NUMERICAL CHARACTERISTICS OF GROUPS AND CORRESPONDING RELATIONS

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Received June 27, 1999.

1. Introduction

Among various ways of "measuring" infinite groups, the most popular is the so called growth, i.e. the asymptotics of the number of words of given length. An enormous amount of works is devoted to this subject. However, there are more deep characteristics, and their relations to the growth and among themselves is maybe the most important subject of the theory of numerical characteristics of algebraic systems. The most difficult problem is to determine and calculate *invariant* characteristics which do not depend on the choice of the system of generators. This field is still little studied. In this short note we present related arguments and examples which seem to be new. A more detailed presentation of these results is to be exposed in author's paper in "Uspekhi matematicheskih nauk".

We study mainly countable exponential groups (i.e. groups of exponential length) which are of the most interest for the applications we have in mind. However, the general principles are valid for other cases as well. The starting point for the author at one time was his studying the entropy of random walks on groups. This notion (see below) was introduced in 1972 by Avez [1], but I did not know this at the time and rediscovered it several years later. However, the original purposes of introducing the entropy were different from these of Avez. For the author, the entropy was first of all an alternative characteristic with respect to the growth, and the purpose was to study their relations. But very soon it became clear that the entropy is a powerful technique for studying random walks, and first of all, the Poisson boundaries which arose in many (not only probabilistic) problems as well as in harmonic analysis on groups, etc. That is why the author together with V. A. Kaimanovich ([2, 3]) started studying random walks from the entropic standpoint. Relations with amenability, Laplacian spectrum, Kesten criterion, Furstenberg hypothesis etc. have united the entropy with many other subjects. Subsequently this subject has intensely developed. Here we return to the original purpose using already established relations and apply the characteristics we introduce to new classes of groups. In particular, as examples we continue studying so called local groups ([6, 7]), some of them (locally free groups) being recently considered in details in a joint work of the author with S. Nechaev and R. Bikbov ([9]).

Partially supported by RFBR, grant 96-15-96060

In fact, the growth is related to only one of many measures on the group, while the entropy enables to describe the asymptotics of arbitrary measures. Relation between these characteristics is similar to that between topological and metric entropy of transformations. The principal fact – *the fundamental inequality* – is similar to the inequality for these entropies. The analogy with dynamical systems is quite far-reaching. Explicit calculation of the entropy is done in few cases and it is by no means easy. The fundamental inequality itself (see below) in some cases was mentioned earlier in V. Kaimanovich's work ([4]). In a recent paper ([5]), a close question was considered for Hausdorf dimensions of walks boundaries on trees. In terms of groups this corresponds to the most studied case of free groups. However, it seems that after this question was posed about twenty years ago, it has not been studied in full generality.

2. Definitions

Let G be a countable group with a finite number of generators, $(x_1, x_2, \dots, x_k)$ be a system of generators, and $(x_1^{-1}, x_2^{-1}, \dots, x_k^{-1})$ be their inverses; denote by

$$S = (x_1, x_2, \dots, x_k, x_1^{-1}, x_2^{-1}, \dots, x_k^{-1}).$$

Generally speaking, we do not assume that the system S is minimal, i.e. it may be that the system still generates the group after removing some elements.

Each element of the group is represented as an irreducible (i.e. mutually inverse generators are not neighbours) words in the alphabet S. We distinguish words and elements of the group – the length of a word is considered literally, and the length of an element g, denoted by $l_S(g) = l(g)$, i.e. the distance (in the words metric) from the group unity in given generators, is the minimal length of the words representing this element. The set of elements of length at most n is denoted by $W_{\le n}$, and the set of elements of length exactly n – by W_n. The number of representations of a given element g as a (maybe reducible) word of length at most n is denoted by $c_n(g)$. In terms of the Cayley graph, $W_{\le n}$ is a "ball", i.e. the set of points which are at distance at most n from the unity, W_n is a "sphere" of radius n, and $c_n(g)$ is the number of paths of length at most n from the unity e to the element g.

