HETEROCLINIC CONTOURS THAT GENERATE STABLE CHAOS

V.E.Chernyshev

June 1998

Abstract. This is an English version of the Introduction to the author's thesis.

Introduction

Statements of the dissertation are numerated as follows. The first numeral (Roman) is the number of chapter, the second numeral is the number of section, and the third numeral is the number of the statement in its section.

Consider a system of differential equations of the form

$$\dot{x} = X(x), \quad x \in \mathbb{R}^3, \quad X \in \mathbb{C}^r(\mathbb{R}^3),\ r \geq 3. \tag{1}$$

Assume that our system generates a flow, i.e., that any its trajectory is defined on the whole axis. Denote this flow by g^t.

Definition. *An invariant set* Γ *is called a heteroclinic contour if* Γ *is connected, compact, and consists of a finite family of trajectories* $\gamma_i,\ i \in 1:m$, *and of their* α*-limit sets and* ω*-limit sets,*

$$\Gamma = \{\cup_{i\in 1:m}\gamma_i\} \cup \{\cup_{i\in 1:m}\alpha(\gamma_i)\} \cup \{\cup_{i\in 1:m}\omega(\gamma_i)\}.$$

It is assumed that any limit set belonging to the contour Γ *is either a hyperbolic rest point or a hyperbolic closed trajectory.*

Definition. *We say that a heteroclinic contour* Γ *is a heteroclinic cycle if it is possible to numerate its trajectories so that* $\omega(\gamma_i) = \alpha(\gamma_{i+1}),\ i \in 1 : m-1,\quad \omega(\gamma_m) = \alpha(\gamma_1),$ *and* $\alpha(\gamma_i) \cap \alpha(\gamma_j) = \emptyset$ *for* $i \neq j (\mathrm{mod}\, m)$.

Definition. *We say that a heteroclinic cycle of a three-dimensional autonomous system belongs to the Lorenz type if any its limit set is either a saddle-node rest point or a closed trajectory with orientable stable and unstable manifolds.*

Definition. *We say that a heteroclinic contour is equidimensional if the dimensions of stable manifolds of all its limit sets are the same.*

Below we consider equidimensional heteroclinic cycles of the Lorenz type such that among their limit sets there are both rest points and closed trajectories. Our goal is to give conditions under which contours of this type generate persistent chaos (in the sense of the following two definitions).

Definition. *We say that an invariant set* J *is chaotic if*

(1) the set J contains a dense trajectory of the flow g^t;

(2) *the union of closed trajectories is dense in J;*

(3) *the set J exhibits sensitive dependence with respect to initial data, i.e., there exists a number $\varepsilon > 0$ such that for any point $x \in J$ and for any number $\delta > 0$ there is a point y and a number $t > 0$ such that $\rho(x, y) < \delta$ and $\rho(g^t x, g^t y) > \varepsilon$, where $\rho(x, y)$ denotes the distance between x and y.*

Definition. *We say that a heteroclinic cycle Γ generates persistent chaos if for any neighborhood $V(\Gamma)$ of the set Γ there exists a number $\delta > 0$ such that any system whose C^1 distance from system (1) is less than δ has a chaotic invariant set belonging to $V(\Gamma)$.*

Now we formulate our conditions I–III. Consider an equidimensional heteroclinic cycle of the Lorenz type consisting of trajectories $\gamma_i,\ i \in 1 : m,\ m \geq 2$, and of their limit sets $\alpha(\gamma_i),\ \omega(\gamma_i),\ i \in 1 : m$. The numeration of the trajectories is chosen so that $\omega(\gamma_i) = \alpha(\gamma_{i+1}),\ i \in 1 : m - 1, ,\ \omega(\gamma_m) = \alpha(\gamma_1)$, and $\alpha(\gamma_i) \cap \alpha(\gamma_j) = \emptyset$ if $i \neq j$. Any limit set $\alpha(\gamma_i),\ i \in 1 : m$, is either a saddle rest point O_i or a saddle closed trajectory P_i such that its stable and unstable manifolds are orientable. We assume that the limit set $\alpha(\gamma_1)$ is a closed trajectory P_1.

