MATHEMATICAL MODELLING OF FAILURE-TIME IN DYNAMIC ENVIRONMENT

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Abstract. New classes of models describing dependance of the lifetime distribution on the time-varying covariate vectors are considered. Such models are used as in survival analysis as in reliability theory analysing results of accelerated life testing.

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0. Introduction

Mathematical models describing dependence of the lifetime distribution on the explanatory variables (stresses) will be considered. . The considered models are used in survival analysis and reliability theory analysing results of accelerated life testing. Really, many manufactured devices have a long life when used under normal conditions. Therefore much time is required to get sufficiently large data for reliability estimation. To avoid this, items can be tested under higher stress conditions. In this case all processes resulting in failures of items elapse more quickly. As a result, failures which in normal conditions would occur only after a long testing, can be observed and the size of data can be enlarged. Such reliability testing is called accelerated life testing. Using information about failures under higher stress conditions, inference about item reliability under normal stress must be made. The solution of this problem requires construction of the mathematical theory of models model relating the distribution of failure time on stress. A number of such models was proposed by engineers who considered physics of failure formation process of certain products or by statisticians, see Andersen, Borgan, Gill & Keiding (1993), Bagdonavičius (1978,1990), Bagdonavičius & Nikulin (1995,1998), Bhattacharyya & Stoejoeti (1989), Cox (1972), Cox & Oakes (1984), Clayton & Cuzick (1985), Dabrowska & Doksum (1988), Elandt-Johnson & Johnson (1980), Gertsbakh & Kordonskiy (1969), Genest, Choudi & Rivest (1995), Greenwood & Nikulin (1996), Harrington & Fleming '1982), Kalbfleisch & Prentice (1980), Lawless (1982), Hougaard (1986), Johnson (1975), Kartashov (1979), Kartashov & Perrote (1968), Lee (1992), Lin & Ying (1994, 1995,1996), Meeker & Escobar (1993, 1998), Mann, Schafer & Singpurwalla (1974), Miner (1945), Meeker & Escobar (1998), Nelson (1990), Robins & Tsiatis (1992), Schabe (1998), Sedyakin (1966), Shaked & Singpurwalla (1983), Singpurwalla (1995), Singpurwalla & Wilson (1999), Schweizer & Sclar (1983), Viertl (1988), Viertl and Spencer (1991), Voinov & Nikulin (1993, 1996) etc.

There is some eclecticism in various definitions of these models that prevents from seeing relations between them. The construction of accelerated life models will be considered now (Bagdonavicius and Nikulin (1994-2000)). This general approach gives the possibility to formulate a number of new models and to show the place of known ones in the proposed classes of models. These models can be considered as parametric, semiparametric or nonparametric. Parametric models, used in accelerated life testing, where thoroughly investigated, see for example, Viertl (1988), Nelson (1990), Nelson & Meeker (1991), Meeker & Escobar (1998), Basu and Ebrahimi (1982), Singpurwalla (1971), etc. These models are the well known: proportional and additive hazards, logistic regression and other models, which are used most often in survival analysis and in reliability. See also some reviews as Rukhin and Hsieh (1987), Meeker and Escobar (1993), Singpurwalla (1987), etc. Statistical analysis of considered models one can find, for example, in Lin and Ying (1994,1995,1996), Meeker and Escobar (1998), Dabrowska and Doksum (1988), Robins and Tsiatis (1992), Bagdonavicius and Nikulin (1995-2000), Gerville-Reache and Nikoulina (1998), Tsiatis (1990), Schmoyer (1991), Sethuraman and Singpurwalla (1982), Ying (1993), etc.

1. Resource

All models of accelerated life will be formulated in unified way, using the notion of the resource. We consider only such models for which statistical estimation or hypothesis testing procedures are given up-to-date.

Suppose at first that stresses are deterministic time functions:

$$x(\cdot) = (x_1(\cdot), ..., x_m(\cdot))^T \ : [0, \infty) \to B \in \mathbf{R}^{\mathbf{m}}.$$

If $x(\cdot)$ is constant in time, we'll write x instead of $x(\cdot)$ in all formulae.

Let Ω be a population of items and suppose that the time-to-failure of items under the stress $x(\cdot)$ is defined by a non-negative absolutely continuous random variable $T_{x(\cdot)} = T_{x(\cdot)}(\omega), \omega \in \Omega$, with the survival function

$$S_{x(\cdot)}(t) = \mathbf{P}\{T_{x(\cdot)} > t\},$$

strictly decreasing on the support of distribution $[0, sp_{x(\cdot)})$.

The moment of failure of a concrete item $\omega_0 \in \Omega$ is given by a nonnegative number $T_{x(\cdot)}(\omega_0)$.

Let $F_{x(\cdot)}(t) = 1 - S_{x(\cdot)}(t)$ be the cummulative distribution function of $T_{x(\cdot)}$. We use the following interpretation of it.

Definition 1. *The proportion $F_{x(\cdot)}(t)$ of items from Ω which fail until the moment t under the stress $x(\cdot)$ is called the uniform resource of population used until the moment t.*

Definition 2. *The random variable*

$$R^U = F_{x(\cdot)}(T_{x(\cdot)}) = 1 - S_{x(\cdot)}(T_{x(\cdot)})$$

is called the uniform resource (of population).

The distribution of the random variable R^U doesn't depend on $x(\cdot)$ and is uniform on $[0, 1)$. This explains the name *uniform* resource. The uniform resource of any *concrete* item $\omega_0 \in \Omega$ is $R^U(\omega_0)$. It shows the proportion of the population Ω which fails until the item's ω_0 failure $T_{x(\cdot)}(\omega_0)$.

The same population of items Ω, observed under different stresses $x_1(\cdot)$ and $x_2(\cdot)$ use different resources until the same moment t when $F_{x_1(\cdot)}(t) \neq F_{x_2(\cdot)}(t)$. In the sense of equality of used resources the moments t_1 and t_2 are equivalent if $F_{x_1(\cdot)}(t_1) = F_{x_2(\cdot)}(t_2)$.

If we denote by

$$G(t) = 1 - t, \, t \in [0, 1),$$

the uniform survival function and

$$H(p) = 1 - p, \, p \in (0, 1]$$

the inverse function of G, then the resource can be written in the form $R^G = H(S_{x(\cdot)}(T_{x(\cdot)}))$.

The considered definition of the resource is not unique. Take any strictly decreasing and continuous function $H : (0, 1] \to \mathbf{R}$ such that the inverse $G = H^{-1}$ of H is a survival function. Then the distribution of the random variable R^G also does not depend on $x(\cdot)$ and the survival function of R^G is G.

Definition 3. *The random variable R^G is called the G- resource and the number $H(S_{x(\cdot)}(t))$ is called the G-resource used until the moment t.*

If identical populations of items operate under different stresses $x_1(\cdot)$ and $x_2(\cdot)$, then *independently of the resource choice* the moments t_1 and t_2 are equivalent (in the sense of equality of the used resources) if $F_{x_1(\cdot)}(t_1) = F_{x_2(\cdot)}(t_2)$, i.e. if the same proportions of items fail until these moments.

Accelerated life models will be formulated in terms of resource usage way, i.e. in terms of the way the survived population proportion changes in time.

To avoid ambiguity, all definitions will be formulated in terms of *exponential resource usage*, when

$$G(t) = e^{-t}, t \geq 0, \quad H(p) = -\ln p, \quad H(S_{x(\cdot)}(t)) = -\ln S_{x(\cdot)}(t).$$

The choice of exponential resource is due to the fact that the *exponential resource usage rate* is the hazard rate

$$\alpha_{x(\cdot)}(t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbf{P}\{T_{x(\cdot)} \in (t, t+h] \mid T_{x(\cdot)} > t\} = -\frac{S'_{x(\cdot)}(t)}{S_{x(\cdot)}(t)}$$

and the used resource is the accumulated hazard rate

$$A_{x(\cdot)}(t) = \int_0^t \alpha_{x(\cdot)}(u) du = -ln\{S_{x(\cdot)}(t)\}$$

under the stress $x(\cdot)$. These notions have good interpretation. Then the (exponential) resource

$$R = A_{x(\cdot)}(T_{x(\cdot)})$$

has the standard exponential distribution with the survival function $G(t) = e^{-t}$, $t \geq 0$. The resource R takes values in the interval $[0, \infty)$ and doesn't depend on $x(\cdot)$:

for any t the number $A_{x(\cdot)}(t) \in [0, \infty)$ is the *exponential resource used until the moment* t under the stress $x(\cdot)$ (see Fig. 1), the rate of exponential resource usage is the hazard rate $\alpha_{x(\cdot)}(t)$ and for any moment t shows the risk of failure just after this moment for items which survived until t.

Sometimes the meaning of certain models in terms of other resources will be discussed. It will be done if formulations of models using non-exponential resource will be simplier or more comprehensive.

