^r, \\ 2^{r-1}, & v = 2^r; \end{cases} \\
\#\{n : 2^{l-1} \le n \le 2^l - 1 \mid S(n) = v\} &= \begin{cases} 2^{r+1} - v, & 2^r < v \le 2^{r+1} - 1, \\ 2^{r-1}, & v = 2^r. \end{cases}
\end{aligned} \tag{6}$$

When finding the limit distribution, we use the principle of equality of the spatial averages and the temporal ones. That is why it must be clear that turning an expression in (4) into 0 corresponds to adding δ-measures with some factors to the tabular measures. Of course, these δ-measures are either of the form $\delta_{u^l_{k\cdot 2^l-1}}$ or $\delta_{u^l_{k\cdot 2^l+2^{l-1}-1}}$.

TABLE 2

	k even	k odd
$a(k)=-1$ & $a(k+1)=-1$	$-R_1^l-R_2^l$	$-R_1^l+R_2^l$
$a(k)=-1$ & $a(k+1)=1$	$R_1^l-R_2^l$	$R_1^l+R_2^l$
$a(k)=1$ & $a(k+1)=-1$	$-R_1^l+R_2^l$	$-R_1^l-R_2^l$
$a(k)=1$ & $a(k+1)=1$	$R_1^l+R_2^l$	$R_1^l-R_2^l$

Form the corresponding table (Table 2), suitable for odd l as well. The numbers included must be interpreted as follows: a number v is assigned to $2^{l-1}\delta_v$ (cf. (5)).

For even l we get that the contribution to the limit measure equals $\dfrac{\delta(1)+\delta(-1)}{4}$.

Now replace every measure in Table 1 by the corresponding limit measure (by the contribution to the total one), denoting beforehand the measure on the segment $[0,1]$ with density $2x$ by μ_1^*, and the measure on the segment $[0,1]$ with density $2(1-x)$ by μ_2^*. We have (see Table 3):

TABLE 3

$\mu_2^*\ {}_{2,-1}$	$\mu_2^*\ {}_{-2,-1}$
$\mu_1^*\ {}_{2,-1}$	$\mu_1^*\ {}_{2,-1}$
$\mu_1^*\ {}_{-2,\ 1}$	$\mu_1^*\ {}_{-2,\ 1}$
$\mu_2^*\ {}_{-2,\ 1}$	$\mu_2^*\ {}_{2,\ 1}$

The total continuous component is the measure on the segment $[-3,3]$ with density

$$\rho_1(x)=\begin{cases}(x+3)/8, & -3\le x\le -1,\\ 3/16, & -1\le x\le 1,\\ (3-x)/8, & 1\le x\le 3.\end{cases}$$

II. l is odd, $R_1=R_2=2^r$. The discrete component, as follows from Table 2, is $\big(\delta(-2)+2\delta(0)+\delta(2)\big)/8$.

$$\#\big\{n:0\le n\le 2^{l-1}-1 \mid S(n)=v\big\}=\begin{cases}v, & 0<v\le 2^r,\\ 2^{r+1}-v, & 2^r<v<2^{r+1},\end{cases}$$

$$\#\big\{n:2^{l-1}\le n\le 2^{l}-1 \mid S(n)=v\big\}=\begin{cases}2v-2^{r+1}, & 2^r<v<2^{r+1},\\ 2^r, & v=2^{r+1}.\end{cases}$$

Here μ_1^* is the measure on $[0,2]$ with density equal to x on $[0,1]$ and equal to $2-x$ on $[1,2]$, while μ_2^* is the measure on $[0,1]$ with density $2x$. The contributions to the continuous component of the measure are presented in Table 4.

TABLE 4

$\mu_2^*\ {}_{2,-2}$	$\mu_2^*\ {}_{-2,0}$
$\mu_1^*\ {}_{2,-2}$	$\mu_1^*\ {}_{2,0}$
$\mu_1^*\ {}_{-2,\ 2}$	$\mu_1^*\ {}_{-2,0}$
$\mu_2^*\ {}_{-2,\ 2}$	$\mu_2^*\ {}_{2,0}$

The resulting measure is concentrated on the segment $[-4,4]$, and its density is

$$\rho_2(x) = \begin{cases} \dfrac{4-|x|}{64}, & 2 < |x| \le 4, \\ \dfrac{3}{32} + \dfrac{2-|x|}{64}, & 0 \le |x| \le 2. \end{cases}$$

