On parametric inference for step-stress models

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Key Words - Accelerated life testing; Additive accumulation of damages; Cox model; Step-stress.

Summary & Conclusions - Applications of the additive accumulation of damages (AAD) and the proportional hazards (PH) models in accelerated life testing with step-stresses are discussed. A new model including both of them is proposed. It is more natural then PH model and wider then the AAD model. Maximum likelihood function construction is discussed.

1. INTRODUCTION

Acronyms

MLE	maximum likelihood estimator(s)
ALT	accelerated life testing
PH	proportional hazards
AAD	additive accumulative of damages
cdf	cumulative distribution function(s)
pdf	probability density function(s)
ahr	accumulated hazard rate
GPH	generalized PH
GLPH	generalized linear PH
TFR	tampered failure rate

Notations

x_0	design stress
k	number of test stresses
j	index for test-stresses : $j = 1, ..., k$ unless otherwise specified
x_j	ordered test-stresses
$x(\cdot)$	time varying stress, particularly a step-stress
n	number of units placed on test
$T_{(i)}$	ordered failure times
t_i	stress-change times, $i = 1, ..., k-1$
t_f	censoring time at x_k
$'$	first derivative

$r,\ \varphi$	functions of a stress
r_j	$r(x_j)$
Δt_j	$t_j - t_{j-1}$
s_i	$\sum_{j=1}^{i} r_j \Delta t_j$
ρ	$r(x_1)/r(x_0)$
$F_{x(\cdot)}$	Cdf of times-to-failure under the stress $x(\cdot)$
F_0	baseline Cdf
$f_{x(\cdot)}$	pdf of times-to-failure under the stress $x(\cdot)$
$\lambda_{x(\cdot)}$	hazard rate under the stress $x(\cdot)$
$\Lambda_{x(\cdot)}$	$\int_0^t \lambda_{x(\cdot)}(u)du$ -ahr under $x(\cdot)$
λ_0	baseline hazard rate
Λ_0	$\int_0^t \lambda_0(u)du$ - baseline ahr
$sp_{x(\cdot)}$	the right end of the support of distribution under $x(\cdot)$
$\beta,\ \delta,\ \gamma,\ \alpha_0$	unknown parameters
$k(t)$	$min\{j : t_j \geq t\}$
θ_i	$r_0\alpha_0/r_i$
$\hat\beta,\ \hat\delta,\ \hat\gamma,\ \hat\alpha_0$	ML estimators of $\beta,\ \delta,\ \gamma,\ \alpha_0$
$T_{x(\cdot)}$	the time-to-failure under the stress $x(\cdot)$
R	$\Lambda_{x(\cdot)}(T_{x(\cdot)})$-the exponential resource

In ALT units are tested at higher-then-usual levels of stress to induce early failures. The results are extrapolated to estimate the life distribution at the design stresss using models which relate the lifetime to the stress.

In step-stress testing a unit is placed on test at an initial low stress and if it does not fail in a predetermined time t_1, the stress is increased and so on.

Denote by $x(\cdot)$ a step-stress of the form

$$x(\tau) = \begin{cases} x_1, & t_0 \leq \tau < t_1, \\ x_2, & t_1 \leq \tau < t_2, \\ \cdots & \cdots \\ x_k, & t_{k-1} \leq \tau < t_k, \end{cases} \tag{1}$$

where $t_0 = 0$, $t_k = \infty$.

The two most known *general* models of ALT with time varying stresses are: the AAD model (Bagdonavičius [1]) and the PH model (Cox [2]).