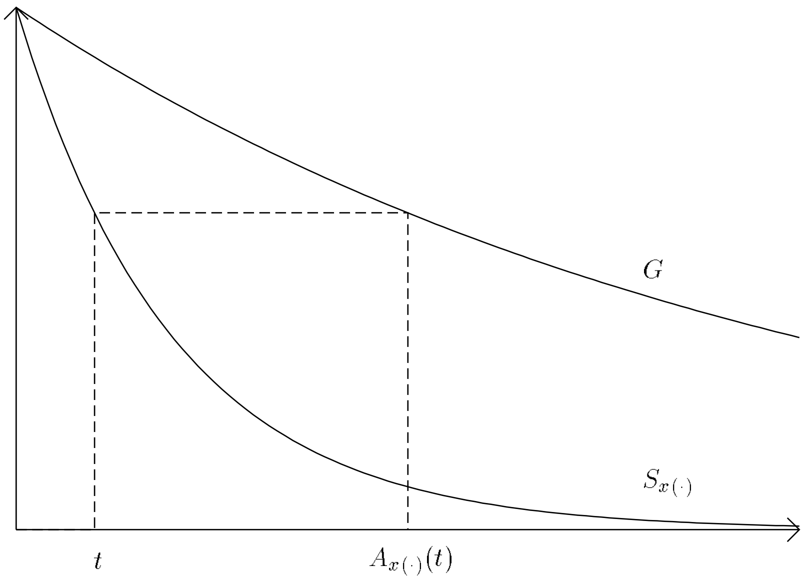

Fig. 1.

Remark 1. We'll give definitions of models for deterministic stresses. If the stress is a stochastic process $X(t)$, $t \geq 0$, and $T_{X(\cdot)}$ is time-to-failure under $X(\cdot)$, then denote by

$$S_{x(\cdot)}(t) = \mathbf{P}\{T_{X(\cdot)} > t | X(s) = x(s), \, 0 \leq s \leq t\},$$

$$\alpha_{x(\cdot)}(t) = -S'_{x(\cdot)}(t)/S_{x(\cdot)}(t), \quad A_{x(\cdot)}(t) = -\ln\{S_{x(\cdot)}(t)\} \tag{0}$$

the conditional survival, hazard rate and accumulated hazard rate functions. In this case definitions of models should be understood in terms of these conditional functions.

2. Generalized Sedyakin's model

Definition of the model

The first idea which comes by modelling the influence of a stress on lifetime distribution is to suppose that the rate of resource usage $\alpha_{x(\cdot)}(t)$ at any moment t depends on the value of the stress $x(t)$ at this moment and the resource $A_{x(\cdot)}(t)$ used until t. It is formalized by the following definition.

Definition 4. *The generalized Sedyakin's (GS) model (Bagdonavičius (1978)) holds on a set of stresses E if there exist a positive on $E \times \mathbf{R}^+$ function g such that for all $x(\cdot) \in E$*

$$\alpha_{x(\cdot)}(t) = g\left(x(t), A_{x(\cdot)}(t)\right). \tag{1}$$

If the stress is a random process, we denote by *E the space of trajectories* of this process and consider the conditional functions (0) in all definitions which follow.

The definition 4 implies that the accumulated hazard rate (or the used resource) verifies the integral equation

$$A_{x(\cdot)}(t) = \int_0^t g\left(x(u), A_{x(\cdot)}(u)\right)\, du. \tag{2}$$

In the following subsections it will be shown that if the GS model holds then the survival functions under the step-stresses can be written in terms of survival functions under constant stresses.

Simple step-stress

Let E_0 be a set of constant in time stresses and E_1 be a set of simple step-stresses of the form

$$x(\tau) = \begin{cases} x_1, & 0 \leq \tau < t_1, \\ x_2, & \tau \geq t_1, \end{cases} \tag{3}$$

where $x_1,\ x_2 \in E_0$.

Proposition 1. *If the GS model holds on E_1 then the survival function and the hazard rate under the stress $x(\cdot) \in E_1$ verify the equalities*

$$S_{x(\cdot)}(t) = \begin{cases} S_{x_1}(t), & 0 \leq \tau < t_1, \\ S_{x_2}(t - t_1 + t_1^*), & \tau \geq t_1, \end{cases} \tag{4}$$

and

$$\alpha_{x(\cdot)}(t) = \begin{cases} \alpha_{x_1}(t), & 0 \leq \tau < t_1, \\ \alpha_{x_2}(t - t_1 + t_1^*), & \tau \geq t_1, \end{cases} \tag{5}$$

respectively; the moment t_1^* is determined by the equality $S_{x_1}(t_1) = S_{x_2}(t_1^*)$.

Proof of the Proposition 1. Put $a = A_{x_1}(t_1) = A_{x(\cdot)}(t_1) = A_{x_2}(t_1^*)$. The equalities (2)-(3) imply that for all $t \geq t_1$

$$A_{x(\cdot)}(t) = a + \int_{t_1}^t g\left(x_2, A_{x(\cdot)}(u)\right)\, du$$

and

$$A_{x_2}(t - t_1 + t_1^*) = a + \int_{t_1^*}^{t - t_1 + t_1^*} g\left(x_2, A_{x_2}(u)\right)\, du = a + \int_{t_1}^t g\left(x_2, A_{x_2}(u - t_1 + t_1^*)\right)\, du.$$

It implies that for all $t \geq t_1$ the functions $A_{x(\cdot)}(t)$ and $A_{x_2}(t - t_1 + t_1^*)$ satisfy the integral equation

$$h(t) = a + \int_{t_1}^t g\left(x_2, h(u)\right)\, du$$

with the initial condition $h(t_1) = a$. The solution of this equation is unique, therefore we have

$$A_{x(\cdot)}(t) = A_{x_2}(t - t_1 + t_1^*), \quad \text{for all} \quad t \geq t_1. \tag{6}$$

It implies the equalities (4) and (5).

The proof is complete.

Corollary 1 *Under conditions of the Proposition 1.*

$$\mathbf{P}\left\{T_{x(\cdot)} \geq t_1 + s \mid T_{x(\cdot)} \geq t_1\right\} = \mathbf{P}\left\{T_{x_2} \geq t_1^* + s \mid T_{x_2} \geq t_1^*\right\}. \tag{7}$$

The model (7) was proposed by Sedyakin (1966).

The equality $S_{x_1}(t_1) = S_{x_2}(t_1^*)$ and the definition of the resource imply that for the two groups of identical items, observed under x_1 and x_2, respectively, the moments t_1 and t_1^*, respectively, are equivalent in the sense of resource usage. The equality (4) implies that for any $s > 0$

$$S_{x(\cdot)}(t_1 + s) = S_{x_2}(t_1^* + s).$$

Thus the GS model implies that if two identical populations of items under different stresses use the same resources until the moments t_1 and t_1^*, respectively, and after these moments both populations operate under the same stress, then the rates of the resource usage of these populations in the intervals $[t_1, t_1 + s]$ and $[t_1^*, t_1^* + s]$, respectively, are the same.

General step-stress

Consider a set E_m of more general stepwise stresses of the form

$$x(\tau) = \begin{cases} x_1, & 0 \leq \tau < t_1, \\ x_2, & t_1 \leq \tau < t_2, \\ \cdots & \cdots \\ x_m, & t_{m-1} \leq \tau < t_m. \end{cases}, \tag{8}$$

where $x_1, \cdots, x_m \in E_0$. Put $t_0 = 0$.

Proposition 2. *If the GS model holds on E_m then the survival function $S_{x(\cdot)}(t)$ verifies the equalities:*

$$S_{x(\cdot)}(t) = S_{x_i}(t - t_{i-1} + t_{i-1}^*), \quad \text{if} \quad t \in [t_{i-1}, t_i),\ (i = 1, 2, \ldots, m), \tag{9}$$

where where t_i^* can be found by solving the equations

$$S_{x_1}(t_1) = S_{x_2}(t_1^*), \ldots, S_{x_i}(t_i - t_{i-1} + t_{i-1}^*) = S_{x_{i+1}}(t_i^*),\ (i = 1, \ldots, m-1). \tag{10}$$

Proof. The Proposition 1 implies that the Proposition 2 is true for $m = 2$, i.e. we have

$$A_{x(\cdot)}(t) = A_{x_2}(t - t_1 + t_1^*), \quad \text{for all} \quad t \in [t_1, t_2),$$

where $A_{x_1}(t_1) = A_{x_2}(t_1^*)$.