Let $m_{\le n}$ (m_n respectively) be the uniform normalized measure on $W_{\le n}$ (on W_n respectively). Assume that μ_n is an arbitrary symmetric ($\mu(g) = \mu(g^{-1})$) measure on the set S. Let μ^{*n} be the nth convolution of the measure μ with itself. It is concentrated on the set $W_{\le n}$. The convolution is defined by the formula

$$\mu^{*n}(g) = \sum_{g = g_1 g_2, \dots, g_n} \prod_{i=1,\dots,n} \mu(g_i).$$

The uniform measure μ_S on S is of particular interest. In this case

$$\mu_S^{*n}(g) = c_n(g)/(2k)^n,$$

thus the measure $\mu_S^{*n}(g)$ of an element g is proportional to the number of representations of this element in the system S.

While the measures $m_{\le n}$ and m_n reflect in a sense the geometry of a group, more exactly, the growth of vertices in the Cayley graph, the measures μ_S^{*n} are related to dynamics, i.e. to the growth of paths in this graph, and the measures μ^{*n} – to the growth of weighted number of paths. All these measures depend on the choice of generators.

Now we can introduce a number of important constants. First of all, the well known "growth" or volume.

Definition 1. *The logarithmic volume of a group G in a given system of generators S is the number*

$$\lim(\log|W_n|)/n = v.$$

The existence of the limit follows easily from the inequality

$$|W_{n+m}| \leq |W_n| \cdot |W_m|.$$

The logarithmic volume is not zero for groups of exponential growth and only for them. Obviously, for these groups one can also calculate it by the formula

$$\lim\ \log(|W_{\leq n}|)/n = v,$$

since the logarithmic asymptotics of balls and spheres are the same.

Recall that the entropy of a measure ν on a finite set K is the number

$$H(\nu) = -\sum_K \nu(k) \cdot \log(\nu(k))$$

(logarithms are binary).

The entropy of a uniform measure is the logarithm of the number of elements of the set.

Definition 2 (see [1, 3])**.** *The entropy of a pair (G, μ) is the limit*

$$\lim H(\mu^{*n})/n = h(G, \mu),$$

i.e. the limit of normalized entropies of convolution powers of the measure μ on the generators.

This limit also exists and equals the infimum, as follows easily from the inequality

$$H(\nu_1 * \nu_2) \leq H(\nu_1) + H(\nu_2).$$

Note that the logarithmic volume is in fact the limit of normalized entropies of uniform measures.

Let us introduce one more important characteristic. Consider the length of an element $l(.)$ as a functional (depending on S) on the set $W_{\leq n}$ with the measure μ^{*n}, and take its mean value with respect to this measure. In other words, consider the expectation of the element length with respect to the random walk for n steps with initial measure μ, i.e. calculate the mean length of a random word $l_n(\mu)$. There exists the limit

$$l(\mu) = \lim l_n(\mu)/n,$$

see Lemma 1.

Definition 3. *The number $l = l(\mu)$ is called the drift or the escape with respect to a given measure, and in case of the measure μ_S – the escape with respect to a given system of generators.*

Theorem 1. Fundamental inequality.

The entropy of an arbitrary symmetric measure on a group concentrated on a given system of generators of this group does not exceed the logarithmic volume in this system multiplied by the escape (drift) with respect to the measure,

$$h \leq l \cdot v.$$

3. Proof of the theorem, corollaries, main problems

Lemma 1. *If the drift is not zero, then the random sequence of words lengths satisfies the Law of Large Numbers, i.e. for each $\epsilon > 0$ there exists N such that for all $n \geq N$ the following relation holds,*

$$\mu^{*n}(g : |l(g)/l - 1| \leq \epsilon) \geq (1 - \epsilon).$$

Proof. Consider the lengths of initial segments of words as cylindric functionals on the infinite product (G^∞, μ^∞). Let $l_n(\omega)$ be the length of the element of the group corresponding to the initial segment of n letters of an infinite sequence ω in the alphabet S. It is easy to see that the sequence $l_n(\omega)$, $n = 1, 2, \ldots$, satisfies the conditions of Kingman's Subadditive Ergodic Theorem with respect to the left shift T as a transformation of G^∞ with the product measure μ^∞, namely