The *AAD model*:

$$F_{x(\cdot)}(t) = F_0(\int_0^t r[x(\tau)]d\tau). \tag{2}$$

The *PH model*:

$$\lambda_{x(\cdot)}(t) = r[x(t)]\lambda_0(t). \tag{3}$$

The AAD and PH models are rather restrictive. In the case of AAD model the stress changes locally only the scale. If the PH model is true, x_0 is a usual stress, x_1 is an accelerated with respect to x_0 stress, $x_0, x_1 \in \mathcal{E}$, i.e. $S_{x_0}(t) \geq S_{x_1}(t)$ for all $t \geq 0$, and

$$x(t) = \begin{cases} x_1, & 0 \leq t < t_1, \\ x_0, & t \geq t_1, \end{cases}$$

is a step-stress, then $\alpha_{x(\cdot)}(t) = \alpha_{x_0}(t)$ for all $t_1 > 0$, $t > t_1$. So if one group of items is tested under the usual stress and the second group - under an accelerated stress x_1 untill the moment t_1 and after this moment both groups are observed under the same usual stress x_0, the failure rate after the moment t_1 is the same for both groups. If items are aging, it is not natural.

2. FORMULATION OF AAD AND PH MODELS FOR CONSTANT AND STEP-STRESSES

In terms of Cdf the PH model (3) is written:

$$F_{x(\cdot)}(t) = 1 - exp\{-\int_0^t r[x(\tau)]d\Lambda_0(\tau)\}. \tag{4}$$

In the particular case of constant in time stress x_j and the step-stress (1) the AAD model (2) implies

$$F_{x_j}(t) = F_0(r_j t),$$

and for $t \in [t_{i-1}, t_i)$, $(i = 1, 2, ..., k)$

$$F_{x(\cdot)}(t) = F_0\left(\sum_{j=1}^{i-1} r_j \Delta t_j + r_i(t - t_{i-1})\right).$$

In terms of F_{x_j},

$$F_{x(\cdot)}(t) = F_{x_i}(s_{i-1} + t - t_{i-1}), \tag{5}$$

where s_i can be determined recurrently from the equations

$$F_{x_{i+1}}(s_i) = F_{x_i}(s_{i-1} + t_i - t_{i-1}). \tag{6}$$

For the step-stress (1) and $k = 2$ the PH model (3) implies

$$\lambda_{x(\cdot)}(\tau) = \begin{cases} \lambda_{x_0}(\tau), & 0 \le \tau \le t, \\ \rho\lambda_{x_0}(\tau), & \tau > t. \end{cases}$$

In this particular case of step-stresses the Cox model is sometimes called the TFR model (Bhattacharyya and Stoejoeti (1989)). If $k \ge 2$, the formula (4) implies $F_{x_j}(t) = 1 - e^{-r_j\Lambda_0(t)}$ and for $t \in [t_{i-1}, t_i)$

$$F_{x(\cdot)}(t) = 1 - \exp\{-\sum_{j=1}^{i-1} r_j(\Lambda_0(t_j) - \Lambda_0(t_{j-1})) - r_i(\Lambda_0(t) - \Lambda_0(t_{i-1}))\}. \tag{7}$$

In the case when the distribution under the design stress x_0 is Weibull :

$$F_{x_0}(t) = 1 - e^{-t^\delta/\alpha_0}, \quad \Lambda_0(t) = \frac{t^\delta}{r_0\alpha_0},$$

we have

$$F_{x_i}(t) = 1 - e^{-t^\delta/\theta_i},$$

and for $t \in [t_{i-1}, t_i)$

$$F_{x(\cdot)}(t) = 1 - \exp\left\{ -\sum_{j=1}^{i-1} \frac{t_j^\delta - t_{j-1}^\delta}{\theta_j} - \frac{t^\delta - t_{i-1}^\delta}{\theta_i} \right\}.$$

This particular case of the PH model was considered by Khamis and Higgins [4].

3. NEW MODEL

Consider generalisation of both the AAD and PH models. For this purpose we formulate these models in other terms. For any $x(\cdot)$ the random variable $R = \Lambda_{x(\cdot)}(T_{x(\cdot)})$ has the standard exponential distribution with the survival function $S_R(t) = e^{-t}$, $t \geq 0$ and is called the exponential resource, the number $\Lambda_{x(\cdot)}(t)$ being the exponentional resource used untill the moment t under the stress $x(\cdot)$. So the failure rate $\lambda_{x(\cdot)}(t)$ is the rate of exponential resource usage.