Suppose that Proposition 2 is true for $m = j - 1$. Then

$$A_{x(\cdot)}(t) = A_{x_i}(t - t_{i-1} + t_{i-1}^*), \text{ if } t \in [t_{i-1}, t_i),\ (i = 1, \cdots, j-1), \tag{11}$$

where

$$A_{x_1}(t_1) = A_{x_2}(t_1^*), \ldots, A_{x_i}(t_i - t_{i-1} + t_{i-1}^*) = A_{x_{i+1}}(t_i^*),\ (t_0 = 0,\ i = 1, \ldots, j-2).$$

We'll prove that the Proposition 2 is true for $m = j$. Continuity of the functions $A_{x(\cdot)}(t)$ and $A_{x_2}(t)$, and the equalities (11) imply that

$$A_{x(\cdot)}(t_{j-1}) = A_{x_{j-1}}(t_{j-1} - t_{j-2} + t_{j-2}^*).$$

So the equation (2) implies that for all $t \in [t_{j-1}, t_j)$

$$A_{x(\cdot)}(t) = A_{x(\cdot)}(t_{j-1}) + \int_{t_{j-1}}^{t} g\left(x_j, A_{x(\cdot)}(u)\right)\, du =$$

$$A_{x_{j-1}}(t_{j-1} - t_{j-2} + t^*_{j-2}) + \int_{t_{j-1}}^{t} g\left(x_j, A_{x(\cdot)}(u)\right)\, du. \tag{12}$$

The definition of t^*_{j-1}, given in (10), and the equation (2) imply that for all $t \in [t_1, t_2)$:

$$A_{x_j}(t - t_{j-1} + t^*_{j-1}) = A_{x_j}(t^*_{j-1}) + \int_{t^*_{j-1}}^{t - t_{j-1} + t^*_{j-1}} g\left(x_j, A_{x_j}(u)\right)\, du =$$

$$A_{x_{j-1}}(t_{j-1} - t_{j-2} + t^*_{j-2}) + \int_{t_{j-1}}^{t} g\left(x_j, A_{x_2}(u - t_{j-1} + t^*_{j-1})\right)\, du. \tag{13}$$

The equalities (2.12) and (2.13) imply that the functions $A_{x(\cdot)}(t)$ and $A_{x_j}(t - t_{j-1} + t^*_{j-1})$ satisfy the integral equation

$$h(t) = a + \int_{t_{j-1}}^{t} g\left(x_j, h(u)\right)\, du \quad \text{for all} \quad t \in [t_{j-1}, t_j),$$

with the initial condition $h(t_{j-1}) = b = A_{x_{j-1}}(t_{j-1} - t_{j-2} + t^*_{j-2})$. The solution of this equation is unique, therefore for all $t \in [t_{j-1}, t_j)$ we have

$$A_{x(\cdot)}(t) = A_{x_j}(t - t_{j-1} + t^*_{j-1}).$$

The proof is complete.

Remark 2. In the statistical literature (see Nelson (1990)) the model (9) is called the *basic cumulative exposure model*

In terms of graphs of the accumulated hazard rate functions $A_{x(\cdot)}(t)$ (thick curve) and $A_{x_i}(t)$ (m=3, i=1,2,3) the result of the proposition 2.2 is illustrated by the Fig. 2.

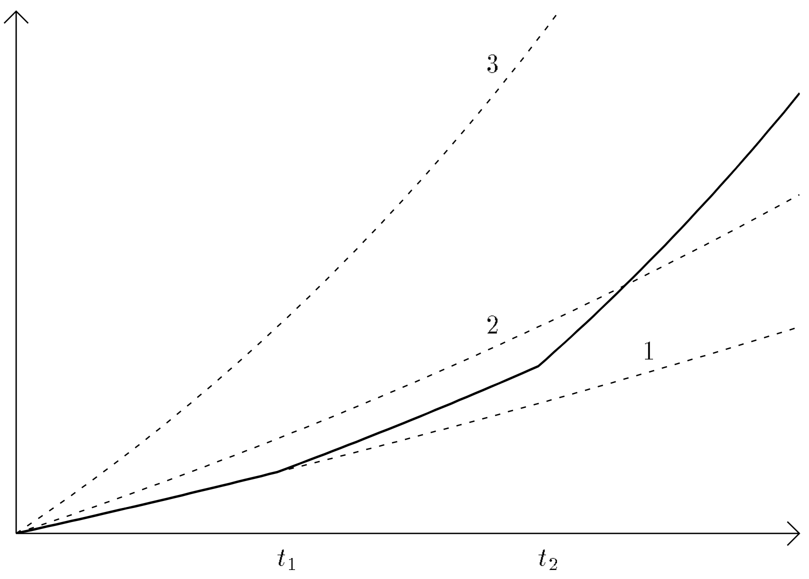

Fig. 2.

Remark 3. The GS model assumes that the failure rate $\alpha_{x(\cdot)}(t)$ at any moment t depends only on the resource accumulated until this moment (or, equivalently, on proportion of items failed until t) and on the value of the stress applied at this moment t. In situations of periodic and quick change of the stress level or when there are many life shortening switch-on's and switch off's of the stress, this model is not appropriate. We'll consider generalizations later.

There are no methods of estimation for this model.

What is the region of applications of this model? Suppose that the model is parametric and it is impossible to obtain the complete sample under the "normal" conditions of functionning of items. When the right censored data is used, the goodness-of-fit tests can test that the left tail of a survival distribution corresponds well the chosen model. But often the estimates of p-quantiles with p near the unity are needed and in the case of bad choice of the model big mistakes can be made. The utilisation of the model of Sedyakin can help to solve this problem. If the stepwise stresses are used, it is possible to obtain failures of items at the end of life under the "normal" conditions and therefore to test if the right tail is from the class of specified distribution.

A test for the Sedyakin's model can be found in Bagdonavičius & Nikoulina (1997).

3. Additive accumulation of damages model

Definition of the model

Consider the following important particular case of the GS model. We now suppose that the rate of resource usage $\alpha_{x(\cdot)}(t)$ at any moment t is proportional to a function of the stress applied at this moment and to a function of the resource used until t. It is formalized by the following definition.

Definition 5. *The additive accumulation of damages (AAD) model (Bagdonavičius (1978)) holds on E if there exists a positive function r on E and a positive on $[0,\infty)$ function q such that for all $x(\cdot)\in E$*

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\, q\{A_{x(\cdot)}(t))\}. \tag{14}$$

Proposition 3. *Suppose that the integral*

$$\int_0^x \frac{dv}{q(v)}$$

converges for all $x \geq 0$.

The AAD model holds on the set of stresses E iff there exists a survival function G such that

$$S_{x(\cdot)}(t) = G\left(\int_0^t r\{x(\tau)\}\, d\tau\right). \tag{15}$$

The inverse $H = G^{-1}$ of the function G is defined by the equality

$$H(u) = \int_0^{-ln\, u} \frac{dv}{q(v)}.$$

Proof. The equation (14) is equivalent to the integral equation

$$\int_0^{A_{x(\cdot)}(t)} \frac{dv}{q(v)} = \int_0^t r\,(x(u))\; du.$$

The result follows immediately.

The name of the AAD model is implied by the following considerations.

Fix any constant in time stress x_0, for example, let x_0 be the usual stress. Then under the AAD model $S_{x_0}(t) = G(r(x_0)t)$ and putting $\rho(x) = r(x)/r(x_0)$ we obtain

$$S_{x(\cdot)}(t) = S_{x_0}\left(\int_0^t \rho\{x(\tau)\}\, d\tau\right).$$

The S_{x_0}-resource is

$$R = S_{x_0}^{-1}(S_{x(\cdot)}(T_{x_0}) = \int_0^{T_{x_0}} \rho\{x(\tau)\}\, d\tau.$$

It's survival function is S_{x_0} and thus it is stochastically equivalent to the time-to-failure under the usual stress.

Decrease of the resource in the interval $[\tau, \tau + d\tau]$ we'll call the damage. For the AAD model damages have the form $\rho\{x(\tau)\}\, d\tau$ and are linear functions of $d\tau$. Thus the resource is used by the linear accumulation of damages.

Remark 4. *The AAD model written in the form (15) is also called the accelerated failure time (AFT) model (Cox and Oakes (1984)).*

Constant stresses

If $x(\tau) \equiv x = const$ then the AAD model gives

$$S_x(t) = G\{r(x)\, t\}. \tag{16}$$

Thus different constant in time stresses change only the scale of distribution. Applicability of this model in accelerated life testing was first noted by Pieruschka (1961). It is the most simple and the most used model in accelerated life testing.

Simple step-stresses.

Consider the properties of the survival functions under the step-stresses. As before, we denote by E_1 and E_m sets of stresses of the form (3) and (8), respectively.

Proposition 4. *If the AAD model holds on E_1 then the survival function under the stress $x(\cdot)$ verifies the equality*

$$S_{x(\cdot)}(t) = \begin{cases} S_{x_1}(t), & 0 \le \tau < t_1, \\ S_{x_2}(t - t_1 + t_1^*), & \tau \ge t_1, \end{cases} \tag{17}$$

where

$$t_1^* = \frac{r(x_1)}{r(x_2)} t_1.$$

Proof. The equality (16) implies that

$$S_{x_1}(t) = S_{x_2}\left(\frac{r(x_1)}{r(x_2)} t\right),$$

therefore the moment t_1^*, done in the Proposition 1, is $t_1^* = \frac{r(x_1)}{r(x_2)} t_1$.