Since one of the limit sets of our equidimensional cycle is a closed trajectory, for any limit set, the dimension of its stable manifold equals two. Denote by $\mu_i > 0 > \lambda_i > \nu_i$ the eigenvalues of the Jacobi matrix $D\,X(O_i)$ at a saddle rest point $O_i \in \Gamma$.

Our first condition has the following form:

I. $\lambda_i > -\mu_i, \quad \lambda_i - \mu_i > \nu_i.$

Our second condition is related to the character of approach of trajectories γ_i of our heteroclinic cycle to their limit sets. This is a general position condition.

II. *There exists a continuous bundle P of the planes $P(x),\ x \in \Gamma$, invariant with respect to the differential $D\,g^t,\ t \in \mathbb{R}$, of the shift g^t for time t along trajectories of system (1). The plane $P(O_i)$ coincides with the plane $\langle v_i^s, v_i^u \rangle$ spanned by the eigenvetors $v_i^s,\ v_i^u$ corresponding to the eigenvalues $\mu_i,\ \lambda_i$. For a point $x \in P_j$, the plane $P(x)$ is the tangent plane at x of the unstable manifold W_j^u of the closed trajectory P_j.*

Let $P_{l_1},\ P_{l_2}, \ldots, P_{l_k}$ be all closed trajectories belonging to the cycle $\Gamma,\ l_1 = 1 < l_2 < \cdots < l_k$. We show in Theorem I.1.1 that condition II implies, in particular, the following property: the unstable manifold $W_{l_i}^u$ of the closed trajectory P_{l_i} intersects transversally the stable manifold $W_{l_i+1}^s$ of the next limit set in the cycle Γ along a trajectory $\gamma_{l_i}:\ W_{l_i}^u \cap W_{l_i+1}^s = \gamma_{l_i}$. In addition, trajectories of the cycle Γ approach rest points along leading directions (the latter term means directions corresponding to the eigenvalues λ_i).

The third condition (similarly to Condition I) is not a general position condition.

III. *It is possible to fix an orientation on the bundle P so that this orientation is continuous on the set $\Gamma \backslash \{\cup \gamma_{l_i},\ i \in 1 : k\}$ and has the following properties. In the planes $P(O_i)$, the orientation is determined by the frame (v_i^s, v_i^u), where v_i^s (v_i^u) is the limit position of the vector $X(x)$ as $x \to O_i, \quad x \in \gamma_{i-1} \quad (x \in \gamma_i)$. In the plane $P(x)$, where a point x belongs to a closed trajectory $P_{l_i},\ i \in 1 : k$, the orientation is determined by the frame $(X(x), v^u(x))$, where $v^u(x)$ is the tangent vector to the unstable manifold $W^u(x)$ of the point $x \in P_{l_i}$, directed toward the intersection of $W^u(x)$ with the trajectory γ_{l_i}.*

Chapter I: "**Heteroclinic contours of the Lorenz type**".

In this chapter, we investigate general properties of heteroclinic contours.

Section 1 of Chapter 1 is devoted to equidimensional contours of the Lorenz type. By analogy with heteroclinic cycles of the Lorenz type, we say that a heteroclinic contour is of the Lorenz type if any its limit set is either a saddle rest point or a saddle closed trajectory. It is assumed that our contour contains both rest points and closed trajectories.

Condition II formulated above for heteroclinic cycles can be stated in the same way for heteroclinic contours. Sometimes other general position conditions are applied in the investigation of dynamics near heteroclinic contours. They are formulated in Section 1. In Theorem I.1.1, we prove that these conditions are equivalent to Condition I in the case of an equidimensional contour of the Lorenz type.

Definition. *We say that a heteroclinic contour is ramification free if it is possible to numerate its trajectories so that* $\omega(\gamma_i) = \alpha(\gamma_{i+1}),\ i \in 1 : m-1,$ *and* $\alpha(\gamma_i) \cap \alpha(\gamma_j) = \emptyset,\ i, j \in 1 : m,\ i \neq j.$

In Section 2, we study ramification free heteroclinic contours of the Lorenz type that satisfy Conditions I and II and the following additional assumption: the limit sets $\alpha(\gamma_i),\ i \in 2 : m$, are saddle rest points, while the limit set $\alpha(\gamma_1)$ is a saddle closed trajectory with orientable stable and unstable manifolds. Such a contour is called a simple contour of the Lorenz type.