It is useful to consider the general case $n_r = S \cdot 4^r$, where S is a fixed natural number. We have (for an arbitrary $l > 1$)

$$\begin{aligned} \sum_{i=2^l k}^{2^l(k+S)-1} a(i) &= R_1 \sum_{j=k}^{k+S-1} a(j) + R_2 \sum_{j=k}^{k+S-1} (-1)^j a(j), \\ \sum_{i=2^{l-1}+2^l k}^{2^{l-1}+2^l(k+S)-1} a(i) &= R_1 \sum_{j=k+1}^{k+S} a(j) + R_2 \sum_{j=k}^{k+S-1} (-1)^j a(j). \end{aligned} \tag{7}$$

In our case ($l = 2r$, S may be even) $R_2 = 0$. Here is an analog of formula (4):

$$\delta^l_{n,S} = a(n + S\cdot 2^l) - a(n) = \begin{cases} a(c)\big(a(k+S) - a(k)\big) = u_1 a(c); & 0 \le c < 2^{l-1}, \\ a(c)\big((-1)^{k+S}a(k+S) - (-1)^k a(k)\big) = u_2 a(c); & 2^{l-1} \le c < 2^l. \end{cases} \tag{8}$$

Since S is fixed, consider a measure μ_S on the set E of triples $\{(\alpha,\beta,\gamma) : \alpha = \pm 1;\ \beta = \pm 1;\ \gamma = -2,0,2\}$ and specified by the equality

$$\mu_S(\alpha,\beta,\gamma) = \mathbf{P}\left(a(k) = \alpha;\ a(k+S) = \beta;\ u_2 = \gamma\right) = \mathbf{P}\left(A_{\alpha,\beta,\gamma}\right).$$

For each collection $(\alpha,\beta,\gamma) \in E$, define a probability measure $\nu_{\alpha,\beta,\gamma}$ on the set of integers that describes the distribution of values of $\sum\limits_{j=k}^{k+S-1} a(j)$ under condition $A_{\alpha,\beta,\gamma}$.

We have the following expression for the required measure:

$$\mu = \frac{1}{2} \sum_{(\alpha,\beta,\gamma)\in E} \mu_S(\alpha,\beta,\gamma)\, \nu_{\alpha,\beta,\gamma} * \left(\mu^*_{1\,\beta-\alpha,0} + \delta_{\beta-\alpha} * \mu^*_{2\,\gamma,0}\right), \tag{9}$$

where μ^*_i, $i = 1,2$, relates to the case $S = 1$ considered above.

§3. Markov representation for the limit measures

In [1], we studied sequences of matrices of generating functions of probability distributions, which were constructed in the following way. Let $A(t)$ be a polynomial matrix such that all the coefficients of all these polynomials are nonnegative and the number matrix $A(1)$ is primitive. Let a sequence of polynomial matrices $B_n(t)$ be defined by the recursion $B_0(t) = E$, $B_{n+1}(t) = A(t)B_n(t^2)$, $B_n(t) = \{b^n_{ij}(t)\}^N_{i,j=1}$,

$$b^n_{i,j}(t) = p^n_{s_n(i,j)}\, t^{s_n(i,j)} + \dots + p^n_{y_n(i,j)}\, t^{y_n(i,j)}.$$

Probability measures ν^n_{ij} were considered, defined as follows:

$$\nu^n_{ij} = \left(b^n_{ij}(1)\right)^{-1} \sum_{k=s_n(i,j)}^{y_n(i,j)} p^n_k\, \delta(k/2^n).$$

It was claimed that these measures converge to limit measures. The proofs (using methods of harmonic analysis and other methods) were carried out for particular cases only (for the sake of simplicity). It turned out that the matrix of measures obtained (in fact, a row of measures, since the limit measure does not depend on the index of the row) satisfies certain relations (in a bivariate particular case presented in [1] by system (4)). Using these relations, we obtain in this paper explicit expressions for the values these measures take on binary segments. In the general case (i.e., for not necessarily nonatomic limit measures), ν_i stands for its purely continuous component.

