Statistical Analysis of Semiparametric Models in Accelerated Life Testing

VILIJANDAS BAGDONAVIČIUS

Department of Statistics, Vilnius University, Lithuania

MIKHAIL S. NIKULIN

Steklov Mathematical Institute, 27 Fontanka, 191011, Saint Petersburg, Russia

Abstract: Several accelerated life models are introduced. Their relationships are investigated. Semiparametric estimation procedures are proposed. Properties of estimators are discussed

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1. Models

Suppose that stresses are deterministic time functions:

$$x(\cdot) = (x_1(\cdot), ..., x_m(\cdot))^T \ : [0, \infty) \to B \in \mathbf{R^m}.$$

Let Ω be a population of items and suppose that the time-to-failure of items under the stress $x(\cdot)$ is defined by a non-negative absolutely continuous random variable $T_{x(\cdot)} = T_{x(\cdot)}(\omega), \omega \in \Omega$, with the survival function $S_{x(\cdot)}(t) = \mathbf{P}\{T_{x(\cdot)} > t\}$ and the cummulative distribution function $F_{x(\cdot)}(t) = 1 - S_{x(\cdot)}(t)$. If $x(\cdot)$ is constant in time, we'll write x instead of $x(\cdot)$ in all formulae. We shall consider here only the deterministic stresses. The case when the stress is a stochastic process X(t),can be done by the same way as in Bagdonavičius and Nikulin (1996, 1997c, 1997d) for the generalized additive and additive-multiplicative semiparametric models with random covariates.

The proportion $F_{x(\cdot)}(t)$ of items from Ω which fail until the moment t under the stress $x(\cdot)$ is called the *resource used until the moment* t.

Let

$$\alpha_{x(\cdot)}(t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbf{P}\{T_{x(\cdot)} \in (t, t+h] \mid T_{x(\cdot)} > t\} = -\frac{S'_{x(\cdot)}(t)}{S_{x(\cdot)}(t)}, \ t \geq 0,$$

be the hazard rate function under the stress $x(\cdot)$. Denote by

$$A_{x(\cdot)}(t) = \int_0^t \alpha_{x(\cdot)}(u) du = -ln\{S_{x(\cdot)}(t)\}, \ t \geq 0,$$

the accumulated hazard rate under $x(\cdot)$.

The first idea which comes by modelling the influence of a stress on lifetime distribution is to suppose that the hazard rate at any moment t depends on the value of the stress at this moment and the resource used until t:

The generalized Sedyakin's (GS) model (Bagdonavičius (1978),Bagdonavičius and Nikulin (1998)) holds on the set of stresses E, if for all $x(\cdot) \in E$

$$\alpha_{x(\cdot)}(t) = g\left(x(t), A_{x(\cdot)}(t)\right).$$

Consider the following important particular case of the GS model. Suppose that the hazard rate at any moment t is proportional to a function of the stress used until this moment and to a function of the resource used until t:

The additive accumulation of damages (AAD) model (Bagdonavičius (1978)) holds on the set of stresses E, if for all $x(\cdot) \in E$

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\, q\{A_{x(\cdot)}(t)\}.$$

Proposition 1. *The AAD model holds on E iff there exists a survival function G such that*

$$S_{x(\cdot)}(t) = G\left(\int_0^t r\{x(\tau)\}\, d\tau\right). \tag{1}$$

The AAD model written in the form (1) is also called the *accelerated failure time* (AFT) model (Cox and Oakes (1984)).

Cox (1972) considered the model which means that the hazard rate at any moment t is proportional to some function of a stress applied at this moment and to a baseline rate which doesn't depend on the stress:

The *proportional hazards* (PH) model holds on E if for all $x(\cdot) \in E$

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\, \alpha_0(t). \tag{2}$$

When the PH model is also the AAD model? The answer is given in the following two propositions.

Proposition 2 (Cox and Oakes (1984)). *Suppose that the PH model holds on a set E_0 of constant stresses such that the set $r(E_0)$ has an interior point.*

The AAD model also holds on E_0 iff the time-to-failure distribution is Weibull for all $x \in E_0$.