Under the PH model we have

$$\Lambda'_{x(\cdot)}(t) = r[x(t)]\lambda_0(t),$$

i.e. the rate of resource usage is proportional to some baseline rate and the constant of proportionality is a function of a stress.

GPH model is true on $\mathcal{E}$ if for all $x(\cdot) \in \mathcal{E}$

$$\lambda_{x(\cdot)}(t) = r\{x(t)\}q\{\Lambda_{x(\cdot)}(t)\}\lambda_0(t).$$

The rate of resource usage at the moment t depends not only on a stress applied at this moment but also on the resource used untill t.

The particular cases of the GPH model are the PH model ($q(u) \equiv 1$) and the AAD model ($\lambda_0(t) \equiv \lambda_0 = const$).

The function r in the GPH model (as in the PH model) can be parametrized in the following way :

$$r[x(\tau)] = e^{\beta^T \varphi(x(\tau))},$$

where φ is some known function of the stress. For example, if $\varphi(x) = x$, $\ln x$, $1/x$, we have generalizations of the log-linear, power rule, Arrhenius models, respectively, see, for example, Nelson (1990). We'll write $x(\cdot) = \varphi(x(\cdot))$ in what follows.

Consider the case when the rate of exponential resource usage with respect to the baseline rate is proportional to a monotone function of the used resource. One of the most natural parametrizations of the function q in this case would be

$$q(u) = e^{\gamma u}, \quad \gamma \in \mathbf{R}.$$

So we consider

GLPH model :

$$\lambda_{x(\cdot)}(t) = e^{\beta x(t) + \gamma \Lambda_{x(\cdot)}(t)} \lambda_0(t). \tag{8}$$

This model implies

$$F_{x(\cdot)}(t) = \begin{cases} 1 - \{1 - \gamma \int_0^t e^{\beta x(u)} d\Lambda_0(u)\}^{1/\gamma}, & \gamma \neq 0 \\ \exp\{-\int_0^t e^{\beta x(u)} d\Lambda_0(u)\}, & \gamma = 0 \end{cases}$$

If $\gamma \leq 0$, the support of distribution is $[0, \infty)$, if $\gamma > 0$, it is $[0, sp_{x(\cdot)})$, where $sp_{x(\cdot)}$ verifies the equation

$$\int_0^{sp_{x(\cdot)}} e^{\beta x(u)} d\Lambda_0(u) = 1.$$

In the particular cases of step-stress (1) we have the formula (7) when $\gamma = 0$; if $\gamma \neq 0$ then for all $t \in [t_{i-1}, t_i)$

$$F_{x(\cdot)}(t) = 1 - \{1 - \gamma \sum_{j=1}^{i-1} e^{\beta x_j}(\Lambda_0(t_j) - \Lambda_0(t_{j-1})) - \gamma e^{\beta x_i}(\Lambda_0(t) - \Lambda_0(t_{i-1}))\}^{1/\gamma} \quad (9)$$

and for the constant stress x_j

$$F_{x_j}(t) = 1 - \{1 - \gamma e^{\beta x_j} \Lambda_0(t)\}^{1/\gamma}.$$

In the case when the distribution of time-to-failure under the design stress x_0 is Weibull, i.e.

$$F_{x_0}(t) = 1 - e^{-t^\delta/\alpha_0}, \quad \lambda_0(t) = \frac{1}{\gamma} e^{-\beta x_0} \left(1 - e^{-\frac{\gamma}{\alpha_0} + \delta}\right),$$

then under the GLPH model (8) and the step-stress (1)

$$F_{x(\cdot)}(t) = 1 - \{1 + \sum_{j=1}^{i-1} e^{\beta(x_j - x_0)}[e^{-\gamma t_j^\delta/\alpha_0} - e^{-\gamma t_{j-1}^\delta/\alpha_0}] +$$

$$e^{\beta(x_i - x_0)}[e^{-\gamma t^\delta/\alpha_0} - e^{-\gamma t_{i-1}^\delta/\alpha_0}]\}^{1/\gamma}. \quad (10)$$