General step-stress

Proposition 5. *If the AAD model holds on E_m then the survival function $S_{x(\cdot)}(t)$ verifies the equalities:*

$$S_{x(\cdot)}(t) = G\left\{\sum_{j=1}^{i-1} r(x_j)(t_j - t_{j-1}) + r(x_i)(t - t_{i-1})\right\} =$$

$$S_{x_i}\left\{t - t_{i-1} + \frac{1}{r(x_i)} \sum_{j=1}^{i-1} r(x_j)(t_j - t_{j-1})\right\}, \quad \text{if} \quad t \in [t_{i-1}, t_i),\ (i = 1, 2, ..., m). \tag{18}$$

Proof. The first equality is implied by the formula (15) and the form (8) of the stress. The second is implied by the formula (16).

The proof is complete.

Remark 5. If the AAD model holds on E_m, the moment t_i^* defined by (10) has the form

$$t_i^* = \frac{1}{r(x_{i+1})} \sum_{j=1}^{i} r(x_j)(t_j - t_{j-1}). \tag{19}$$

Relations between the means and the quantiles

Suppose that $x(\cdot)$ is a time-varying stress and denote by $t_{x(\cdot)}(p)$ the p-quantile of the random variable $T_{x(\cdot)}$, $0<p<1$. In the next proposition we shall write $x(\tau)$ to note as the value of the stress $x(\cdot)$ at the point τ as the the constant in time stress $x(\tau)\mathbf{1}(\cdot)$.

Proposition 6. *Suppose that the AAD model holds on E and $x(\cdot)$, $x(\tau)\in E$ for all $\tau\geq 0$. Then*

$$\int_0^{t_{x(\cdot)}(p)} \frac{d\tau}{t_{x(\tau)}(p)} = 1. \tag{20}$$

If the means $\mathbf{E}T_{x(\cdot)}$, $\mathbf{E}T_{x(\tau)}$ exist then

$$\mathbf{E}\left(\int_0^{T_{x(\cdot)}} \frac{d\tau}{\mathbf{E}T_{x(\tau)}}\right) = 1. \tag{21}$$

The model (21) is the *model of Miner* (1949).

Proof. If the AAD model holds, the equality (2.15) implies that the G-resource used until t is

$$f_{x(\cdot)}(t) = H\left(S_{x(\cdot)}(t)\right) = \int_0^t r\{x(s)\}ds. \tag{22}$$

The resource R has the form :

$$R = \int_0^{T_{x(\cdot)}} r\left\{x(s)\right\}\, ds. \tag{23}$$

The cumulative distribution function G of R doesn't depend on $x(\cdot)$. Taking the constant in time stress $x(\tau)$, equal to the value of the stress $x(\cdot)$ at the moment τ, we obtain

$$R = \int_0^{T_{x(\tau)}} r\left\{x(\tau)\right\}\, ds = r\left\{x(\tau)\right\} T_{x(\tau)}. \tag{24}$$

Taking the means of both sides we'll get

$$\mathbf{E}R = r\left\{x(\tau)\right\} \mathbf{E}T_{x(\tau)}. \tag{25}$$

The equalities (22) and (23) imply

$$\mathbf{E}R = \mathbf{E}\left(\int_0^{T_{x(\cdot)}} r(x(\tau))\, d\tau\right) = \mathbf{E}\left(\int_0^{T_{x(\cdot)}} \frac{\mathbf{E}R}{\mathbf{E}T_{x(\tau)}}\, d\tau\right)$$

$$= \mathbf{E}R\cdot\mathbf{E}\left(\int_0^{T_{x(\cdot)}} \frac{d\tau}{\mathbf{E}T_{x(\tau)}}\right)$$

and the equality (21) is obtained.

Denote by $t(p)$ the p-quantile of the resource R.

If τ is fixed, the equality (16) implies

$$t(p) = r\{x(\tau)\}\, t_{x(\tau)}(p). \tag{26}$$

Using the definition of the resource we have

$$p = \mathbf{P}\{T_{x(\cdot)} \le t_{x(\cdot)}(p)\} = \mathbf{P}\left\{ \int\limits_0^{T_{x(\tau)}} r\{x(\tau)\}\, d\tau \le \int\limits_0^{t_{x(\tau)}(p)} r\{x(\tau)\}\, d\tau \right\} =$$

$$\mathbf{P}\left\{ R \le \int\limits_0^{t_{x(\tau)}(p)} r\{x(\tau)\}\, d\tau \right\}. \tag{27}$$

The equalities (27) and (26) imply that

$$t(p) = \int_0^{t_{x(\cdot)}(p)} r\{x(\tau)\}\, d\tau = t(p) \int_0^{t_{x(\cdot)}(p)} \frac{d\tau}{t_{x(\tau)}(p)}$$

and hence the equality (20) is obtained.

The proof is completed.

Corollary 2. *For the stress (8) the formula (21) implies the equality*

$$\sum_{k=1}^{m} \frac{\mathbf{E}T_k}{\mathbf{E}T_{x_k}} = 1, \tag{28}$$

where

$$T_k = \begin{cases} 0, & T_{x(\cdot)} < t_{k-1}, \\ T_{x(\cdot)} - t_{k-1}, & t_{k-1} \le T_{x(\cdot)} < t_k, \\ t_k - t_{k-1}, & T_{x(\cdot)} \ge t_k, \end{cases}$$

is the life of an item, tested under the stress $x(\cdot)$, in the interval $[t_{k-1}, t_k)$.

For the stress (2.8) the formula (20) implies that for $t_{x(\cdot)}(p) \in [t_{k-1}, t_k)$ the following equality is true

$$\sum_{i=1}^{k-1} \frac{t_i - t_{i-1}}{t_{x_i}(p)} + \frac{t_{x(\cdot)}(p) - t_{k-1}}{t_{x_k}(p)} = 1. \tag{29}$$

The model (29) is the *model of Peshes-Stepanova* (see, Kartashov, 1979). So *all of the models (16), (18), (20),(21), (28), (29) are implied by the AAD model and illustrate properties of this model.*

In the case $m = 2$, the formula (28) can be written in the form

$$\frac{\mathbf{E}T_1}{\mathbf{E}T_{x_1}} + \frac{\mathbf{E}T_2}{\mathbf{E}T_{x_2}} = 1, \tag{30}$$

and the formula (29) can be written in the form

$$\frac{t_1}{t_{x_1}(p)} + \frac{t_{x(\cdot)}(p) - t_1}{t_{x_2}(p)} = 1. \tag{31}$$

So

$$\mathbf{E}T_{x_1} = \frac{\mathbf{E}T_1}{1 - \frac{\mathbf{E}T_2}{\mathbf{E}T_{x_2}}} \tag{32}$$

and

$$t_{x_1}(p) = \frac{t_1}{1 - \frac{t_{x(\cdot)}(p) - t_1}{t_{x_2}(p)}}, \text{ if } t_{x(\cdot)}(p) \ge t_1. \tag{33}$$

Thus, if the AAD model holds on E_1, then $\mathbf{E}T_1$, $\mathbf{E}T_2$ and $\mathbf{E}T_{x_2}$ determine $\mathbf{E}T_{x_1}$ and $t_{x(\cdot)}(p)$ and $t_{x_2}(p)$ determine $t_{x_1}(p)$.

4. Proportional hazards model

Definition of the model

In survival analysis the most used model describing the influence of covariates on the lifetime distribution is the *proportional hazards* or Cox model, introduced by Cox (1972). In terms of stresses it is formulated as follows.

Definition 6. *The proportional hazards (PH) model holds on a set of stresses E if for all $x(\cdot) \in E$*

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\ \alpha_0(t), \tag{34}$$

where $\alpha_0(t)$ is the baseline hazard rate.

The model means that the rate of resource usage at any moment t is proportional to some function of a stress applied at this moment and to a baseline rate which does not depend on the stress.

Under the PH model the resource used until the moment t has the form

$$A_{x(\cdot)}(t) = \int_0^t r\{x(u)\} dA_0(u),$$

where

$$A_0(t) = \int_0^t \alpha_0(u) du.$$

In terms of survival functions the PH model is written :

$$S_{x(\cdot)}(t) = \exp\left\{ -\int_0^t r\{x(u)\} dA_0(u) \right\}. \tag{2.35}$$

Relations between the PH and the AAD models

When the PH model is also the AAD model? The answer is given in the following two propositions.