Suppose that E_0 is the set of constant stresses defined in Proposition 2, $x_1, x_2 \in E_0$ are two fixed constant stresses and a step-stress $x_s(\cdot)$ has the form

$$x_s(\tau) = \begin{cases} x_1, & 0 \le \tau < s, \\ x_2, & \tau \ge s, \end{cases} \tag{3}$$

where s is a fixed positive number.

Proposition 3. *Suppose that the PH model holds on the set E including E_0 and $x_s(\cdot)$. The AAD model also holds on E iff the time to-failure is exponential for all $x \in E_0$.*

Proof. The Proposition 2 implies that for all $x \in E_0$

$$S_x(t) = e^{-(\frac{t}{\theta(x)})^\alpha}. \tag{4}$$

Put $\theta_i = \theta(x_i)$, $i = 1, 2$. Then

$$S_{x_i}(t) = e^{-(\frac{t}{\theta_i})^\alpha}, \quad \alpha_{x_i}(t) = \frac{\alpha}{\theta_i^\alpha} t^{\alpha-1}.$$

The PH model implies that

$$\alpha_{x_s(\cdot)}(t) = \begin{cases} \alpha_{x_1}(t), & 0 < \tau < s, \\ \alpha_{x_2}(t), & \tau \ge s, \end{cases}$$

and for all $t > s$

$$S_{x_s(\cdot)}(t) = \exp\{-\int_0^t \alpha_{x_s(\cdot)}(u)du\} = \exp\{-\int_0^s \alpha_{x_1}(u)du - \int_s^t \alpha_{x_2}(u)du\} =$$

$$\exp\left\{-\left(\frac{s}{\theta_1}\right)^\alpha - \left(\frac{t}{\theta_2}\right)^\alpha + \left(\frac{s}{\theta_2}\right)^\alpha\right\}. \tag{5}$$

1) Suppose that both the PH and the AAD models hold on E. Then (1) and (4) imply that for all $t > s$

$$S_{x_s(\cdot)}(t) = \exp\left\{-\left(\frac{s}{\theta_1} + \frac{t-s}{\theta_2}\right)^\alpha\right\} \tag{6}$$

The equalities (5) and (6) imply that for all $t > s$

$$\exp\left\{-\left(\frac{s}{\theta_1}\right)^\alpha - \left(\frac{t}{\theta_2}\right)^\alpha + \left(\frac{s}{\theta_2}\right)^\alpha\right\} = \exp\left\{-\left(\frac{s}{\theta_1} + \frac{t-s}{\theta_2}\right)^\alpha\right\}. \tag{7}$$

If $\alpha = 1$, this equality is verified. Suppose that $\alpha \neq 1$. For all $t > s$ put

$$g(t) = \exp\left\{-\left(\frac{s}{\theta_1}\right)^\alpha - \left(\frac{t}{\theta_2}\right)^\alpha + \left(\frac{s}{\theta_2}\right)^\alpha\right\} - \exp\left\{-\left(\frac{s}{\theta_1} + \frac{t-s}{\theta_2}\right)^\alpha\right\}.$$

This function is strictly monotone for all $t > s$ with fixed $\theta_1 \neq \theta_2$ and $\alpha \neq 1$. So the function g is not constant in t which contradicts to the equality (7). The assumption $\alpha \neq 1$ was false. Thus $\alpha = 1$, and the lifetime distribution under any $x \in E_0$ is exponential.

2) Suppose that the PH model holds on E and the time-to-failure is exponential for all $x \in E_0$. The formula (2) implies that for all $x \in E_0$

$$S_x(t) = \exp\{-r(x)\, A_0(t)\}, \quad A_0(t) = \int_0^t \alpha_0(v)dv.$$

Exponentiality of the times-to-failure under $x \in E_0$ and the last formula imply that $A_0(t) = ct$. The constant c can be included in $r(x)$, so we have $A_0(t) = t$. The formula (2) implies that

$$S_{x(\cdot)}(t) = \exp\left\{-\int_0^t r\{x(u)\}du\right\},$$

i.e. the AAD model holds on E.

The proof is complete.

The GS model is more general then the AAD model. When the PH model is also the GS model? It is given in the following proposition.