$$l_{n+m}(\omega) \leq l_n(\omega) + l_m(T^n\omega).$$

Thus the sequence $l_n(\omega)/n$ converges almost everywhere. Its limit is constant as follows from the 0–1 law for a sequence of independent variables and from the fact that this limit does not change under any change of the initial segment of the sequence. The statement of the Lemma is equivalent to the convergence in measure of the normalized lengths of elements and thus follows from the almost everywhere convergence. □

Remark. If the drift is not zero, then it follows from the Lemma that the lengths of almost all words grow linearly in n with the same rate. On the other hand, it is well known that for abelian and many other groups the drift is zero and the rate of growth of the words lengths is of order $\sqrt{n}$. The question posed in this connection is whether this rate can for some groups be intermediate between $\sqrt{n}$ and n. As far as we know, hitherto this question was not studied. The answer is positive, the corresponding examples were constructed recently by a student A. Djubina on a basis of wreath products which are quite useful in the theory of random walks and were considered earlier (see [8]). See in this volume her result on properties of the drift on wreath products.

Now we pass to the proof of the Theorem.

Proof. Take an arbitrary positive ϵ and choose a positive integer $n = n(\epsilon)$ satisfying the condition of Lemma 1. Decompose the measure μ^{*n} into a convex combination of two measures, one of them μ_1 being the normalized restriction of this measure on the set V of elements whose lengths are in the interval $[(1-\epsilon)l \cdot n, (1+\epsilon)l \cdot n]$, and the other μ_2 being the normalized restriction of the measure μ^{*n}on the complement of the set V in $W_{\leq n}$. By the Lemma, the measure of the set V is greater than $1-\epsilon$, and the number of elements of the second set does not exceed the total number of elements in $W_{\leq n}$.

Then the entropy $H(\mu^{*n})$ may be estimated as

$$H(\mu^{*n}) \leq (1-\epsilon)H(\mu_1) - \log(1-\epsilon) + \epsilon \cdot \log(W_{\leq n})$$

(the last summand estimates from above the entropy of the measure μ_2 by the entropy of the uniform measure on $W_{\leq n}$). Divide by n and take the limit on n, taking into account that the last summand divided by n tends to the logarithmic volume v multiplied by ϵ (see the definition). Thus we obtain

$$h \leq (1-\epsilon)H(\mu_1)/n + \epsilon \cdot v.$$

But $H(\mu_1)/n$ may be estimated from above by the expression $(\log |V|)/n$ which gives $l \cdot v$ when n tends to infinity, i.e.

$$h \leq (1-\epsilon)l \cdot v + \epsilon \cdot v,$$

and since ϵ is arbitrary, we obtain the desired inequality

$$h \leq l \cdot v.$$

□

Corollary. *If $l = 0$, i.e. the growth of words is slower than linear, then the entropy is zero, too. Thus the positiveness of the entropy implies the linear growth of words typical with respect to this measure.*

Perhaps, the inverse statement is true, too[1].

An analogy of the fundamental inequality with the inequality between topological and metric entropy of dynamical systems (see, for example, [11]) inevitably comes to mind. It is convenient to write the inequality in the form

$$h/l \leq v.$$

The left hand side depends on the measure, and the right hand side depends only on the set of generators (compare with [5], where for trees the right hand side is interpreted as the Hausdorf dimension of some measure). The ratio in the left hand side is the *normalized entropy* (i.e. the entropy on one step, or one letter, in the real increment of the word). Denote it by $\widehat{h}$.