Theorem I.2.2. *A simple contour* Γ *of the Lorenz type has a neighborhood* $V(\Gamma)$ *with the following property: system (1) has in* $V(\Gamma)$ *a smooth invariant surface* Q *containing the contour* Γ. *In addition, the bundle* $P(x),\ x \in \Gamma$, *from Condition II coincides with the bundle of tangent planes* $T_xQ,\ x \in \Gamma$, *of the surface* Q.

Further, we describe in Section 2 the behavior of trajectories of system (1) on the surface Q. The description is given in terms of the Poincaré mapping h sending points of a transversal $s_1 \subset Q$ of the trajectory γ_1 along trajectories of system (1) to a transversal $s_{m+1} \subset Q$ of the trajectory γ_m. The contour $\Gamma \subset Q$ divides the curve s_1 into two parts s_1^+ and s_1^-.

It is shown in Theorem I.2.3 that under Condition III, the mapping h is defined on one of the curves s_1^+ or s_1^- and can be written (in proper coordinates) in the following form:

$$h(\eta_1) = C\eta_1^E + q(\eta_1^E),$$
$$h'(\eta_1) = CE\eta_1^{E-1} + r(\eta_1^{E-1}),$$

where $E < 1$, and the functions q, r are infinitesimally small as $\eta_1 \to 0$.

Since $E < 1$, the mapping h is expanding for small η_1. Consider the part of the surface Q covered by trajectories intersecting the curve on which the mapping h is defined. It follows from our reasons that it is natural to call this part of Q the unstable surface of the contour Γ.

Below, in Chapter II, we show that simple cycles of the Lorenz type satisfying Codition III generate persistent chaos. By definition, a saddle-focus rest point cannot be a limit set in a simple cycle of the Lorenz type. We extend the class of heteroclinic cycles generating persistent chaos. In Section 3, we consider a class of heteroclinic contours having saddle-focus rest points as their limit sets. This class includes simple heteroclinic contours of the Lorenz type satisfying Conditions I – III considered above. It is shown in Theorem I.3.1 that any neighborhood of a contour of this class contains a simple heteroclinic contour of the Lorenz type satisfying

Conditions I – III. If the unperturbed contour is a cycle, then the generated simple contour is also a cycle. It follows that cycles of the introduced class generate chaos.

Chapter II. "**Persistent chaotic invariant sets generated by cycles of the Lorenz type**".

In this chapter, we prove the main results of the dissertation. Everywhere in this chapter, we assume that the unperturbed system (1) has a heteroclinic cycle Γ of the Lorenz type satisfying Conditions I, II, and III. We also assume that all systems of differential equations (both unperturbed and perturbed) are of class C^4.

Denote by g_ε^t the flow of the perturbed system

$$\dot{x} = X(x) + Y(x), \quad Y \in \mathbb{C}^r(\mathbb{R}^3), \quad r \geq 4, \quad ||Y||_{\mathbb{C}^1} < \varepsilon, \qquad \mathbf{(2)}$$

let $D\, g_\varepsilon^t$ be its derivative.

The first step in the proof of the main result of Chapter II is to establish the following statement (having an independent interest). In this statement, we construct an invariant strongly stable fiber bundle over a neighborhood of a cycle Γ.

heorem II.2.1. *Assume that an equidimensional cycle Γ of the Lorenz type satisfies Conditions I, II, and III. Then there exist numbers $\varepsilon_0 > 0,\ C > 0,\ \lambda^i < 0,\ \Lambda^i < 0,\ \lambda^i < \Lambda^i,\ i \in 1:m,$ a neighborhood $V(\Gamma)$ of the heteroclinic cycle Γ, and neighborhoods $V_i \subset V(\Gamma)$ of the limit sets $\alpha(\gamma_i),\ i \in 1:m$, such that for $\varepsilon < \varepsilon_0$ there exists a continuous $D\, g_\varepsilon^t$ - invariant decomposition*