Proposition 4. *Suppose that the PH model holds on the set E including E_0 and all the stresses of the form (3) with $s < \delta$, where δ is any positive number. The GS model also holds on E iff the time to-failure is exponential for all $x \in E_0$.*

Proof. The PH model implies that for all $s < \delta$

$$\alpha_{x_s(\cdot)}(t) = \alpha_{x_2}(t), \; t > s.$$

1) If the GS model also holds on E, then for all $s < \delta$

$$\alpha_{x_s(\cdot)}(t) = \alpha_{x_2}(t - s + \varphi(s)), \; t > s,$$

where
$$\varphi(s) = A_{x_2}^{-1}(A_{x_1}(s))$$
is an increasing function. It implies that if both the GS and PH models hold on E then for all $s_1 < \delta$ and $s_2 < \delta$

$$\alpha_{x_2}(t - s_1 + \varphi(s_1)) = \alpha_{x_2}(t - s_2 + \varphi(s_2)),\ t > max(s_1, s_2).$$

Any function $\alpha_{x_2}(t)$ $\equiv$const verifies this. Assume that the function $\alpha_{x_2}(t)$ is not constant. Then $\varphi(s) - s = c =$const for all $s > 0$, because the function $\alpha_{x_2}(t)$ cannot be two or more-periodic. Note that $c \neq 0$, because

$$A_{x_2}(\varphi(s)) = A_{x_1}(s) \neq A_{x_2}(s).$$

The equalities
$$\lim_{s\to 0} A_{x_2}(\varphi(s)) = \lim_{s\to 0} A_{x_1}(s) = 0$$
and the monotonicity of $\varphi(s)$ imply that $\lim_{s\to 0}\varphi(s) = 0$. So exists $\delta_0 \in (0, \delta)$ such that $\mid \varphi(s) - s \mid < \mid c \mid$ when $0 < s < \delta_0$. It contradicts the implication that $\varphi(s) - s = c$ for any $s > 0$. It means that the assumption, that $\alpha_{x_2}(t)$ is not constant, was false. So $\alpha_{x_2}(t) = \alpha =$const which implies that $S_{x_2}(t) = e^{-\alpha_2 t}$. The PH model implies that for all $x \in E_0$
$$S_x(t) = S_0(t)^{r(x)} = e^{-r(x)t},$$
i.e. the time-to-failure distribution is exponential for all $x \in E_0$.

2) Suppose that for all $x \in E_0$ the time-to-failure distribution is exponential and the PH model holds. The proof of the Proposition 2.8 implies that the AAD model and consequently the GS model also holds on E.

The proof is complete.

The AAD and the PH models are rather restrictive. In the case of the AAD model the stress changes locally only the scale.

Under the PH model the hazard rate under the stress $x(\cdot)$ at the moment t *doesn't depend on the resource used until* t. It is not very natural if items are aging. Indeed, let $E_0 \subset E$ be a set of constant in time stresses, x_0 be an usual stress, x_1 be an accelerated with respect to x_0 stress, $x_0, x_1 \in E_0$, i.e. $S_{x_0}(t) > S_{x_1}(t)$ for all $t \geq 0$, and E_1 be a set of simple step-stresses of the form

$$x(t) = \begin{cases} x_1, & 0 \leq t < t_1, \\ x_0, & t \geq t_1. \end{cases}$$

If the PH model holds on $E_0 \cup E_1$ then for all $t_1 > 0$, $t > t_1$ we have $\alpha_{x(\cdot)}(t) = \alpha_{x_0}(t)$.

If two groups of items are tested under the usual stress x_0 and the accelerated stress x_1, respectively, until a moment t_1 and after this moment both groups are observed under the same usual stress x_0, the failure rate after the moment t_1 is the same for both groups under the PH model. It is natural only for aging items.

A generalization of AAD and PH models is obtained by supposing that the hazard rate at any moment t is proportional to some function of the stress applied at this moment, to some function of the resource, used until t, and to some baseline rate. This is formalized by the following definition:

The first generalized proportional hazards (GPH1) model (Bagdonavičius and Nikulin (1999c)) holds on E if for all $x(\cdot) \in E$

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\, q\{A_{x(\cdot)}(t)\}\, \alpha_0(t). \tag{8}$$

The particular cases of the GPH1 model are the PH model ($q(u) \equiv 1$) and the AAD model ($\alpha_0(t) \equiv \alpha_0 = const$).