Since $A(t) = B_1(t)$, we can interpret the measures $\mu_{ij} = b^1_{ij}(1)\,\nu^1_{ij}$ as measures corresponding to the matrix $A(t)$, considered as a matrix of generating functions. Thus, we have a vector of limit measures ν_i such that for each segment $[e,f]$

$$\varphi_j\,\nu_j([e,f]) = \Big(\frac{1}{\lambda}\sum_{i=1}^{N}\varphi_i\,\mu_{ij} * \nu_i\Big)([2e,2f]),\quad 1 \le j \le N, \tag{10}$$

where λ is the Perron–Frobenius eigenvalue, and $\{\varphi_i\}$ is the corresponding eigenvector-row of the number matrix $A(1)$.

For any measure ν_i we denote the intersection of all closed segments of ν_i-full measure by $[a_i,b_i]$. Then the following relations hold:

$$\max_i\Big(b_i + \frac{y_1(i,j)}{2}\Big) = 2b_j,\quad \min_i\Big(a_i + \frac{s_1(i,j)}{2}\Big) = 2a_j, \tag{11}$$

as evidently follows from (10). This, in turn, implies that all a_i and b_i are rational ([1], §2).

Take the maximal number τ such that $1/(2\tau)$, $k_i = (b_i - a_i)/\tau$, and a_i/τ are integers for all i. Consider all possible segments of the form

$$\big[a_i + k\tau,\ a_i + (k+1)\tau\big],$$

$0 \le k \le k_i - 1$, $1 \le i \le N$. We index these segments by the corresponding pairs (i,k) (without identification of the coinciding segments). Denote by N^* the number of all such segments, equal to $\sum_{i=1}^{N} k_i$, and by M the set of these segments.

Let v be a natural number, and $u_1 \dots u_v$ be an arbitrary sequence of zeros and ones. Denote by $\mu_{(i,k),u_1,\dots,u_v}$ the value of

$$\nu_i\Big\{x \in \big[a_i + k\tau,\ a_i + (k+1)\tau\big] \ \big|\ \text{the first digits of the binary expansion of } \frac{x - a_i}{\tau} - k \text{ are } u_1 \dots u_v\Big\}.$$

The system (10) allows one to get all the values $\mu_{(i,k),u_1,\dots,u_v}$ sequentially for $v = 1,2,\dots$. Thus, matrices $A_0 = \big\{a^0_{(i_1,k_1)(i_2,k_2)}\big\}$ and $A_1 = \big\{a^1_{(i_1,k_1)(i_2,k_2)}\big\}$ can be defined such that

$$\mu_{(i,k),\varkappa} = \sum_{(i_1,k_1)} a^{\varkappa}_{(i,k)(i_1,k_1)}\,\mu_{(i_1,k_1)},\quad \varkappa = 0,1. \tag{12}$$

These matrices are completely specified by the coefficients of system (10), i.e., by the values of λ, $\{\varphi_i\}$, and discrete measures $\{\mu_{ij}\}$.

Further applications of system (10) prove that, similarly for any collection $\alpha = u_1,\dots,u_v$ of zeros and ones there exists (and can be constructed in a natural way) a matrix $A_\alpha = \big\{a^\alpha_{(i_1,k_1)(i_2,k_2)}\big\}$ such that

$$\mu_{(i,k),\alpha} = \sum_{(i_1,k_1)\in M} a^\alpha_{(i,k)(i_1,k_1)}\,\mu_{(i_1,k_1)}.$$

The matrix A_α can be expressed in terms of A_0 and A_1. Equality (10) implies the following simple remark. If $\alpha' = \alpha\beta$ is the concatenation of words consisting of zeros and ones, then

$$\mu_{(i,k),\alpha'} = \sum_{(i_1,k_1)} a^\alpha_{(i,k)(i_1,k_1)}\,\mu_{(i_1,k_1),\beta}.$$

Thus, $\{\mu_{(i,k),\alpha 0}\} = A_\alpha\{\mu_{(i_1,k_1),0}\} = A_\alpha A_0\{\mu_{(i_1,k_1)}\}$, and we have $A_{\alpha 0} = A_\alpha A_0$. Similarly, $A_\alpha = A_\alpha A_1$. Hence, if $\alpha = u_1 \dots u_v$, then $A_\alpha = A_{u_1} \dots A_{u_v}$, and the collection of numbers $\{\mu_{i,k}\}$ in combination with matrices determines completely all the measures ν_i. It follows from (12) that the vector-column $\{\mu_{(i,k)}\}$ is invariant with respect to the matrix $A_0 + A_1$. If this matrix is primitive, then the last property of the vector $\{\mu_{(i,k)}\}$ specifies it uniquelly. (Recall that ν_i are normalized.)