If we consider the product measure on the infinite product of supports of the measure μ with factors μ, then the normalized entropy $\widehat{h}$ may be easily interpreted

[1]V. Kaimanovich has brought to my attention that the inverse statement indeed follows from one estimation of N. Varopoulos.

as the limit entropy on one step of partitions of infinite words on equivalence classes with respect to the equality of initial segments of these words in the group, and v becomes the limit entropy of uniform measures on the classes. Thus our inequality has not literally the same meaning as in topological dynamics (we have not a transformation, but only a sequence of partitions).

However, this analogy is quite useful, first of all because of similarity of main questions. Here they are.

Let the set of generators, and hence the value of v, be fixed. We call the measure supported by a given set of generators and their inverses, that provides the greatest $\widehat{h}$, the *measure of maximal normalized entropy* with respect to a given set of generators.

A. WHEN FOR A GIVEN SET OF GENERATORS THERE EXISTS A SYMMETRIC MEASURE OF MAXIMAL NORMALIZED ENTROPY? WHEN THIS MEASURE IS UNIQUE?

As the most natural measure μ one should take the uniform measure on generators of the group and their inverses, but it seems that this measure does not always give the maximal ratio $\widehat{h} = h/l$, since generally speaking the generators may be not equal in rights.

It may be that the measure of maximal normalized entropy exists, but for a given set of generators the maximal value of $\widehat{h} = h/l$ is strictly less than v. A symmetric set of generators in an exponential group is called *extremal*, if there exists a symmetric measure on this set such that for this measure the fundamental inequality in reduced form turns into equality: $\widehat{h} = v$.

B. FOR WHICH GROUPS THERE EXISTS AN EXTREMAL SET OF GENERATORS?

Now we shall vary the set of generators too, and look for its invariant characteristics.

Consider the expression

$$\frac{h}{l \cdot v} = \frac{\widehat{h}}{v} \equiv q = q(S, \mu)$$

(assuming that $l \neq 0$) and its supremum with respect to all symmetric measures μ supported by a finite symmetric set S:

$$q(S) = \sup_{\mu} q(S, \mu).$$

By Theorem 1, the number $q(S)$ does not exceed 1, and equals 1 only for extremal systems.

Thus we obtain an *objective method of comparing systems of generators: the greater $q(S)$ is, the more effective the given system of generators is.*

C. DESCRIBE INTRINSICALLY EXTREMAL SYSTEMS OF GENERATORS. CHOOSE THE ONE WITH THE MAXIMAL VALUE OF v.

Note that the value of v in itself does not tell much about the group, but together with the following invariant it gives more deep information on the group.

For an arbitrary exponential group, denote

$$q_G = \sup_S q(S).$$

This number is an invariant of the group. The system of generators which provides the maximal value of $q(S)$ we call *maximal.* Groups with $q_G = 1$ we call *free-like.* For the present one does not know examples of groups with $q_G < 1$.

Let us elucidate the meaning of these notions. The greater the constant q is, the greater the asymptotic fraction of elements of the group obtained by the random walk (with a given measure on generators) for n steps among all elements of the corresponding length is. In particular, if $q = 1$ for some choice of generators and a proper measure on them (for example, uniform), then typical random words exhaust asymptotically the whole group. In the opposite case, they constitute only an exponentially small part.

4. Examples. Local groups

Let us collect known facts on the fundamental inequality and examples of calculating the introduced constants. There are quite little of them. Recall that we consider only exponential groups. As to the growth (logarithmic volume), there are dozens of works devoted to it. However, the entropy and the drift are calculated in very few cases. The most effective way of calculating the entropy is to use an analogue of Shannon's theorem proved in [3] and [10]. We shall return to this calculation elsewhere. Of course, it is interesting to do these calculations for classic groups – $SL(n, \mathbf{Z})$, other matrix groups, solvable groups, etc. We concentrate our attention on the class of so called local groups. But first we give an example which is known for a long time.

Example 1. Free groups. Calculations in this case are well known.