A generalization of the GS and PH models is the following model:

The second generalized proportional hazards (GPH2) model holds on E if for all $x(\cdot) \in E$

$$\alpha_{x(\cdot)}(t) = u\{x(t), A_{x(\cdot)}(t)\}\, \alpha_0(t). \tag{9}$$

We have inclusions

$$AAD \subset GS \subset GPH1 \subset GPH2, \quad PH \subset GPH1 \subset GPH2.$$

If $AAD = PH$ on E_0 then the lifetime distribution is Weibull under any stress $x \in E_0$. If $AAD = PH$ on E_1 then the lifetime distribution is exponential under any stress $x \in E_0$

Particular cases of the GPH models give important accelerated life models. Numerous examples of real data show that taking two constant in time stresses, say x_1 and x_2, the ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ (which is constant under the PH model) can be increasing or decreasing in time and even a cross-effect of hazard rates can be observed, see Bagdonavičius and Nikulin (1999e).

Such data can be modelled by submodels of the GPH1 or of more general the GPH2 model.

Example 1. Consider the following parametrization of q in the GPH1 model: $q(u) = (1+u)^{\gamma+1}$, where $\gamma \in \mathbb{R}$ is an unknown scalar parameter. We have the model

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}(1 + A_{x(\cdot)}(t))^{\gamma+1}\alpha_0(t). \tag{10}$$

If $\gamma = -1$, it is the PH model.

Suppose that $\gamma < 0$ and $c_0 = r(x_2)/r(x_1)$. Then

$$\alpha_{x_2}(t)/\alpha_{x_1}(t) = c_0 \left\{ \frac{1 - \gamma\, r(x_2)\, A_0(t)}{1 - \gamma\, r(x_1)\, A_0(t)} \right\}^{-1-\frac{1}{\gamma}}.$$

The ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ has the following properties:

a) if $-1 < \gamma < 0$, then the ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ increases from the value c_0 until the value $c_\infty = \lim_{t\to\infty} \alpha_{x_2}(t)/\alpha_{x_1}(t)$, where the constant c_∞ can take any value in the interval (c_0, ∞);

b) if $\gamma = -1$ (PH model), the ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ is constant in time.

c) if $\gamma < -1$, then the ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ decreases from the value c_0 until the value $c_\infty \in (1, c_0)$.

Example 2. To obtain a cross-effect of hazard rates consider the following submodel of GPH2:

$$\alpha_{x(\cdot)}(t) = r(x(t))(1 + A_{x(\cdot)}(t))^{\gamma^T x(t)+1}\, \alpha_0(t). \tag{11}$$

Suppose that $c_0 = r(x_2)/r(x_1) > 1$ and $\gamma^T x_2 < \gamma^T x_1 < 0$. Then

$$\alpha_{x_2}(t)/\alpha_{x_1}(t) = c_0 \frac{(1 - \gamma^T x_2\, r(x_2) A_0(t))^{-1-\frac{1}{\gamma^T x_2}}}{(1 - \gamma^T x_1\, r(x_1)\, A_0(t))^{-1-\frac{1}{\gamma^T x_1}}}$$

and

$$\alpha_{x_2}(0)/\alpha_{x_1}(0) = c_0 > 1, \quad \lim_{t\to\infty} \alpha_{x_2}(t)/\alpha_{x_1}(t) = 0.$$

So we have the cross-effect of the hazard rates.

Example 3. Taking parametrization $q(u) = e^{\gamma u}$, $\gamma \in \mathbf{R}$, we obtain the model

$$\alpha_{x(\cdot)}(t) = r(x(t))\, e^{\gamma A_{x(\cdot)}(t)}\, \alpha_0(t). \tag{12}$$

This model is a generalization of the gamma frailty model (Vaupel et al. (1979)). If $\gamma = 0$, it becomes the usual PH model.

Example 4. Taking $q(u) = 1/(1+\gamma u), \quad \gamma > 0$, we obtain the model

$$\alpha_{x(\cdot)}(t) = r(x(t))\, \frac{1}{1+\gamma A_{x(\cdot)}(t)}\, \alpha_0(t).$$

Consider now an important model which doesn't lie in the class of the GPH models but includes the AAD model as the particular case:

Locally changing shape and scale (LCSS) model (Bagdonavičius and Nikulin (1998, 1999a)) holds on E if

$$S_{x(\cdot)}(t) = G\left(\int_0^t r\{x(\tau)\}\tau^{\nu(x(\tau))-1} d\tau\right). \tag{13}$$

Schabe and Viertl (1995) considered an axiomatic approach to model building.