Proposition 7. *Let E_0 be a set of constant stresses such that the set $r(E_0)$ has an interior point. Suppose that the PH model holds on E_0.*

The AAD model also holds on E_0 iff the time-to-failure distribution is Weibull for all $x \in E_0$

Proof. 1) If the time-to-failure distribution is Weibull and the PH model holds on E_0 then for all $x \in E_0$

$$S_x(t) = e^{-(\frac{t}{\theta(x)})^{\alpha(x)}} = S_0(t)^{r(x)}, \tag{36}$$

Taking two times the logarithms of both sides, we obtain that for all $t > 0$

$$\alpha(x)(\ln t - \ln \theta(x)) = \ln r(x) + \ln(-\ln S_0(t)). \tag{37}$$

The function $\ln(-\ln S_0(t))$ doesn't depend on x, so $\alpha(x) = \alpha = const$ for all $x \in E_0$, which implies

$$S_x(t) = e^{-(\frac{t}{\theta(x)})^{\alpha}}, \tag{38}$$

i.e. the AAD model holds on E_0.

2) Suppose that both the PH and AAD models hold on E_0. It means that there exist functions S_0, S_1, r and ρ such that for all $x \in E_0$

$$S_1(\rho(x)t) = S_0(t)^{r(x)}. \tag{39}$$

Taking two times the logarithms of both sides, we obtain that for all $t > 0$

$$\ln\{-\ln S_1(\rho(x)t)\} = \ln r(x) + \ln(-\ln S_0(t)). \tag{40}$$

Put

$$g_1(v) = \ln(-\ln S_1(e^v)), \quad g_0(v) = \ln(-\ln S_0(e^v)), \quad \alpha(x) = \ln \rho(x), \quad \beta(x) = \ln r(x).$$

The equality (40) can be written in the following way: for all $u \in \mathbf{R}$, $x \in E_0$

$$g_1(u + \alpha(x)) = \beta(x) + g_0(u).$$

The set $r(E_0)$ has an interior point, i.e. contains an interval, so the set $\rho(E_0)$ also has an interior point. Take $x_1, x_2, x_3 \in E_0$ such that

$$\rho(x_2)/\rho(x_1) \neq \rho(x_3)/\rho(x_2).$$

For all $i, j = 1, 2, 3$

$$g_1(u + \alpha(x_i)) - g_1(u + \alpha(x_j)) = \beta(x_i) - \beta(x_j). \tag{41}$$

Put

$$k_1 = \alpha(x_2) - \alpha(x_1), \quad k_2 = \alpha(x_3) - \alpha(x_2), \quad l_1 = \beta(x_2) - \beta(x_1), \quad l_2 = \beta(x_3) - \beta(x_2).$$

For all $v \in \mathbf{R}$

$$g_1(v + k_i) = g_1(v) + l_i \ (i = 1, 2, \ k_1 \neq k_2). \tag{42}$$

It implies that

$$g_1(v) = av + b, \ S_1(t) = \exp\{-e^b t^a\}, \tag{43}$$

and consequently

$$S_x(t) = \exp\{-e^b(\rho(x)t)^a\}. \tag{44}$$

So the lifetime distribution is Weibull for all $x \in E_0$.

The proof is complete.

Suppose that E_0 is the set of constant stresses defined in Proposition 7, $x_1, x_2 \in E_0$ are two fixed constant stresses and a step-stress $x_s(\cdot)$ has the form

$$x_s(\tau) = \begin{cases} x_1, & 0 \leq \tau < s, \\ x_2, & \tau \geq s, \end{cases} \tag{45}$$

where s is a fixed positive number.

Proposition 8. *Suppose that the PH model holds on the set E including E_0 and $x_s(\cdot)$. The AAD model also holds on E iff the time to-failure is exponential for all $x \in E_0$.*

Proof. The Proposition 7 implies that for all $x \in E_0$

$$S_x(t) = e^{-(\frac{t}{\theta(x)})^\alpha}. \tag{46}$$

Put $\theta_i = \theta(x_i)$, $i = 1, 2$. Then

$$S_{x_i}(t) = e^{-(\frac{t}{\theta_i})^\alpha}, \quad \alpha_{x_i}(t) = \frac{\alpha}{\theta_i^\alpha} t^{\alpha - 1}. \tag{47}$$

The PH model implies that

$$\alpha_{x_s(\cdot)}(t) = \begin{cases} \alpha_{x_1}(t), & 0 < \tau < s, \\ \alpha_{x_2}(t), & \tau \geq s, \end{cases}$$

and for all $t > s$

$$S_{x_s(\cdot)}(t) = \exp\{-\int_0^t \alpha_{x_s(\cdot)}(u)du\} = \exp\{-\int_0^s \alpha_{x_1}(u)du - \int_s^t \alpha_{x_2}(u)du\} =$$

$$\exp\left\{-\left(\frac{s}{\theta_1}\right)^\alpha - \left(\frac{t}{\theta_2}\right)^\alpha + \left(\frac{s}{\theta_2}\right)^\alpha\right\}. \tag{48}$$

1) Suppose that both the PH and the AAD models hold on E. Then (46) and (16) imply that

$$S_0(t) = e^{-t^\alpha}, r(x_i) = 1/\theta_i$$

and (18) implies that for all $t > s$

$$S_{x_s(\cdot)}(t) = \exp\left\{-\left(\frac{s}{\theta_1} + \frac{t-s}{\theta_2}\right)^\alpha\right\}$$

The equalities (33) and (34) imply that for all $t > s$

$$\exp\left\{-\left(\frac{s}{\theta_1}\right)^\alpha - \left(\frac{t}{\theta_2}\right)^\alpha + \left(\frac{s}{\theta_2}\right)^\alpha\right\} = \exp\left\{-\left(\frac{s}{\theta_1} + \frac{t-s}{\theta_2}\right)^\alpha\right\}. \tag{49}$$

If $\alpha = 1$, this equality is verified. Suppose that $\alpha \neq 1$. For all $t > s$ put

$$g(t) = -\left(\frac{s}{\theta_1}\right)^\alpha - \left(\frac{t}{\theta_2}\right)^\alpha + \left(\frac{s}{\theta_2}\right)^\alpha + \left(\frac{s}{\theta_1} + \frac{t-s}{\theta_2}\right)^\alpha.$$

The derivative of $g(t)$ is

$$g'(t) = \frac{\alpha}{\theta_2^\alpha} t^{\alpha-1}\left(1 - \left(\frac{\theta_2 - \theta_1}{\theta_1}\frac{s}{t} + 1\right)^{\alpha-1}\right) \neq 0$$

and for all $t > s$ has the same sign for fixed $\theta_1 \neq \theta_2$ and $\alpha \neq 1$. So the function g is increasing or decreasing but not constant in t which contradicts to the equality (49). The assumption $\alpha \neq 1$ was false. So $\alpha = 1$, and the equality (45) implies that the lifetime distribution under any $x \in E_0$ is exponential:

$$S_x(t) = \exp\{-\frac{t}{\theta(x)}\}, \quad t \geq 0.$$

2) Suppose that the PH model holds on E and the time-to-failure is exponential for all $x \in E_0$. The formula (35) implies that for all $x \in E_0$

$$S_x(t) = \exp\{-r(x)A_0(t)\}.$$

Exponentiality of the times-to-failure under $x \in E_0$ and the last formula imply that $A_0(t) = ct$. The constant c can be included in $r(x)$, so we have $A_0(t) = t$. The formula (35) implies that

$$S_{x(\cdot)}(t) = \exp\left\{-\int_0^t r\{x(u)\}d(u)\right\},$$

i.e. the AAD model holds on E.

The proof is complete.

Relations between the GS and the PH models

The GS model is more general then the AAD model. When the PH model is also the GS model? It is given in the following proposition.

Proposition 9. *Suppose that the PH model holds on the set E including E_0 and all the stresses of the form (30) with $s < \delta$, where δ is any positive number. The GS model also holds on E iff the time to-failure is exponential for all $x \in E_0$.*

Proof. The PH model implies that for all $s < \delta$

$$\alpha_{x_s(\cdot)}(t) = \alpha_{x_2}(t), \; t > s.$$

1) If the GS model also holds on E, then for all $s < \delta$

$$\alpha_{x_s(\cdot)}(t) = \alpha_{x_2}(t - s + \varphi(s)), \; t > s,$$

where

$$\varphi(s) = A_{x_2}^{-1}(A_{x_1}(s))$$

is an increasing function. It implies that if both the GS and PH models hold on E then for all $s_1 < \delta$ and $s_2 < \delta$

$$\alpha_{x_2}(t - s_1 + \varphi(s_1)) = \alpha_{x_2}(t - s_2 + \varphi(s_2)), \; t > max(s_1, s_2).$$

Any function $\alpha_{x_2}(t) \equiv$ *const* verifies this. Assume that the function $\alpha_{x_2}(t)$ is not constant. Then

$$\varphi(s) - s = c = \quad const \quad \text{for all} s > 0,$$

because the function $\alpha_{x_2}(t)$ cannot be two or more-periodic. Note that $c \neq 0$, because

$$A_{x_2}(\varphi(s)) = A_{x_1}(s) \neq A_{x_2}(s).$$

The equalities

$$\lim_{s \to 0} A_{x_2}(\varphi(s)) = \lim_{s \to 0} A_{x_1}(s) = 0$$

and the monotonicity of $\varphi(s)$ imply that $\lim_{s \to 0} \varphi(s) = 0$. So exists $\delta_0 \in (0, \delta)$ such that

$$| \varphi(s) - s | < | c |, \quad \text{if} \, 0 < s < \delta_0.$$

It contradicts the implication that $\varphi(s) - s = c$ for any $s > 0$. It means that the assumption, that $\alpha_{x_2}(t)$ is not constant, was false. So $\alpha_{x_2}(t) = \alpha = \quad const$ which implies that

$$S_{x_2}(t) = e^{-\alpha t}.$$

The PH model implies that for all $x \in E_0$

$$S_x(t) = S_0(t)^{r(x)} = e^{-r(x) t},$$

i.e., the time-to-failure distribution is exponential for all $x \in E_0$.