Now we study measures defined in the space $\{(i,k)\} \times \cdots \times \{0,1\} \times \cdots = \{\{s, u_1, \dots\}\}$. Compute the values of measures of cylindrical sets, which have the form

$$\{\, s', u_1, \dots \mid u_{p+1} \dots u_{p+k} = B,\ s' = s = (i,k) \,\} = C^s_{B,p},$$

where $p \geq 0$, and B is a word of length k consisting of zeros and ones. Certainly, one can interpret the model described above as a measure μ defined on the union N^* of disjoint abstract segments of length τ. The restrictions of μ to the corresponding subsets are the measures ν_i, multivariate distributions $\mu_{(i,k),u_1,\dots,u_v}$ being coordinated in the usual sense. Consider the vector $\{\,\mu C^s_{B,p}\,\}_{s=(i,k)\in M} = \{\,\nu_{i(s)} C^s_{B,p}\,\}_{s\in M}$, where B and p are fixed. From what has been proved above, it follows that

$$\{\mu C^s_{B,p}\} = (A_0 + A_1)^p A_B \times \{\mu_{(i,k)}\}.$$

Continuing with the supposition of primitivity, denote some invariant vector-row of the matrix $A_0 + A_1$ by $\{l_{(i,k)}\}$. According to the Perron–Frobenius theorem in the formulation of [6] for an arbitrary vector ξ, we have

$$\lim_{n\to\infty} (A_0 + A_1)^p \xi = (\{l_{(i,k)}\}, \xi)\, \frac{\{\mu_{(i,k)}\}}{(\{l_{(i,k)}\}, \{\mu_{(i,k)}\})},$$

where the parentheses mean scalar products. Hence,

$$\lim_{p\to\infty} \left\{\mu C^{(i,k)}_{B,p}\right\} = (\{l_{(i,k)}\}, A_B\{\mu_{(i,k)}\})\, \frac{\{\mu_{(i,k)}\}}{(\{l_{(i,k)}\}, \{\mu_{(i,k)}\})}. \tag{13}$$

It is evident that in $\{0,1\} \times \{0,1\} \dots$ a shift-invariant probability measure μ^* can be introduced such that

$$\mu^*\Big(\bigcup_s C^s_{B,p}\Big) = \frac{(\{l_s\}, A_B\{\mu_s\})}{(\{l_s\}, \{\mu_s\})},$$

and for every pair (i,k)

$$\lim_{p\to\infty} \mu C^{(i,k)}_{B,p} = \mu_{(i,k)} \cdot \mu^*\Big(\bigcup_s C^s_{B,0}\Big).$$

The direct product of measures, which is described by the right-hand side of the last formula, is, of course, equivalent to the measure μ because of the uniform boundedness from above and from below of the ratios of measures for cylinders of the form $C_{B,0}$. It is easy to understand that the shift in the space with measure μ^* mixes. This facilitates checking singularity if the matrices A_0 and A_1 are known.

Consider now a block matrix $\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$. The matrices of such a type may often occur in various problems. For example, in [5], which is devoted to construction of C^*-algebras with various properties, such a matrix (more exactly, a matrix of order $2m\times 2n$ with (2×2)-blocks of the form $\begin{pmatrix} p_{ij} & q_{ij} \\ q_{ij} & p_{ij} \end{pmatrix}$) is called twice symmetrical (Definition 6.2.3), and a natural analog of the matrix $A_0 + A_1$ is called its symmetrical collapse.