Theorem 2. *The uniform measure on natural generators and their inverses in a free group with k ($k \geq 2$) generators has the maximal entropy, the fundamental inequality turns into equality, and the system of generators is extremal, i.e. a free group is free-like. The values of constants in this case are the following:*

$$v = \log(2k-1),\ l = (k-1)/k,\ h = l \cdot v.$$

Proof. Calculation of the logarithmic volume is trivial: the number of words of length $n > 0$ equals exactly $2k \cdot (2k-1)^{n-1}$, hence $v = \log(2k-1)$. The drift, i.e. the normalized on n expectation of the word length, is the mean range of the random walk on $\mathbf{N}$ with transition probabilities $\mathrm{Prob}\,[i \to i+1] = 2k - 1/2k$ and $\mathrm{Prob}\,[i \to i-1] = 1/2k - 1$ for $i \neq 0$, hence $l = k - 1/k$. The entropy of the measure μ^{*n} is equal to the entropy of the mixture of uniform distributions on words of length $s = 0, \dots, n$ with binomial distribution as mixture coefficients, and since the support of this distribution grows linearly and not exponentially with n, its contribution to the entropy is zero. Hence, the entropy is equal to the entropy of the uniform distribution on words of the typical length, i.e. $\ln \cdot \log(2k-1)$, and after normalizing on n we obtain $h = k - 1/k \cdot \log(2k-1)$, i.e. the equality in the fundamental inequality. $\square$

For extensions, factor groups obtained from free groups by subexponential groups (so called virtually free groups) the situation is the same. However, in case of general groups, the situation, and in particular the calculation of the entropy, which is the most technically difficult part, is far more difficult.

The following question is initiated by similarity of hyperbolic and free groups.

D. ARE HYPERBOLIC IN SENSE OF GROMOV GROUPS FREE-LIKE?

If they are, then for which generators and measures the fundamental inequality turns into equality? Also the following question is of interest:

E. ARE THE FUNDAMENTAL GROUPS OF SURFACES OF GENUS $g \geq 2$ FREE-LIKE?

Example 2. Locally free groups, braid groups.

We now recall the definition of local groups ([6, 7]).

Definition 4. *A group (semi-group, algebra) is called local, if it is generated by a finite or countable well-ordered set of generators $z_1, z_2, \dots$, the generators with numbers differing by more than s (in the most interesting case $s = 1$) commute, and the neighbour ones may be connected by some relations. If these relations depend only on the difference of numbers of generators, then the local group (semi-group, algebra) is called local stationary.*

We shall consider here only the case $s = 1$, i.e. commutation at distance 2 and greater. The term "local" in algebra is evidently overloaded, but here the analogy is rather physical – commutation of remote elements correspond to their "non-interaction".

Coxeter groups, braid groups, Hecke algebra etc. are examples of local groups and algebras. More generally, one can consider as z_i not generators but subgroups or subalgebras. In the most recent time, such objects become popular in mathematical physics (see [12, 13]). Even more general notion is the localness with respect to a graph. In our case the graph is a chain. Another interesting case is a cycle, when generators are indexed with roots of unity.

Definition 5. *A locally free group LF_k (semi-group LF_k^+, algebra) is a group (semi-group, algebra respectively) such that the neighbour generators $z_i, z_{i+1}, i = 1 \dots k$, $k \leq \infty$ generate free subgroups with two generators, and the other pairs of generators commute.*

The locally free group LF_k is in a sense an approximation of the braid group B_{k+1}: it is simultaneously a subgroup and a supergroup of the braid group.

The paper of S. Nechaev, R. Bikbov and the author [9] considers in details the case of locally free group and semi-group from the above standpoint. The results may be summed up as follows.

Theorem 3. *The natural generators in a locally free group (semi-group) are not extremal. More exactly, for an arbitrary number of generators (including infinite!) k the following bounds hold.*

For a locally free semi-group LF_k^+:

$$v = \log 4 + o(k^{-1}),$$

$$l = 1,$$

since in the semi-group LF_k^+ there are no cancellations,

$$h = \log 3 + \epsilon(n, k),$$

where $\epsilon(n,k) \to 0$, n, $k \to \infty$, k is the number of generators (maybe infinite), and n is the length of elements.