Proposition 4 (Schabe and Viertl (1995)). *Suppose that exists a functional* $a : E \times E \times [0,\infty) \to [0,\infty)$ *such that for any* $x_1(\cdot), x_2(\cdot) \in E$ *it is differentiable and increasing in* t, $a(x_1(\cdot), x_2(\cdot), 0) = 0$ *and* $T_{x_2(\cdot)} \sim a(x_1(\cdot), x_2(\cdot), T_{x_1(\cdot)})$, *where* $\sim$ *denotes equality in distribution.*

For any differentiable on $[0,\infty)$ *survival function* S *exists a functional* $b : E \times [0,\infty) \to [0,\infty)$ *such that for all* $x(\cdot) \in E$

$$S_{x(\cdot)}(t) = S\left(\int_0^t b\{x(\cdot), u\}\, du\right).$$

All above considered models are particular cases of this general Schabe-Viertl model. See also Bagdonavičius and Nikulin (1999d).

The GS and AAD models are not appropriate when stresses are periodic with quick change of their values. Greater is the number of stress cycles, shorter is the life of items. So the effect of cycling must be included in the model.

Suppose that a periodic stress is differentiable. Then the number of cycles in the interval $[0,t]$ is

$$n(t) = \int_0^t \mid d\mathbf{1}\{x'(u) > 0\} \mid .$$

Generalization of the GPH1 model has the form

$$S_{x(\cdot)}(t) = G\left\{\int_0^t r_1\{x(u)\}\, dH(S_0(u)) + \int_0^t r_2\{x(u)\} \mid d\mathbf{1}\{x'(u) > 0\}\right\}.$$

The second integral includes the effect of cycling on resource usage.

Taking $H = S_0^{-1}$, we obtain a generalization of the AAD model which includes the effect of cycling. The GS and AAD models are also not appropriate if $x(\cdot)$ is a step stress with many switch on's and switch off's which shorten the life of items. In this case the following model can be considered:

$$S_{x(\cdot)}(t) = \int_0^t r_1\{x(u)\}\, dH(S_0(u)) + \int_0^t r_2\{x(u)\}\mathbf{1}(\Delta x(u) > 0)\frac{\mid dx(u) \mid}{\mid \Delta x(u) \mid}$$

$$+ \int_0^t r_3\{x(u)\}\mathbf{1}(\Delta x(u) < 0)\frac{\mid dx(u) \mid}{\mid \Delta x(u) \mid}.$$

The second and the third terms include the effect of switch-on's and switch-off's on resource usage. Taking $H = S_0^{-1}$, we obtain a generalization of the AAD model which includes this effect.

2. Semiparametric estimation

Semiparametric estimation for the AAD model was considered by Basu and Ebrahimi (1982), Sethuraman and Singpurwalla (1982), Shaked and Singpurwalla (1983), Schmoyer (1986, 1991), Robin and Tsiatis (1992), Lin and Ying (1995), Bagdonavičius and Nikulin (1999a,b).

Estimation for the PH model was developed by Cox (1972), Tsiatis (1981), Andersen and Gill (1982).

Particular cases of the GPH1 model with parametrization of q was considered by Andersen, Borgan, Gill and Keiding (1993), with completely known q by Dabrowska and Doksum (1988a,b), Cheng and others (1995), Murphy and others (1997), Bagdonavičius and Nikulin (1997a,b).

Estimation in LCSS model was discussed by Bagdonavičius and Nikulin (1998, 1999a,b,c).

We consider the most general GPH2 model

$$\alpha_{x(\cdot)}(t) = u\{x(t), A_{x(\cdot)}(t), \theta\}\, \alpha_0(t). \tag{14}$$

with any specified parametrization $u = u(x, s, \theta)$ via the parameters θ and unknown baseline function $\alpha_0(t)$.