$$\mathbb{R}^3 = E_\varepsilon^{ss}(x) \oplus E_\varepsilon^v(x), \quad x \in V(\Gamma), \qquad \mathbf{(3)}$$

where $E_\varepsilon^{ss}(x)$ is a line and $E_\varepsilon^v(x)$ is a plane for which the following estimates hold:

$$||D\, g_\varepsilon^t(x)v|| \leq C \exp(\lambda^i t)||v||, \quad t \geq 0, \quad v \in E_\varepsilon^{ss}(x),\ [x,\ g_\varepsilon^t(x)] \in V_i, \qquad \mathbf{(4)}$$

$$||D\, g_\varepsilon^t(x)v|| \leq C \exp(\Lambda^i t)||v||, \quad t \leq 0, \quad v \in E_\varepsilon^v(x),\ [x,\ g_\varepsilon^t(x)] \in V_i. \qquad \mathbf{(5)}$$

Here $[x, g_\varepsilon^t(x)]$ denotes the arc of the trajectory of a point x with ends x and $g_\varepsilon^t(x)$. In addition, the bundle of lines $E_\varepsilon^{ss}(x),\ x \in V(\Gamma)$, is locally Lipschitz continuous.

The heteroclinic cycle Γ does not have a neighborhood invariant with respect to the mapping g_ε^t, hence we have to give a special definition.

Definition. *The bundles $E_\varepsilon^{ss}(x),\ E_\varepsilon^v(x),\ x \in V(\Gamma)$, over a neighborhood $V(\Gamma)$ of the cycle Γ are invariant with respect to $D\, g_\varepsilon^t$ if*

$$D\, g_\varepsilon^t(x) E_\varepsilon^{ss}(x) = E_\varepsilon^{ss}(g_\varepsilon^t(x)), \quad D\, g_\varepsilon^t E_\varepsilon^v(x) = E_\varepsilon^v(g_\varepsilon^t(x)),$$

provided $g_\varepsilon^\tau(x) \in V(\Gamma)$ for $\tau \in [0, t]$.

We prove Theorem II.1.1 in two steps. In Section 1, we construct decomposition (3) over the heteroclinic cycle Γ for the unperturbed system (1). In this proof, we show that both terms in (3) are locally Lipschitz continuous and satisfy estimates (4) and (5).

In Section 2, we construct the needed decomposition (3) over a neighborhood $V(\Gamma)$ of the heteroclinic cycle Γ. First we extend the decomposition constructed in Section 1 to a neighborhood $W(\Gamma)$ of Γ preserving its local Lipschitz continuity. The extension

$$\mathbb{R}^3 = \widetilde{E}_1^{ss}(x) \oplus \widetilde{E}_1^v(x), \quad x \in W(\Gamma),$$

is not necessarily invariant with respect to $D\,g_\varepsilon^t$. We obtain invariant terms in (3) by analogous reasons, let us describe them for the first term.

Consider a bundle $\mathfrak{L}$ over a neighborhood $W(\Gamma)$ of the heteroclinic cycle Γ such that its fiber $\mathfrak{L}(x)$ over $x \in W(\Gamma)$ is the set of all lines transverse to the plane $\widetilde{E}_1^v(x)$. Any such line is the graph of a linear mapping $P(x): \widetilde{E}_1^{ss}(x) \to \widetilde{E}_1^v(x)$, we identify the mapping and the line.

Take a neighborhood $V(\Gamma)$ of Γ such that it belongs to $W(\Gamma)$ together with its closure. Consider the space L of continuous sections of the bundle $\mathfrak{L}$, i.e., the family of continuous mappings $P: V(\Gamma) \to \mathfrak{L}$ such that any $P(x)$ is a linear mapping from $\widetilde{E}_1^{ss}(x)$ to $\widetilde{E}_1^v(x)$.