Note that if the rate of resource usage depends on used resource (i.e. $\gamma \neq 0$), the distribution of the time-to-failure under constant in time stresses $x_j \neq x_0$ is not Weibull :

$$F_{x_j}(t) = 1 - \{1 - e^{\beta(x_j - x_0)}[1 - e^{-\gamma t^\delta/\alpha_0}]\}^{1/\gamma}.$$

Under the GLPH model distributions of times-to-failure under constant in time stresses are from the same family of distributions if we take one of the following two families of distributions :

The family K_1 :

$$F_{x_0}(t) = 1 - (1 + \alpha_0 t^\delta)^{1/\gamma}, \quad t \geq 0 \quad (\alpha_0 > 0,\, \delta > 0,\, \gamma < 0).$$

If $0 < \delta \leq 1$, the failure rate $\lambda_{x_0}(t)$ is *decreasing*. If $\delta > 1$, then the failure rate is increasing until the moment $t = \left(\frac{\delta - 1}{\alpha_0}\right)^{1/\delta}$ and decreasing after this moment i.e. has the $\bigcap$-*shape*.

The family K_2 :

$$F_{x_0}(t) = 1 - (1 + \alpha_0 t^\delta)^{1/\gamma}, \quad t \in [0, \left(-\frac{1}{\alpha_0}\right)^{1/\delta}) \quad (\alpha_0 < 0,\, \delta > 0,\, \gamma > 0).$$

If $0 < \delta < 1$, the hazard rate is decreasing in the interval $[0, \left(\frac{\delta-1}{\alpha_0}\right)^{1/\delta})$ and increasing in the interval $(\left(\frac{\delta-1}{\alpha_0}\right)^{1/\delta}, \left(-\frac{1}{\alpha_0}\right)^{1/\delta})$. The hazard rate has $\bigcup$-*shape* . So this family is very appealing. If $\delta > 1$, the hazard rate is *increasing* on $(0, \left(-\frac{1}{\alpha_0}\right)^{1/\delta})$.

Suppose that the time-to-failure distribution is from one of the above mentioned families. If $F_{x_0}(t) \in K_i$ (i=1,2) and the GLPH model is satisfied then

$$\Lambda_0(t) = -\frac{\alpha_0}{\gamma} e^{-\beta x_0} t^{\delta}$$

and

$$F_{x_j}(t) = 1 - (1 + \alpha_j t^{\delta})^{1/\gamma},$$

where $\alpha_j = \alpha_0 e^{\beta(x_j - x_0)}$. If $F_{x_0} \in K_2$, the support of the distribution is finite : $[0, \left(\frac{1}{-\alpha_j}\right)^{\frac{1}{\delta}})$. If $x(\cdot)$ is a step stress of the form (1) then

$$F_{x(\cdot)}(t) = 1 - \{1 + \alpha_0 \varphi(t, \beta, \delta)\}^{1/\gamma}, \quad t < sp_{x(\cdot)}, \tag{11}$$

where

$$\varphi(t, \beta, \delta) = \sum_{j=1}^{k(t)-1} e^{\beta(x_j - x_0)}(t_j^{\delta} - t_{j-1}^{\delta}) + e^{\beta(x_{k(t)} - x_0)}(t^{\delta} - t_{k(t)-1}^{\delta}).$$

If $\alpha_0 > 0$, $\gamma < 0$, then $sp_{x(\cdot)} = +\infty$. If $\alpha_0 < 0$, $\gamma > 0$, the right limit of the support $(0, sp_{x(\cdot)})$ verifies the equation $\varphi(sp_{x(\cdot)}, \beta, \delta) = -1/\alpha_0$, i.e.