The invariant vectors (a row and a column) of our block matrix are, respectively, $\{\mu_{(i,k)}\}\{\mu_{(i,k)}\}$ and $\{l_{(i,k)}\}\{l_{(i,k)}\}$. Recall a formula expressing the measure with maximal entropy for the topological Markov chain ([7, 8] and others). If the state space of the topological Markov chain is $[1:n]$, and the matrix of multiplicities of edges is (h_{ij}), then the measure of an arbitrary cylinder, corresponding to the word $i_1 \dots i_k$, is $\theta^{1-k}\varphi^*_{i_1}\varphi_{i_k}\prod_{j=1}^{k-1} h_{i_j,i_{j+1}}$, where θ is the maximal eigenvalue of the matrix of multiplicities, $\{\varphi^*_i\}_1^n$ is an eigenvector-row, and $\{\varphi_i\}_1^n$ is an eigenvector-column, such that $(\{\varphi^*_i\}, \{\varphi_i\}) = 1$. Note here that the central measure of the adic stationary transformation of the Markov compact described by this matrix has the same transition probabilities $p_{ij} = h_{ij}\,\varphi_j/\theta\varphi_i$, but the initial distribution is a normalized vector column ([7] and others). Now let $\Omega = \{\{\omega_i\}_1^\infty\}$ be the space of one-sided sequences of numbers $1, \dots, 2N^*$. Consider a measure η on it, specified by the following definition. Its multivariate distributions can be expressed in terms of the matrix $\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ and its eigenvectors $\{\varphi_i\}_1^{2N^*}$, $\{\varphi_i^*\}_1^{2N^*}$ (which are proportional to $\{\mu_{(i,k)}\}_1^{N^*}\{\mu_{(i,k)}\}$ and $\{l_{(i,k)}\}_1^{N^*}\{l_{(i,k)}\}$) by the same formulas by which multivariate distributions of the measure with maximal entropy are expressed in terms of the matrix $\{h_{ij}\}$ and its Perron–Frobenius eigenvectors. Of course, this is only a formal analog of the topological Markov chain, because the matrix $\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ is not necessarily integer-valued and even rational. In order to describe a natural correspondence between the sets of the form $C_{B,0}$ and subsets of Ω, we consider the events

$$\begin{aligned}
V_1^m &= (1 \le \omega_m \le N^*) \cap (1 \le \omega_{m+1} \le N^*);\\
V_2^m &= (N^* + 1 \le \omega_m \le 2N^*) \cap (N^* + 1 \le \omega_{m+1} \le 2N^*);\\
V_3^m &= (1 \le \omega_m \le N^*) \cap (N^* + 1 \le \omega_{m+1} \le 2N^*);\\
V_4^m &= (N^* + 1 \le \omega_m \le 2N^*) \cap (1 \le \omega_{m+1} \le N^*);\\
U_0^m &= V_1^m \vee V_2^m; \quad U_1^m = V_3^m \vee V_4^m.
\end{aligned}$$

For an arbitrary word B consisting of zeros and ones, $B = b_1 \dots b_v$, denote by U_B the event $U_{b_1}^1 \wedge \dots \wedge U_{b_v}^v$. It follows from the definition of $U_{b_i}^m$ that U_B is a union of not more than 2^v intersections of V_i^m. But, in fact, every U_B consists of precisely two such intersections, for example, $U_{1011} = (V_3^1 \wedge V_2^2 \wedge V_4^3 \wedge V_3^4) \vee (V_4^1 \wedge V_1^2 \wedge V_3^3 \wedge V_4^4)$. If $a = 0, 1$ and $b : 2a < b \le 2a + 2$ are fixed, and to any binary sequence $U = c_1 \dots c_{n-1}a$ one of two events corresponding to it, namely $V_{i_1(u)}^1 \cdots V_{i_{n-1}(u)}^{n-1} V_b^n$, is assigned, then when enumerating the sequences U in the natural order, parities $i_1(u)$ form a segment of the Morse sequence. The following is an immediate consequence of the definitions.

Proposition 3. *For any binary word B we have*

$$\mu^* C_{B,p} = \eta U_B.$$

In fact, we have found the multivariate distributions for a process with consolidated states [9]. Using system (10), it is not difficult to write down an explicit expression for the matrices A_0, A_1:

$$\text{if } r^0_{(i,k)(i_1,k_1)} = 2a_i + 2k\tau - a_{i_1} - k_1\tau \in \left[\frac{y_1(i_1,i)}{2},\ \frac{s_1(i_1,i)}{2}\right], \text{ then } a^0_{(i,k)(i_1,k_1)} = \frac{\varphi_{i_1}}{\varphi_i\lambda}\, p^1_{2r^0_{(i,k)(i_1,k_1)}},$$

$$\text{if } r^1_{(i,k)(i_1,k_1)} = 2a_i + (2k+1)\tau - a_{i_1} - k_1\tau \in \left[\frac{y_1(i_1,i)}{2},\ \frac{s(i_1,i)}{2}\right], \text{ then } a^1_{(i,k)(i_1,k_1)} = \frac{\varphi_{i_1}}{\varphi_i\lambda}\, p^1_{2r^1_{(i,k)(i_1,k_1)}}.$$

The remaining coefficients of matrices are equal to 0.