Suppose that m groups of items are tested. The ith group of k_i items is tested under the accelerated stress $x_i(\cdot)$. Denote by T_{ij} and C_{ij} the failure and censoring times for the jth item of the ith group and let

$$X_{ij} = T_{ij} \wedge C_{ij},\ \delta_{ij} = I_{\{T_{ij} \leq C_{ij}\}},\ N_{ij}(t) = I_{\{T_{ij} \leq t, \delta_{ij}=1\}},\ Y_{ij}(t) = I_{\{X_{ij} \geq t\}},$$

where I_A denotes the indicator of the event A. Then

$$N_i(t) = \sum_{j=1}^{k_i} N_{ij}(t) \quad \text{and} \quad Y_i(t) = \sum_{i=1}^{k_i} Y_{ij}(t)$$

are the numbers of observed failures in the interval $[0, t]$ and items "at risk" just prior to the moment t, respectively, for the ith group of items.

We suppose that times-to-failures T_{ij} are absolutely continuous random variables and censoring is independent right censoring.

Suppose that maximal time given for experiment is $\tau \in (0, \infty)$ and all items which did not fail and were not censored before τ, are censored at this moment.

The partial likelihood function (see Andersen and others (1993))

$$L(\theta) = \prod_{i=1}^{m} \prod_{j=1}^{k_i} \left(\int_0^{\tau} \frac{u\{x_i(v), A_i(v), \theta\}}{\sum_{l=1}^{m} Y_l(v)\, u\{x_l(v), A_l(v), \theta\}} dN_{ij}(v) \right)^{\delta_{ij}} \tag{15}$$

depend on unknown accumulated hazard rates $A_i(t) = \int_0^t \alpha_{x_i}(u) du$.

Modify the partial likelihood function in the following manner: replace the functions $A_i(t)$ in (15) by the Nelson-Aalen estimators:

$$\hat{A}_i(t) = \int_0^t \frac{dN_i(s)}{Y_i(s)}.$$

The modified partial likelihood function

$$\tilde{L}(\theta) = \prod_{i=1}^{m} \prod_{j=1}^{k_i} \left(\int_0^{\tau} \frac{u\{x_i(v), \hat{A}_i(v), \theta\}}{\sum_{l=1}^{m} Y_l(v)\, u\{x_l(v), \hat{A}_l(v), \theta\}} dN_{ij}(v) \right)^{\delta_{ij}} \tag{16}$$

Put $g(x, t, \theta) = \frac{\partial}{\partial \theta} u(x, t, \theta)$. The score function has the form

$$U(\theta) = \sum_{i=1}^{m} \int_0^{\tau} \left(\frac{g\{x_i(v), \hat{A}_i(v), \theta\}}{u\{x_i(v), \hat{A}_i(v), \theta\}} - \frac{\sum_{l=1}^{m} Y_l(v)\, g\{x_l(v), \hat{A}_l(v), \theta\}}{\sum_{l=1}^{m} Y_l(v)\, u\{x_l(v), \hat{A}_l(v), \theta\}} \right) dN_i(v). \tag{17}$$

If k_i are small, the Nelson-Aalen estimators can be not very efficient and the functions A_i should be replaced by some other estimator which is a function of all data. This can be done in the following manner.

The model (14) implies that

$$dA_i(t) = u\{x_i(t), A_i(t), \theta\}\, dA_0(t).$$

Using considerations similar to considerations used to obtain the Nelson-Aalen estimator for the PH model we obtain a pseudoestimator $\tilde{A}_0(t, \theta)$ (depending on θ) for $A_0(t)$:

$$\tilde{A}_0(t, \theta) = \int_0^t \frac{dN(u)}{\sum_{i=1}^{m} u\{x_i(u), \tilde{A}_i(u-; \theta), \theta\}\, Y_i(u)}.$$

The last two equalities give the recurrent equations for a pseudoestimator $\tilde{A}_i(t; \theta)$ of $A_i(t)$:

$$\tilde{A}_i(t; \theta) = \int_0^t \frac{u\{x_i(u), \tilde{A}_i(u-, \theta), \theta\}\, dN(u)}{\sum_{l=1}^{m} u\{x_l(u), \tilde{A}_l(u-; \theta), \theta\}\, Y_l(u)}, \quad \tilde{A}_i(0; \theta) = 0. \tag{18}$$