We take the neighborhood $V(\Gamma)$ to be closed, hence L is a Banach space in the $\mathbb{C}$ norm. For ε small enough, we define a mapping H_ε taking a closed convex subset of L to itself and contracting on this subset. This mapping is a graph transformation of linear mappings $P(x),\ x \in V(\Gamma)$, under $D\,g_\varepsilon^{-l\tau}$ for some fixed $l \in N,\ \tau \in \mathbb{R}$. It follows that the bundle $E_\varepsilon^{ss}(x),\ x \in V(\Gamma)$, corresponding to the fixed point of H_ε is invariant with respect to $D\,g_\varepsilon^{-l\tau}$. Now the uniqueness of a fixed point of the contraction H_ε implies the invariance of $E_\varepsilon^{ss}(x),\ x \in V(\Gamma)$, under $D\,g_\varepsilon^t$ for all $t \in \mathbb{R}$. Further, we prove that the bundle $E_\varepsilon^{ss}(x),\ x \in V(\Gamma)$, is locally Lipschitz in $V(\Gamma)$.

Similarly one constructs the second term $E_\varepsilon^v(x),\ x \in V(\Gamma)$, in decomposition (3). Note that the family $E_\varepsilon^v(x),\ x \in V(\Gamma)$, may be not Lipschitz in a neighborghood of Γ.

We apply the bundle $E_\varepsilon^{ss}(x),\ x \in V(\Gamma)$, in Section 3 (Theorem II.3.1) to construct a strongly stable one-dimensional lamination $W_\varepsilon^{ss}(x),\ x \in V(\Gamma)$, in a neighborhood of the cycle Γ. Let us give a definition.

Definition. *We define a one-dimensional lamination on a neighborhood $V(\Gamma)$ as a decomposition of this neighborhood into a union of smooth one-dimensional submanifolds called laminae such that for any point $x \in V(\Gamma)$ there is a neighborhood U of this point, a subset $W \subset I^{n-1}$, and a homeomorphism $\varphi: I \times I^{n-1} \to U$ mapping $I \times w,\ w \in W$, diffeomorphically onto a connected component of the intersetion of some lamina with U. In addition, in some local coordinates $v,\ w$, the derivative $\frac{\partial \varphi}{\partial v}(v, w)$ is continuous in v, w.*

In the definition above, I denotes $(-1, 1)$, and we use coordinates $v,\ w,\ v \in I,\ w \in I^{n-1}$ in the n-dimensional cube $I^n = I \times I^{n-1}$.

The tangent bundle $W_\varepsilon^{ss}(x),\ x \in V(\Gamma)$, of our lamination is the bundle of lines

$$E_\varepsilon^{ss}(x),\ x \in V(\Gamma).$$

Since this last bundle is Lipschitz continuous, it is easy to show that the homeomorphism φ from the definition of a lamination satisfies the Lipschitz condition with respect to w locally in our case. In addition, it is shown in Sect. 3 that the family of curves $W_\varepsilon^{ss}(x),\ x \in V(\Gamma)$, is invariant under the mapping g_ε^τ, i.e., $g_\varepsilon^\tau W_\varepsilon^{ss}(x) \cap V(\Gamma) \subset W_\varepsilon^{ss}(g_\varepsilon^\tau(x)),\ x \in V(\Gamma), \tau \in \mathbb{R}$, and the mapping g_ε^τ contracts along laminae. The last statement has the following meaning: there exist numbers $\lambda > 0,\ K > 0,\ \sigma > 0$, such that for $z, y \in W_\varepsilon^{ss}(x) \cap B_\sigma(x),\ x \in V(\Gamma)$, the inequality

$$d(g_\varepsilon^t(z), g_\varepsilon^t(y)) \le K \exp(-\lambda t)\, d(z, y),$$

holds if the points $g_\varepsilon^\tau(y),\ g_\varepsilon^\tau(z) \in V(\Gamma)$ for $\tau \in [0, t]$. (Here $d(z, y)$ denotes the distance between the points z, y, and $B_\sigma(x)$ is the ball of radius σ centered at x.)

For a segment $\widetilde{\gamma}$ of a trajectory of system (2), different from a rest point and such that $\widetilde{\gamma} \subset V(\Gamma)$, we define its strongly stable set

$$W_\varepsilon^{ss}(\widetilde{\gamma}, \delta) = \bigcup_{x \in \widetilde{\gamma}} W_\varepsilon^{ss}(x, \delta),$$

where $W_\varepsilon^{ss}(x, \delta) = W_\varepsilon^{ss}(x) \cap B_\delta(x)$. For δ small enough, this set is a two-dimensional smooth manifold.