Thus for k great enough and for infinite k we obtain $v \cdot l \neq h$.

For a locally free group LF_k:

$$v = \log 7 + o_k(1),$$

$$l = 2/3 - \beta,$$

where β is a non-negative constant (maybe zero for infinite k),

$$h = \log(3 - \alpha) + o(k^{-1}),$$

where α is a constant which does not exceed $1/2$ by absolute value.

Thus in natural generators of a locally free group (semi-group), the fundamental inequality is strict, $v \cdot l > h$.

The detailed calculations in [9] are related to combinatorics which is interesting in itself (so called heaps and their bases) and is similar to the combinatorics of Young diagrams and tableaus. We emphasize that for the present only satisfactory bounds on constants are found which enable us to deduce non-maximality etc. But it seems that is is far more difficult to find exact values of these constants.

Remarks.

1. It is not known whether the uniform measure is the measure of maximal entropy.
2. It is interesting that the logarithmic volume and the constants l and h in a locally free group *stabilize* with growth of the number of generators k, and do not tend to infinity as one could suppose.
3. The paper [9] contains also calculations for locally free groups with generators of finite order. If the order tends to infinity, all the constants tend to their values for a locally free group.
4. From the Theorem and the above mentioned relation between the locally free group and the braid group we obtain two-sided bounds on the same constants for the braid group. It turns out that in this case, for the classical system of generators of this group and the uniform measure, the fundamental inequality is not strict either, and exactly as in a locally free group all constants stabilize to a finite value when the number of generators tends to infinity. Thus *in braid groups the set of elements which are typical random words in natural generators with the uniform measure constitutes only an exponentially small part of the set of all elements.*

Note also that generators in local groups are not equal in rights. More exactly, a non-identity substitution which is not a reflection ($i \to k - i$) does not generate an automorphism of the group. Thus one could suppose that the uniform measure on generators and their inverses is not quite natural. However, if we introduce a cycle locally free group (i.e. identify the first and the last generators in a locally free group), then our estimations will not change, though now a cycle group acts transitively on generators and the uniform measure becomes natural.

Example 3. Locally M-free groups. We now introduce the following class of local groups. Consider some manifold of groups M and a free group with two generators $F_M(2)$ in this manifold. *Locally M-free group $LF_M(k)$* with k generators $(z_1, \dots, z_k)$ is the group generated by these generators and the following relations: $z_i \cdot z_j = z_j \cdot z_i, |i-j| \geq 2$, and (z_i, z_{i+1}) are the generators of a subgroup which is canonically isomorphic to $F_M(2)$. Of course, the existence of such groups for an arbitrary manifold M is not evident, and generally speaking the groups $LF_M(k)$ themselves do not belong to the manifold for $k \geq 3$. For example, consider a locally free solvable group of level 2 with two generators and construct a locally solvable free group in this way. The structure of this group is rather complicated (for $k \geq 3$) and it is already not solvable. However, it follows from the above considerations of the locally free group that the growth in this group stabilizes. These groups seem to arise in connection with crystal bases.

A more simple class of local groups is that of locally nilpotent free groups. It arose in works on the theory of integrable systems. The simplest variant of this group is the following. The neighbour generators (z_i, z_{i+1}) generate canonically the discrete Heisenberg group, and all the other pairs commute. Though nilpotent groups have polynomial growth, locally nilpotent free groups are already exponential for $k \geq 3$. In all these cases the fundamental inequality remains presumably strict. Relations with the Poisson boundary, whose non-triviality is equivalent to positiveness of the entropy, with the spectral theory and harmonic analysis on groups, in particular on local groups, are to be considered elsewhere.

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Vershik A. M. Numerical characteristics of the countable groups and corresponding relations.
The entropy of the random walk on the discrete countable group could be used for comparison of the system of the generators. Fundamental inequality between growth, entropy and escape gives the possibility to define "the best" system of the generators. We formulate a new circle of the problems related to the various growth and asymptotics on the groups.