Modify the score function (17) by considering the score function

$$\tilde{U}(\theta) = \sum_{i=1}^{m} \int_0^{\tau} \left(\frac{g\{x_i(v), \tilde{A}_i(v, \theta), \theta\}}{u\{x_i(v), \tilde{A}_i(v, \theta), \theta\}} - \frac{\sum_{l=1}^{m} Y_l(v)\, g\{x_l(v), \tilde{A}_l(v, \theta), \theta\}}{\sum_{l=1}^{m} Y_l(v)\, u\{x_l(v), \tilde{A}_l(v, \theta), \theta\}} \right). \tag{19}$$

Note that to find an initial estimator $\hat{\theta}^{(0)}$ verifying the equations (19) you need initial estimators $\tilde{A}_i^{(0)}(t, \theta)$ and *vice versa*: to find initial estimators $\tilde{A}_i^{(0)}(t, \theta)$ you need an initial estimator $\hat{\theta}^{(0)}$. This problem can be solved by taking the initial estimator $\hat{\theta}^{(0)}$

as the solution of the equations $U(\theta)=0$, where the function $U(\theta)$ is defined by the formula (17). Then the initial estimators $\tilde{A}_i^{(0)}(t,\theta^{(0)})$ are obtained recurrently from the equation (18). The estimator $\hat{\theta}^{(1)}$ is obtained by solving the equations $\tilde{U}(\theta)=0$, where $\tilde{U}(\theta)$ is given by (19) with $\tilde{A}_i^{(0)}(t,\theta^{(0)})$ taking place of $\tilde{A}_i(t,\theta)$. And so on. Given the estimator $\hat{\theta}^{(l)}$, the estimator $\tilde{A}_i^{(l)}(t,\theta^{(l)})$ is right-continuous step-function with jumps at the observed failure moments $T_{(j)}$ (whi ch are the jump points of $N(t)$).

The estimator of the survival function $S_{x_0}(t)$, $t\le\tau$ under the usual stress is $\hat{S}_{x_0}(t)=\exp\{-\hat{A}_{x_0}(t)\}$, where $\hat{A}_{x_0}(t)$ is found recurrently from the equation

$$\hat{A}_{x_0}(t)=\int_0^t \frac{u\{x_0,\hat{A}_{x_0}(u-),\hat{\theta}\}\,dN(u)}{\sum_{l=1}^m u\{x_0,\tilde{A}_l(u-;\hat{\theta}),\hat{\theta}\}\,Y_l(u)},\quad \hat{A}_{x_0}(0)=0.$$

3. Properties of the estimators

Suppose that the covariates $x_i(\tau)\equiv x_i$ are constant in time. Denote by θ_0 the true value of θ under the GPH2 model, $\|A\|=\sup_{i,j}|a_{ij}|$ - the norm of the matrix $A=(a_{ij})$, $A^{\otimes 2}$ the product AA^T,

$$w_j(v;\theta)=\frac{g\{x_j,A_j(v),\theta\}}{u\{x_j,A_j(v),\theta\}}\quad S^{(0)}(v;\theta)=\sum_{j=1}^m Y_j(v)\,u\{x_j,A_j(v),\theta\},$$

$$S^{(1)}(v;\theta)=\sum_{j=1}^m Y_j(v)\,g\{x_j,A_j(v),\theta\},\quad E(v;\theta)=\frac{S^{(1)}(v;\theta)}{S^{(0)}(v;\theta)},$$

$$S_*^{(0)}(v;\theta)=\sum_{i=1}^n Y_i(v)\,u\{x_j,A_j(v),\theta\}u_2\{x_j,A_j(v),\theta\},$$

$$S_*^{(1)}(v;\theta)=\sum_{j=1}^n g\{x_j,A_j(v),\theta\}\,u_2\{x_j,A_j(v),\theta\},$$

$$S^{(2)}(v;\theta)=\sum_{j=1}^n \frac{\partial w_j(v;\theta)}{\partial\theta}Y_j(v)\,u\{x_j,A_j(v),\theta\},$$

$$S_*^{(2)}(v;\theta)=\sum_{j=1}^n u\{x_j,A_j(v),\theta\}\,g_2\{x_j,A_j(v),\theta\},$$

where u_2 denotes the partial derivative of u with respect to the second argument.