The set $W_\varepsilon^{ss}(\widetilde{\gamma}, \delta)$ *is called the strongly stable manifold of the segment* $\widetilde{\gamma}$ *of a trajectory of system* (2) *of size* δ.

Let S be an arbitrary transversal to trajectories of system (2) belonging to the neighborhood $V(\Gamma)$. For a point $x \in S$, the connected component of the intersection $W_\varepsilon^{ss}(\widetilde{\gamma}, \delta) \cap S$, containing the point x lying on a small enough segment $\widetilde{\gamma}$ of its trajectory does not depend on the segment $\widetilde{\gamma}$ if the number δ is small enough. If S is a smooth transversal, then the component $w_\varepsilon^{ss}(x, \delta)$ is a smooth curve.

We say that $w_\varepsilon^{ss}(x, \delta)$ *is the strongly stable manifold (or a strongly stable curve) of the point* x *of size* δ *on the transversal* S.

It is shown at the end of Sect. 3 that if a Poincare mapping is defined for two transversals, then this mapping contracts along strongly stable curves.

In Sect. 4, we construct a chaotic invariant set J for the unperturbed system (1). The following statement is proved.

Theorem II.4.2. *For any neighborhood* $\widetilde{V}$ *of the heteroclinic cycle* Γ, *satisfying conditions* I, II, *and* III, *there exists a chaotic invariant set* $J \subset \widetilde{V}$.

The invariant set J is defined as the set of all trajectories of system (1) through points of an invariant set I of the Poincare mapping F of a transversal $S(1) \subset \widetilde{V}$.

We construct the invariant set I in Lemmas II.4.1 – II.4.8. This set is constucted as the image of the space Ω whose elements are two-sided infinite sequences $\omega = \{\omega_s\}$, $s \in Z$, under a one-to-one mapping Ψ. Elements of the sequences above belong to an infinite set E of integer k-dimensional vectors. If the space Ω is endowed with a standard metric, then the mapping Ψ is a homeomorphism $\Psi : \Omega \to \Psi\Omega = I$. This statement is proved in Lemma II.4.9. Consider the shift homeomorhism σ on the space Ω. It follows from the construction of the mapping Ψ that this mapping conjugates the mapping σ on the space Ω with the mapping F on I. Thus, a symbolic dynamics for the mapping $F|_I$ is constructed. This construction is applied in Theorem II.4.1 to show that the invariant set I is chaotic. This method for establishing chaotic structure of an invariant set is well known [22, 2]. Note that in our case, the alphabet E used in construction of symbolic sequences is infinite. As a result, we see that both the space Ω and the set I are not compact, and it is impossible to extend the mapping F to the closure of the set I preserving its continuity.

Coding of points of the set I by sequences $\{\omega_s\}$, $s \in Z$, has a simple geometric interpretation. If a vector ω_s has coordinates $(e_1^s, \dots, e_k^s)$, $e_i^s \in N$, then the trajectory of the point x makes, after its sth crossing of the transversal $S(1)$, e_1^s turns around the closed trajectory P_{l_1} not leaving a small neighborhood of this closed trajectory, then it makes e_2^s turns around the closed trajectory P_{l_2} and so on, then it makes e_k^s turns around $P_{l_k} \subset \Gamma$, and then it crosses the transversal $S(1)$ at $(s+1)$th time.

It is easy to show that the shift σ has the same basic properties as the shift on the space of sequences with finite number of symbols. Thus, we can prove that the mapping σ is chaotic.

In Theorem II.4.1, we apply the topological conjugacy of the mappings $F|_I$ and σ to show that the set I is chaotic.

By this theorem, the invariant set J defined above has a dense half-trajectory, and closed trajectories are dense in J.

The third property from the definition of a chaotic set (the sensitive dependence on the initial point) is established in Theorem II.4.2. Finally, we see that the invariant set J of system (1) is chaotic. The invariant set $\overline{J}$ is also chaotic. Since the set I is not compact, $\overline{J} \neq J$.