2) Suppose that for all $x \in E_0$ the time-to-failure distribution is exponential and the PH model holds. The proof of the Proposition 8 implies that the AAD model and consequently the GS model also holds on E.

The proof is complete.

Constant stresses

If $x \in E_0$ is a constant stress, then the PH model gives

$$\alpha_x(t) = r(x)\, \alpha_0(t), \quad S_x(t) = S_0(t)^{r(x)},$$

where

$$S_0(t) = e^{-A_0(t)}.$$

For any $x_0 \in E_0$

$$\alpha_x(t) = \rho(x_0, x)\, \alpha_{x_0}(t), \quad S_x(t) = S_{x_0}^{\rho(x_0, x)}(t),$$

where

$$\rho(x_0, x) = r(x)/r(x_0).$$

Simple step-stresses

If $x(\cdot) \in E_1$ is a simple step-stress (3) then the PH model gives

$$\alpha_{x(\cdot)}(t) = \begin{cases} r(x_1)\, \alpha_0(t), & 0 \leq t < t_1, \\ r(x_2)\, \alpha_0(t), & t \geq t_1, \end{cases}$$

$$S_{x(\cdot)}(t) = \begin{cases} S_0(t)^{r(x_1)}, & 0 \leq t < t_1, \\ S_0(t_1)^{r(x_1)} \left(\frac{S_0(t)}{S_0(t_1)} \right)^{r(x_2)}, & t \geq t_1. \end{cases}$$

It implies that

$$\alpha_{x(\cdot)}(t) = \begin{cases} \alpha_{x_1}(t), & 0 \leq t < t_1, \\ \rho(x_1, x_2)\, \alpha_{x_1}(t), & t \geq t_1. \end{cases}$$

$$S_{x(\cdot)}(t) = \begin{cases} S_{x_1}(t), & 0 \leq t < t_1, \\ S_{x_1}(t_1) \left(\frac{S_{x_1}(t)}{S_{x_1}(t_1)} \right)^{\rho(x_1, x_2)}, & t \geq t_1. \end{cases} \tag{50}$$

The PH model in the form (50) is called the *tampered failure rate* (TFR) model (Bhattacharyya & Stoejoeti (1989)).

For any $x_0 \in E_0$

$$\alpha_{x(\cdot)}(t) = \begin{cases} \rho(x_0, x_1)\, \alpha_{x_0}(t), & 0 \le t < t_1, \\ \rho(x_0, x_2)\, \alpha_{x_0}(t), & t \ge t_1. \end{cases}$$

$$S_{x(\cdot)}(t) = \begin{cases} S_{x_1}(t), & 0 \le t < t_1, \\ S_{x_1}(t_1) \left(\frac{S_{x_1}(t)}{S_{x_1}(t_1)} \right)^{\rho(x_1, x_2)}, & t \ge t_1. \end{cases}$$

General step-stresses

If $x \in E_m$ is a general step-stress, then PH model can written in the following forms : for any $t \in [t_{i-1}, t_i)$

$$\alpha_{x(\cdot)}(t) = r(x_i)\, \alpha_0(t),\ (t_0 = 0, i = 1, ..., m),$$

$$S_{x(\cdot)}(t) = \prod_{j=1}^{i-1} \left(\frac{S_0(t_j)}{S_0(t_{j-1})} \right)^{r(x_j)} \left(\frac{S_0(t)}{S_0(t_{i-1})} \right)^{r(x_i)}.$$

For any $x_0 \in E_0$ and $t \in [t_{i-1}, t_i)$

$$\alpha_{x(\cdot)}(t) = \rho(x_0, x_i)\alpha_{x_0}(t), \quad S_{x(\cdot)}(t) = \prod_{j=1}^{i-1} \left(\frac{S_{x_0}(t_j)}{S_{x_0}(t_{j-1})} \right)^{\rho(x_0, x_j)} \left(\frac{S_{x_0}(t)}{S_{x_0}(t_{i-1})} \right)^{\rho(x_0, x_i)}.$$

5. Generalized proportional hazards models

The AAD and the PH models are rather restrictive. In the case of the AAD model the stress changes locally only the scale.

Under the PH model the hazard rate under the stress $x(\cdot)$ at the moment t *doesn't depend on the resource, used until* t. It is not very natural if items are aging. Indeed, let $E_0 \subset E$ be a set of constant in time covariates, x_0 be an usual stress, x_1 be an accelerated with respect to x_0 stress, $x_0, x_1 \in E_0$, i.e. $S_{x_0}(t) > S_{x_1}(t)$ for all $t \ge 0$, and E_1 be a set of simple step-stresses of the form

$$x(t) = \begin{cases} x_1, & 0 \le t < t_1, \\ x_0, & t \ge t_1. \end{cases}$$

If the PH model holds on $E_0 \cup E_1$ then for all $t_1 > 0$, $t > t_1$

$$\alpha_{x(\cdot)}(t) = \alpha_{x_0}(t).$$

If one population of items is tested under the usual stress and the second identical population - under the accelerated stress x_1 until a moment t_1 and after this moment both populations are observed under the same usual stress x_0, the failure rate after the moment t_1 is the same for both populations. These populations use different resources until the moment t_1, nevertherless, after the moment t, when both populations operate under the stress x_0, the resource usage rate is the same. It is not natural for aging items.

Definitions of the generalized proportional hazards models

A generalization of AAD and PH models is obtained by supposing that the rate of resource usage at any moment t is proportional not only to a function of the stress applied at this moment and to a baseline rate, but also to a function of the resource used until t. This is formalized by the following definition.

Definition 7 *The first generalized proportional hazards* (GPH1) model (Bagdonavičius & Nikulin (1998)) holds on E if for all $x(\cdot) \in E$

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\, q\{A_{x(\cdot)}(t)\}\, \alpha_0(t). \tag{51}$$

The particular cases of the GPH1 model are the PH model ($q(u) \equiv 1$) and the AAD model ($\alpha_0(t) \equiv \alpha_0 = const$).

A generalization of the GS and PH models is the following model.

Definition 8 *The second generalized proportional hazards* (GPH2) model holds on E if for all $x(\cdot) \in E$

$$\alpha_{x(\cdot)}(t) = u\{x(t), A_{x(\cdot)}(t)\}\, \alpha_0(t). \tag{52}$$

The particular cases of the GPH2 model are the GS model ($\alpha_0(t) \equiv \alpha_0 = const$) and GPH1 model ($u(x,s) = r(x)\, q(s)$)

Models of different levels of generality can be obtained by completely specifying q, parametrizing q or considering q as unknown.

Relations with generalized multiplicative models

The GPH1 models can be formulated in terms of resources other then exponential. It helps to choose the function q.

Denote by

$$f^G_{x(\cdot)}(t) = H(S_{x(\cdot)}(t)), \tag{53}$$

the G-resource used until the moment t.

Definition 9 *The generalized multiplicative (GM) model (Bagdonavičius & Nikulin (1994)) with the resource survival function G holds on E if there exist a positive function r and a survival function S_0 such that for all $x(\cdot) \in E$*

$$\frac{\partial f^G_{x(\cdot)}(t)}{\partial t} = r\{x(t)\} \frac{\partial f^G_0(t)}{\partial t}, \tag{54}$$

where $f^G_0(t) = H(S_0(t))$. The equality (54) implies that for all $x(\cdot) \in E$

$$S_{x(\cdot)}(t) = G\left\{ \int_0^t r(x(\tau))\, dH(S_0(\tau)) \right\}. \tag{55}$$

If E_0 a set of constant in time covariates then for all $x \in E_0$

$$S_x(t) = G\{r(x) H(S_0(t))\}. \tag{2.56}$$

For any $x_1, x_2 \in E_0$

$$S_{x_2}(t) = G\{\rho(x_1, x_2)\, H(S_{x_1}(t))\}, \tag{57}$$

where $\rho(x_1, x_2) = r(x_2)/r(x_1)$.