Assumptions A:

a). Suppose that exist a neighbourhood θ of θ_0 and continuous on θ uniformly in $t\in[0,\tau]$ and bounded on $\theta\times[0,\tau]$ functions $s^{(i)}(v;\theta)$, $s_*^{(i)}(v;\theta)$, such that $s^{(0)}(v;\theta_0)>0$ on $[0,\tau]$ and

$$\sup_{\theta\in\theta,\,v\in[0,\tau]}\|\frac{1}{n}S^{(i)}(v;\theta)-s^{(i)}(v;\theta)\|\to 0\quad\text{as}\quad n\to\infty,,$$

$$\sup_{\theta\in\theta,\,v\in[0,\tau]}\|\frac{1}{n}S_*^{(i)}(v;\theta)-s_*^{(i)}(v;\theta)\|\to 0\quad\text{as}\quad n\to\infty,\ (i=0,1,2),$$

$$\sup_{\theta\in\theta,\, v\in[0,\tau]} \left\| \frac{\partial E(v;\theta)}{\partial\theta} - \frac{\partial e(v;\theta)}{\partial\theta} \right\| \xrightarrow{\mathbf{P}} 0, \quad e(v;\theta) = \frac{s^{(1)}(v;\theta)}{s^{(0)}(v;\theta)}.$$

b). $A_0(\tau) < \infty$.

Put $e_*(v;\theta) = s_*^{(0)}(v;\theta)/s^{(0)}(v;\theta)$, $h(t;\theta) = exp\{\int_0^t e_*(v;\theta)dA_0(v)\}$,

$$h_1(v;\theta) = \frac{s^{(1)}(v;\theta)s_*^{(0)}(v;\theta) - s^{(0)}(v;\theta)s_*^{(1)}(v;\theta)}{s^{(0)}(v;\theta)} - s_*^{(2)}(v;\theta),$$

$$w(v;\theta) = e(v;\theta) - \frac{1}{h(v;\theta)s^{(0)}(v;\theta)} \int_v^\tau h_1(s;\theta)h(s;\theta)dA_0(s).$$

c). $\frac{1}{n}\sum_{i=1}^n \int_0^\tau J(v)(w_i(v;\theta) - w(v;\theta)^{\otimes 2} u\{x_j, A_j(v), \theta\}Y_i(v)dA_0(u) \xrightarrow{\mathbf{P}} \mathbf{\Sigma}(\theta)$.

d).The matrix

$$\mathbf{\Sigma}_1(\theta_0) = -\int_0^t \left(s^{(2)}(u;\theta_0) - \frac{\partial e(u;\theta_0)}{\partial\theta} s^{(0)}(u;\theta_0) \right) dA_0(u)$$

is positive definite.

Theorem 1. *Under Assumptions A*

$$n^{1/2}(\hat\theta - \theta_0) \xrightarrow{\mathcal{D}} N(0, \mathbf{\Sigma}_1^{-1}(\theta_0)\mathbf{\Sigma}(\theta_0)\left(\mathbf{\Sigma}_1^{-1}(\theta_0)\right)^T).$$

Consider the asymptotic distribution of the survival function $S_{x_0}(t)$ under the usual stress x_0. Put

$$C(\theta_0) = -\int_0^t \frac{\partial}{\partial\theta} \ln S^{(0)}(v;\theta_0)dA_0(v),$$

$$H_i(v;\theta_0) = J(v)\{C(\theta_0)\mathbf{\Sigma}_1^{-1}(\theta_0)(w_i(v;\theta_0) - w(v;\theta_0)) + \frac{1}{s^{(0)}(v;\theta_0)}\},$$

Assumptions B.

$$\frac{1}{n}\sum_{i=1}^m \int_0^t H_i^2(v;\theta_0)\, Y_i(v)\, u\{x_i, A_i(v), \theta\}\, dA_0(v) \xrightarrow{\mathbf{P}} \sigma^2(t).$$

Put

$$\sigma_{x_0}^2(t) = S_{x_0}^2(t)\, u^2\{x_0, A_{x_0}(t), \theta\}\, \sigma^2(t).$$

Theorem 2. *Under Assumptions A and B*

$$\sqrt{n}(\hat S_{x_0}(t) - S_{x_0}(t)) \xrightarrow{\mathcal{D}} N(0, \sigma_{x_0}^2(t)).$$

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