Theorem II.4.3 and its corollaries describe the structure of invariant sets $\overline{J} \cap W^u_{l_i}$, $i \in 1 : k$. In particular, it is shown that the heteroclinic cycle Γ belongs to the closure of the invariant set J.

In the last Sect. 5, we study the perturbed system (2) such that the perturbation $Y(x)$ is $\mathbb{C}^1$-small. Let us formulate the main result.

Theorem II.5.1. *Assume that system (1) has a heteroclinic cycle* Γ *of the Lorenz type satisfying conditions* I, II, III. *For any neighborhood* $V(\Gamma)$ *of the cycle* Γ *there exists a number* ε_0 *such that if* $\varepsilon < \varepsilon_0$, *then the perturbed system (2) has a chaotic set lying in* $V(\Gamma)$.

The proof of this theorem is contained in Lemmas II.5.1 –II.5.7. This proof is to some extent parallel to the proof of Theorem II.4.2. Indeed, the chaotic set $J(\varepsilon)$ is constructed as the union of trajectories of system (2) through points of a chaotic invariant set $I(\varepsilon)$ for the mapping $F(\varepsilon)$ defined by first return to the transversal $S(1)$ of trajectories of system (2) belonging to a neighborhood $W(\Gamma)$ of the heteroclinic cycle Γ. Here and below, dependence on ε means that we consider the perturbed system (2) such that $||Y||_{\mathbb{C}^1} < \varepsilon$.

The mapping $F(\varepsilon)|_{I(\varepsilon)}$ is topologically conjugate to the shift σ on an invariant subset of the space $\Omega(\varepsilon)$; this space consists of infinite sequences $\omega = \{\omega_s\}_{s=-\infty}^{+\infty}$, $\omega_s \in E(\varepsilon)$, $s \in Z$. The subset invariant under σ on which the shift σ is chaotic is determined by a condition on regularity of intersection of some sets. This last condition is quite complicated and similar to condition from [2].

The main difference with the case of the nonperturbed system is that the alphabet $E(\varepsilon)$ applied for coding of points of the set $I(\varepsilon)$ may be finite for some perturbations. In this case, the invariant set $I(\varepsilon)$ is compact and locally maximal (see Lemma II.5.8.) In the next lemma, we show that if the alphabet $E(\varepsilon)$ is finite, then the invariant set $J(\varepsilon)$ constructed in Theorem II.5.1 is compact and locally maximal. By definition, the invariant set $J(\varepsilon)$ does not contain rest points of the perturbed system belonging to the neighborhood $W(\Gamma)$; it also does not contain trajectories double-asymptotic to rest points. Since the set $J(\varepsilon)$ is compact, in the case of finite alphabet $E(\varepsilon)$, the chaotic set is separated from rest points of the perturbed system.

It is shown in Theorem II.5.2 that, under some additional condition that guarantees that the mapping $F(\varepsilon)|_{I(\varepsilon)}$ is topologically conjugate to the shift σ on the whole space $\Omega(\varepsilon)$, the set $J(\varepsilon)$ is a maximal compact chaotic invariant set belonging to the neighborhood $W(\Gamma)$ and containing neither rest points of system (2) nor their double-asymptotic trajectories.

In Theorem II.5.3, we describe the structure of the invariant set $\overline{J(\varepsilon)} \setminus J(\varepsilon)$ in the case where the first condition of Theorem II.5.2 (the condition of finiteness of the set $E(\varepsilon)$) is violated. The proof of this theorem is contained in Lemmas II.5.10 – II.5.14.

Conditions of Theorem II.5.2 providing the maximality of the invariant set $J(\varepsilon)$ in the neighborhood $W(\Gamma)$ are formulated in terms of the perturbation $Y(x)$. At the end of the last section, we construct a perturbation $Y(x)$ satisfying the conditions of Theorem II.5.2 and having arbitrarily small $\mathbb{C}^1$-norm.

This construction is contained in Theorem II.5.5. This theorem shows that it is possible to separate a chaotic invariant set $\overline{J(\varepsilon)}$ from rest points by an arbitrarily $\mathbb{C}^1$-small perturbation of system (1).

Translated into English by S. Yu. Pilyugin.