Proposition 10. *Suppose that the integral*

$$\int_0^x \frac{dv}{q(v)}$$

converges for all $x \geq 0$. The GPH1 model (51) holds on E iff there exists a survival function G such that the GM model (54) holds on E.

Proof. Suppose that the GPH1 model (51) holds on E. Define the function $H(u)$ by the formula

$$H(u) = \int_0^{-\ln u} \frac{dv}{q(v)}.$$

Then the used resource $f^G_{x(\cdot)}(t)$ is defined by the formula (53), verifies the equalities

$$\frac{\partial f^G_{x(\cdot)}(t)}{\partial t} = \frac{1}{q\{A_{x(\cdot)}(t)\}} \alpha_{x(\cdot)}(t) = r\{x(t)\} \frac{\partial f^G_0(t)}{\partial t}.$$

So the GM model holds on E.

Vice versa, if there exists a survival function G such that the GM model holds on E then (55) implies that for all $x(\cdot) \in E$

$$S_{x(\cdot)}(t) = G\left[\int_0^t r\{x(\tau)\} dH(S_0(\tau))\right]$$

and

$$\alpha_{x(\cdot)}(t) = e^{A_{x(\cdot)}(t)} G'(H(e^{-A_{x(\cdot)}(t)})) r\{x(t)\} H'\{S_0(t)\}\alpha_0(t).$$

Put

$$q(u) = -e^u G'(H(e^{-u})), \quad \alpha_0^*(t) = -H'\{S_0(t)\}\, \alpha_0(t).$$

Then for all $x(\cdot) \in E$

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\, q\{A_{x(\cdot)}(t)\}\, \alpha_0^*(t).$$

So the GPH1 holds.

The proof is complete.

Corollary 3 The GPH1 with specified q is equivalent to the GM model with the survival function of the resource $G = H^{-1}$ and the following relations hold:

$$H(u) = \int_0^{-\ln u} \frac{dv}{q(v)}, \quad q(u) = -e^u G'(H(e^{-u})). \tag{58}$$

Put $A_0(u) = H(S_0(u))$. In terms of survival functions the GPH1 model is written

$$S_{x(\cdot)}(t) = G\left(\int_0^t r\{x(u)\} dA_0(u)\right), \tag{59}$$

where

$$G = H^{-1}, \quad H(p) = \int_0^{-\ln p} \frac{dv}{q(v)}, \quad A_0(u) = \int_0^t \alpha_0(u) du.$$

Remark 6 *The GPH1 model is the PH model when $q(u) \equiv 1$. The equality (58) implies that in this case $H(u) = \ln u$, $G(t) = e^{-t}$, i.e. the PH model is the GPH1 model with standard exponential distribution of the resource.*

Remark 7 Let x_0 be the usual stress. In terms of the survival functions the AAD model can be written in the form

$$S_{x(\cdot)}(t) = S_{x_0}\left(\int_0^t r\{x(u)\} du\right).$$

Hence

$$f^{S_{x_0}}(t) = S_{x_0}^{-1}(S_{x(\cdot)}(t)) = \int_0^t r\{x(u)\} du$$

and

$$\frac{\partial f_{x(\cdot)}^{S_{x_0}}(t)}{\partial t} = r\{x(t)\},$$

i.e., the AAD model is GM-model with S_{x_0}-resource.

Constant stresses

If the GPH1 model holds on a set of constant stresses E_0, then

$$\alpha_x(t) = r(x)\, q(A_x(t))\, \alpha_0(t), \quad S_x(t) = G\left(r(x)\, A_0(t)\right).$$

For all $x_1, x_2 \in E_0$

$$S_{x_2}(t) = G\left(\rho(x_1, x_2)\, H(S_{x_1}(t))\right).$$

Simple step-stresses

If the GPH1 model holds on a set of simple step-stresses E_1, then for any $x(\cdot) \in E_1$ of the form (3)

$$S_{x(\cdot)}(t) = \begin{cases} G(r(x_1)A_0(t)), & 0 \leq t < t_1, \\ G(r(x_1)A_0(t) + r(x_2)(A_0(t) - A_0(t_1)))), & t \geq t_1. \end{cases}$$

The survival function $S_{x(\cdot)}$ can be written in terms of the function S_{x_1} :

$$S_{x(\cdot)}(t) = \begin{cases} S_{x_1}(t), & 0 \leq t < t_1, \\ G\{H(S_{x_1}(t)) + \rho(x_1, x_2)(H(S_{x_1}(t) - S_{x_1}(t_1))\}, & t \geq t_1. \end{cases}$$

General step-stresses

If the GPH1 model holds on a set of general step-stresses E_m, then for any $x \in E_m$ of the form (8) and for any $t \in [t_{i-1}, t_i)$

$$S_{x(\cdot)}(t) = G\left\{\sum_{j=1}^{i-1} r(x_j)(A_0(t_j) - A_0(t_{j-1})) + r(x_i)(A_0(t) - A_0(t_{i-1}))\right\},$$

$$S_{x(\cdot)}(t) = G\left\{(H(S_{x_i}(t)) - H(S_{x_i}(t_{i-1}))) + \sum_{j=1}^{i-1} \rho(x_i, x_j)(H(S_{x_i}(t_j)) - H(S_{x_i}(t_{j-1})))\right\}.$$

Relations between survival functions under constant and non-constant stresses.

Similarly as in the case of the AAD model, consider some useful relations between survival functions under constant and time-varying stresses.

Proposition 11 *Suppose that* $x(\cdot)$, $x(\tau)$, $x_0 \in E$ for all $\tau \geq 0$.

If the GM model holds on E*, then*

$$S_{x(\cdot)}(t) = G\left(\int_0^t \frac{H(S_{x(\tau)}(t))}{H(S_{x_0}(t))}\, dH(S_{x_0}(\tau))\right) \tag{60}$$

$$= G\left(\int_0^t H(S_{x(\tau)}(\tau))\, d\, log\, H(S_{x_0}(\tau))\right).$$

Corollary 4 *If* $x(\cdot) \in E_1$ *is a simple step-stress of the form (3),* $x_0 \in E_0$*, then*

$$S_{x(\cdot)}(t) = \begin{cases} G\{\rho(x_0, x_1)(H(S_{x_0}(t)),\ 0 \leq t < t_1, \\ G\{\rho(x_0, x_1)H(S_{x_0}(t_1)) + \rho(x_0, x_2)(H(S_{x_0}(t)) - H(S_{x_0}(t_1)))\},\ t \geq t_1. \end{cases} \tag{61}$$

If $x(\cdot) \in E_m$ *is a general step-stress of the form (8),* $x_0 \in E_0$*, then for all* $t \in [t_{i-1}, t_i)$

$$S_{x(\cdot)}(t) = G\left\{\rho(x_0, x_i)H(S_{x_0}(t)) + \sum_{j=1}^{i-1} \rho(x_0, x_j)(H(S_{x_0}(t_i)) - H(S_{x_0}(t_{i-1})))\right\}. \tag{62}$$

Proof.

The equality (55) implies that there exists the functional $r_1 : E \to [0, \infty)$ such that

$$S_{x(\cdot)}(t) = G\left\{\int_0^t r_1[x(\tau)] dH(S_{x_0}(\tau))\right\}. \tag{63}$$

So for all fixed τ, $\tau \leq t$

$$r_1\{x(\tau)\} = H(S_{x(\tau)}(t))/H(S_{x_0}(t)). \tag{64}$$

Putting $r_1\{x(\tau)\}$ in the equality (63), the first of the equalities (60) is obtained. Putting $t=\tau$ in (64) and the obtained expression of $r_1\{x(\tau)\}$ in (63), the second of equalities (60) is obtained.

Characterization of the GM model with constant in time stresses

At first glance it looks like there are too many GM models. It appears that it is not so.

Indeed, assume that a function G is continuous and strictly decreasing on $[0,\infty[$ and

$$G_1(u)=G((u/\theta)^p).$$

Let $E_0=[x_0,x_1]\subset \mathbf{R}$ be an interval of constant in time stresses, $\{S_x,\ x\in[x_0,x_1]\}$ be a class of continuous survival functions, such that $S_x(t)>S_y(t)$ for all $x,y\in E_0,\ x<y,\ t>0$,

$$H=G^{-1}:]0,1]\to[0,\infty] \text{ and } H_1=G_1^{-1}$$

be the inverse functions of G and G_1, respectively. If the GM model with the resource survival function G holds on E_0, then the equality (57) implies that

$$H(S_x(t))=\lambda(x)H(S_{x_0}(t)),\ t>0,\ x\in[x_0,x_1], \tag{65}$$

where $\lambda(x)=\rho(x_0,x)$. Then

$$H_1(S_x(t))=\lambda^{1/p}(x)H_1(S_{x_0}(t)),\ t>0,\ x\in[x_0,x_1]. \tag{66}$$

The inverse result also takes place :

Proposition 12 *Assume that a function G is continuous and strictly decreasing on $[0,\infty[$ and the equality (65) holds. Then the equality (66) also holds iff*

$$G_1(u)=G((u/\theta)^p),\ u\in[0,\infty),$$

for some positive constants θ and p.