As an example we consider the polynomial matrix $A(t) = \begin{pmatrix} t+1 & t^{-1}+t \\ 1+t^{-1} & t^{-1} \end{pmatrix}$ from [1]. In [1], a_i, b_i were found by solving a system similar to (11). (Note that because of the uniqueness of the solution of (11) one can assume that the segments $[a_i, b_i]$ are minimal in the sense described above, if the scale is fixed.) We have $a_1 = a_2 = -1/2$, $b_1 = b_2 = 1/2$. One can take $1/2$ as τ. We have $r^0_{(i,k)(i_1,k_1)} = \dfrac{-1+2k-k_1}{2}$. Moreover, the expression for $a^0_{(i,k)(i_1,k_1)}$ turns into 0:

for $i = 1$, $i_1 = 1$, if $2k \ne k_1 + 1,\ k_1 + 2$;
for $i = 1$, $i_1 = 2$, if $2k \ne k_1,\ k_1 + 1$;
for $i = 2$, $i_1 = 1$, if $2k \ne k_1,\ k_1 + 2$;
for $i = 2$, $i_1 = 2$, if $2k \ne k_1$.

TABLE 5

	(1,0)	(1,1)	(2,0)	(2,1)
(1,0)	0	0	ω_1/λ	0
(1,1)	$1/\lambda$	$1/\lambda$	0	ω_1/λ
(2,0)	ω_2/λ	0	$1/\lambda$	0
(2,1)	ω_2/λ	0	0	0

TABLE 6

	(1,0)	(1,1)	(2,0)	(2,1)
(1,0)	$1/\lambda$	0	ω_1/λ	ω_1/λ
(1,1)	0	$1/\lambda$	0	0
(2,0)	0	ω_2/λ	0	$1/\lambda$
(2,1)	0	ω_2/λ	0	0

Here all the nonzero $P^{s}_{2r^{s}_{(i,k)(i_1,k_1)}}$ are equal to 1. We get the matrix A_0 (see Table 5)

The matrix A_1 arises in a similar way (see Table 6)

Here $(1, \omega_1) = \big(1, (\sqrt{17}-1)/4\big)$ is the eigenvector corresponding to the eigenvalue $\lambda = (3+\sqrt{17})/2$ of the matrix $A(1)$, $\omega_2 = \omega_1^{-1}$.

§4. ESTIMATE OF THE EXCESS FOR SUBSTITUTIONS

A simple corollary of the estimates from [2] is obtained for primitive substitution of the general form such that its matrix has two eigenvalues > 1. Generally speaking, this corollary does not imply the topological mixing (the corresponding algorithmic investigation of the asymptotic behavior of the excess for admissible words of a special kind requires a rather complicated treatment). Still, for a particular case considered in [9], this corollary is identical to "the fact 2" proved there by other methods.

The following result is obtained in [2].

Theorem. *Let σ be a substitution with primitive matrix M such that for the second (by ordering by magnitude) eigenvalue θ_2 of it we have $\theta_2 < \theta$ (where θ is the Perron–Frobenius eigenvalue), θ_2 is real, majorizing 1 and the modules of other eigenvalues. Let the eigenvalue θ_2 be of order $\alpha+1$ in the minimal polynomial of the matrix M and $\beta = \log_\theta \theta_2$. Then for an arbitrary mapping f of the alphabet of σ to R such that the vector of its values is orthogonal to the eigensubspace of M corresponding to θ, and for each fixed point of σ $\{u_i\}_{i\geq 1}$ there exists a real function $F_f \in C\big([1,+\infty]\big)$ such that for all $x \in [1,+\infty)$ we have $F_f(\theta x) = F_f(x)$ and*

$$S^f(N) = \sum_{1\leq i\leq N} f(u_i) = (\log_\theta N)^\alpha N^\beta F_f(N) + O\big(\Psi(N)\big),$$

where $\Psi = o\big((\log N)^\alpha N^\beta\big)$.

We suppose that F_f is not a constant (this follows, however, from the nondifferentiability, which is also proved in [2] for the case $F_f \not\equiv 0$). We prove that there exist constants $a, b, N_0 > 0$ such that if $n > N_0$, then

$$\inf_N\big(S^{(f)}(N+n) - S^f(N)\big) < -an^\beta(\log_\theta n)^\alpha << bn^\beta(\log_\theta n^\alpha) < \sup_N\big(S^{(f)}(N+n) - S^f(N)\big).$$

Lemma. *Let on an interval $[u,v]$ a continuous function $h(x)$ be defined. Then one can find δ such that if $\Delta < \delta$, then there exists x such that $u < x < x+\delta < v$ and $\dfrac{h(x+\Delta)-h(x)}{\Delta} = \dfrac{h(v)-h(u)}{v-u}$.*

The assertion of the lemma is evident (the simplest case: $x \in (u,v) \Rightarrow h(x) > 0 = h(v) = h(v)$).