$$sp_{x(\cdot)} = \{t_{k(t)-1}^{\delta} - \frac{1}{\alpha_0} e^{\beta(x_{k(t)} - x_0)} - \sum_{j=1}^{k(t)-1} e^{\beta(x_j - x_0)}(t_j^{\delta} - t_{j-1}^{\delta}).$$

The probability density is

$$f_{x(\cdot)}(t) = \frac{\delta}{\gamma} e^{\beta(x_{k(t)} - x_0)} t^{\delta - 1} \{1 + \alpha_0 \varphi(t, \beta, \delta)\}^{1/\gamma - 1}, \quad t < sp_{x(\cdot)},$$

and the failure rate

$$\lambda_{x(\cdot)}(t) = \frac{\delta}{\gamma} e^{\beta(x_{k(t)} - x_0)} \frac{t^{\delta - 1}}{(1 + \alpha_0 \varphi(t, \beta, \delta))}$$

has the same shape as in the case of constant in time stresses (decreasing or $\bigcap$-shape for the family K_1, increasing or $\bigcup$-shape for the family K_2.

4. ESTIMATION

Suppose that all units are initially placed on test at x_1 and run until t_1 when the stress is changed to $x_2 > x_1$. At x_2, testing continues until t_2 when stress is changed to $x_3 > x_2$, and so on, until x_k. Under x_k, testing continues until all remaining units fail or until t_f whichever comes first.

Denote by D the number of failures during the experiment, $T_{(1)} \leq ... \leq T_{(D)}$ the observed moments of failures. In the case of a finite support the following reparametrization can be introduced: $\sigma = sp_{x(\cdot)}$, β, γ, δ. Then $\alpha_0 = -1/\varphi(\sigma, \beta, \delta)$.
The likelihood function:
for the family K_1

$$L(\alpha_0, \beta, \gamma, \delta) = \left(\frac{\delta}{\gamma}\right)^D e^{\beta \sum_{i=1}^{D} (x_{k(T_{(i)})} - x_0)} \times$$

$$\prod_{i=1}^{D} T_{(i)}^{\delta-1}\{1+\alpha_0\varphi(T_{(i)};\beta,\delta)\}^{\frac{1}{\gamma}-1}\{1+\alpha_0\varphi(t_f;\beta,\delta)\}^{\frac{n-D}{\gamma}}$$

and for the family K_2

$$L(\sigma,\beta,\gamma,\delta)=\left(\frac{\delta}{\gamma}\right)^{D} e^{\beta\sum_{i=1}^{D}(x_{k(T_{(i)})}-x_0)}\times$$

$$\prod_{i=1}^{n} T_{(i)}^{\delta-1}\left(1-\frac{\varphi(T_{(i)},\beta,\delta))}{\varphi(\sigma,\beta,\delta)}\right)^{\frac{1}{\gamma}-1}\left(1+\frac{\varphi(t_f,\beta,\delta))}{\varphi(\sigma,\beta,\delta)}\right)^{\frac{n-D}{\gamma}}\mathbf{1}\{T_{(D)}<\sigma\}.$$

The Cdf $F_{x_0}(t)$ is estimated by

$$\hat{F}_{x_0}(t)=1-(1+\hat{\alpha}_0 t^{\hat{\delta}})^{\frac{1}{\hat{\gamma}}}.$$

In the case of the class K_1 approximate confidence intervals for $F_{x_0}(t)$ and other reliability characteristics under the design stress x_0 are obtained by standard ML methods using the estimated Fisher information matrix. In the case of the class K_2 when the support is finite, rigorous asymptotic theory is to be established. If the time-to-failure under the design stress x_0 has the Weibull (or another) distribution, then estimation under the GLPH model can be done similarly taking the formula (10) (or (9) with respective Λ_0) for $F_{x(\cdot)}$ instead of (11) in all formulas.

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