Our assumption about the function F_f and its properties implies that there exist $x_1 < x_2$, $x_1 > 1$ such that F_f is of constant sign on the closed interval $[x_1, x_2]$ (without loss of generality, one can suppose that $F_f > 0$ on $[x_1, x_2]$) and $F_f(x_1) < F_f(x_2)$. Let $0 < \varepsilon < \frac{F_f(x_2) - F_f(x_1)}{x_2 - x_1} = \varepsilon_1$. Denote $\varepsilon_1 - \varepsilon$ by ε_0. By the lemma applied to F_f and $[x_1, x_2]$ we have the following: there exist constants $c_1, c_2 > 0$, such that if n is large enough, then there exist natural l, N', $N' \in (c_1 n, c_2 n)$ such that $\delta/\theta \le n/\theta^l < \delta$, $F_f(N'+n) - F_f(N') > \varepsilon_0 \delta/\theta$, $\Psi(N')$ is small enough in comparison with $S^f(N')$ and $S^f(N'+n)$, and

$$S^f(N'+n) - S^f(N') = (\log_\theta N')^\alpha N'^\beta (F_f(N'+n) - F_f(N'))$$
$$+F_f(N'+n)\left((\log_\theta (N'+n))^\alpha (N'+n)^\beta - (\log_\theta N')^\alpha N'^\beta\right) + \omega, \tag{14}$$

where ω is small enough and the second item is positive. This proves the existence of the constant b.

The proof of the existence of the constant a (revised in proofreading) is connected with additional difficulties. It is clear that (14) implies the existence of a if for any c, $0 < c < \sup_x F_f(x)$, $A > 0$ there exist $d, y \in [1, \theta]$, $d < y$, $\frac{F_f(d) - F_f(y)}{y - d} > A$, $\inf_{x \in [d,y]} F_f(x) > c$, $\Delta < \delta$ (by the lemma, δ corresponds to the segment $[d, y]$), $c_1, c_2 > 0$, such that if n is large enough, then there exist l, N', $N' \in (c_1 n,\ c_2 n)$, such that $\Delta/\theta < n/\theta^l < \Delta$, $F_f(N'+n) - F(N') > A\Delta/\theta$, $|\Psi(N')|$ is a number small enough in comparison with $S^f(N')$ and $S^f(N'+n)$, and (14) is valid with ω small enough. But the existence of d, y (which implies all the rest) follows from the representation of real numbers in the substitutional system of numeration obtained in [2]. Namely, from the expression for the increments of the function $\Phi(x) = x^\beta F_f(x)$ on the admissible intervals ([2], n°4) in the case $F_f \not\equiv 0$ one can get the existence of constants D, E, $G > 0$ such that for any natural l there exists a system of disjoint segments $I_1^l, \dots, I_{v(l)}^l \subset [1, \theta]$; $I_j^l = [u_j^l, v_j^l]$; $(v_j^l - u_j^l) < D\theta^{-l}$; $|F_f(v_j^l) - F(u_j^l)| > G\theta_2^{-l}$; $j = 1, \dots, v(l)$; $\sum_j (v_j^l - u_j^l) > E(\theta - 1)$, which is an $\varepsilon(l)$-net in $[1, \theta]$ with $\varepsilon(l) \underset{l \to \infty}{\longrightarrow} 0$.

Either for any segment $J \subset [1, \theta]$ and for any l large enough one can find $I_j^l \subset J$ with $F_f(v_j^l) - F_f(u_j^l) < 0$ (this would be enough for our purposes), or, supposing that for some A (c being fixed) the required d, y do not exist, we obtain a contradiction with the boundedness of F_f.

In this case, the condition for the constants c_1, c_2 to obey is the following one: $\left(c_1\Delta, \frac{c_2+1}{\theta}\Delta\right) \supset [d, y]$.

Remark. After this paper was sent to the editorial board, the paper [11] appeared. It turned out that our equation (10) is a particular case of condition (2.3) of [11], so our family of measures proves to be self-similar from the point of view of the definition suggested there.

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