Proof.

1) It was just shown that if the GM model holds for the survival function G and $G_1(t)=G((t/\theta)^p)$ then the GM model holds for the survival function G_1.

2) Suppose that the GM model holds for the survival functions G and G_1, i.e. the equalities (65) and (66) hold. Introduce a function $D:[0,\infty[\to[0,\infty[$ such that $D(u)=H_1(G(u)),\ u\in[0,\infty[$. In this case $H_1(p)=D(H(p)),\ p\in]0,1]$, and the relation (66) can be rewritten as follows:

$$D(H(S_x(t)))=\lambda^{1/p}(x)D(H(S_{x_0}(t)),\ t>0,\ x\in[x_0,x_1].$$

Using (65) we obtain that

$$D(\lambda(x)H(S_x(t)))=\lambda^{1/p}(x)D(H(F_{x_0}(t))),\ t>0,\ x\in[x_0,x_1]$$

with the initial conditions $D(0)=0$ and $\lim_{u\to\infty}D(u)=\infty$. Putting $y=H(S_{x_0}(t))$ we obtain that

$$D(\lambda(x)y)=\lambda^{1/p}(x)D(y),\ y\in[0,\infty[,\ x\in[x_0,x_1],$$

or for $v=\ln y$

$$Q(\ln\lambda(x)+v)=\frac{1}{p}\ln(\lambda(x))+Q(v),\ v\in\mathbf{R},\ x\in[x_0,x_1],$$

where $Q(v)=\ln(D(e^v)))$. This equality leads to the equality

$$Q(v)=av+b,\quad a=\frac{1}{p}.$$

It implies that $D(y)=\theta\, y^a$, where $\theta=e^b$. Consequently,

$$G(y)=G_1(D(y))=G_1(\theta y^a)\quad\text{and}\quad G_1(u)=G((u/\theta)^p),\ u\in[0,\infty[.$$

The proof is completed.

This proposition implies that, for example, the PH model is a submodel of the GM model when G is not only standard exponential but when it is any exponential or two-parameter Weibull survival function. So submodels of the GM model form classes generated by classes of resource distributions which differ only by shape and scale parameters.

Relations with the frailty models

A method of the function q choice is obtained by using relations between the GPH1 models and the *frailty models with covariates.*

The hazard rate can be influenced not only by the observable stress $x(\cdot)$ but also by a non-observable positive random covariate Z, called the *frailty variable*, see Hougaard (1986). Suppose that for all $x(\cdot) \in E$

$$\alpha_{x(\cdot)}(t|Z=z) = z\, r(x(t))\, \alpha_0(t). \tag{67}$$

Then

$$S_{x(\cdot)}(t|Z=z) = exp\{-z\int_0^t r(x(\tau))\, dA_0(\tau)\}$$

and

$$S_{x(\cdot)}(t) = \mathbf{E}exp\{-Z\int_0^t r(x(\tau))\, dA_0(\tau)\} = G\{\int_0^t r(x(\tau))dA_0(\tau)\}, \tag{68}$$

where $G(s) = \mathbf{E}e^{-sZ}$. If we put $S_0(t) = G(A_0(t))$, the equality (2.68) implies that for all $x(\cdot) \in E$

$$S_{x(\cdot)}(t) = G\{\int_0^t r(x(\tau))\, dH(S_0(\tau))\}, \tag{69}$$

where $H = G^{-1}$. We obtained that the *frailty model* defined by a frailty variable Z, the GM *model* with the survival function of the resource $G(s) = \mathbf{E}e^{-sZ}$, and the GPH1 *model* with the function q defined by (58) give the same survival function under any stress $x(\cdot) \in E$.

Relations with the linear transformation models

Under constant in time stresses the GPH1 model is related with the *linear transormation* (LT), Dabrowska & Doksum (1988b), Cheng, Wei, Ying (1995).

Let consider the set E_0 of constant in time stresses and let T_x denote the time-to-failure under the covariate $x \in E_0$. The LT model holds on E_0 if for all $x \in E_0$

$$h(T_x) = -\beta^T x + \varepsilon, \quad \text{for all} \quad x \in \mathcal{E}, \tag{70}$$

where $h : [0,\infty) \to [0,\infty)$ is a strictly increasing function, and ε is a random error with distribution function Q. The relation (70) implies that for all $x \in E_0$

$$S_x(t) = G\{e^{\beta^T x + h(t)}\} = G\{e^{\beta^T x} H(S_0(t))\},$$

where $G(t) = 1 - Q(\ln t)$, $S_0(t) = G\{e^{h(t)}\}$. Therefore, in the case of *constant in time* stresses, the frailty model defined by the frailty variable Z, the GM model with the survival function of the resource $G(s) = Ee^{-sZ}$, the GPH1 model with the function q defined by (58) and the LT model with the distribution function $Q(x) = 1 - G(\ln x)$ of the random error ε give the same expression of survival functions.

6. The main classes of GPH models.

Particular classes of the GPH models are very important for survival analysis and accelerated life testing. The numerous examples of real data show that taking two constant in time covariates, say

x_1 and x_2, the ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ (which is constant under the PH model), can be *increasing or decreasing* in time and even a *cross-effect* of hazard rates can be observed.

Such data can be modelled by submodels of the GPH1 or more general GPH2 model.

Consider possible parametrizations in the GPH models.

GPH model with a monotone ratio of hazard rates

Consider the GPH1 model with parametrization

$$q(u) = (1+u)^{\gamma+1}. \tag{71}$$

where $\gamma \in I\!\!R$ is an unknown scalar parameter. We have the model

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}(1 + A_{x(\cdot)}(t))^{\gamma+1}\alpha_0(t). \tag{72}$$

The particular case of this model is the PH model when $\gamma = -1$.

Suppose that $\gamma < 0$ and $c_0 = r(x_2)/r(x_1), (x_1, x_2 \in E_0)$. Then

$$\alpha_{x_2}(t)/\alpha_{x_1}(t) = c_0 \left\{ \frac{1 - \gamma\, r(x_2)\, A_0(t)}{1 - \gamma\, r(x_1)\, A_0(t)} \right\}^{-1-\frac{1}{\gamma}}.$$

The ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ has the following properties:

a) if $-1 < \gamma < 0$, then the ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ *increases* from the value c_0 until the value $c_\infty = \lim_{t\to\infty} \alpha_{x_2}(t)/\alpha_{x_1}(t)$, where the constant c_∞ can take any value in the interval (c_0, ∞);

b) if $\gamma = -1$ (PH model), the ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ is *constant* in time.

c) if $\gamma < -1$, then the ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ *decreases* from the value c_0 until the value $c_\infty \in (1, c_0)$.

GPH model with cross-effects of hazard rates

To obtain a cross-effect of hazard rates consider the following submodel of GPH2:

$$\alpha_{x(\cdot)}(t) = r(x(t))(1 + A_{x(\cdot)}(t))^{\gamma^T x(t)+1}\,\alpha_0(t). \tag{73}$$

Suppose that $c_0 = r(x_2)/r(x_1) > 1, (x_1, x_2 \in E_0)$, and $\gamma^T x_2 < \gamma^T x_1 < 0$. Then

$$\alpha_{x_2}(t)/\alpha_{x_1}(t) = c_0 \frac{\left(1 - \gamma^T x_2\, r(x_2) A_0(t)\right)^{-1-\frac{1}{\gamma^T x_2}}}{\left(1 - \gamma^T x_1\, r(x_1)\, A_0(t)\right)^{-1-\frac{1}{\gamma^T x_1}}}$$

and

$$\alpha_{x_2}(0)/\alpha_{x_1}(0) = c_0 > 1, \quad \lim_{t\to\infty} \alpha_{x_2}(t)/\alpha_{x_1}(t) = 0.$$

So we have a cross-effect of the hazard rates.

Generalization of the gamma frailty model with covariates

Consider the GPH1 model with parametrization

$$q(u) = e^{\gamma u},\ \gamma \in \mathbf{R}. \tag{74}$$

We have the model

$$\alpha_{x(\cdot)}(t) = r(x(t))\, e^{\gamma A_{x(\cdot)}(t)}\,\alpha_0(t). \tag{75}$$

If $\gamma = 0$, it becomes the usual PH model.

Suppose that $x_0 < x(\cdot)$ and that the support of $S_{x(\cdot)}$ is $I_{x(\cdot)} = [0, sp_{x(\cdot)})$. The model (75) implies for all $x(\cdot)$ and $t \in I_{x(\cdot)}$

$$S_{x(\cdot)}(t) = \begin{cases} \{1 + \frac{1}{r(x_0)} \int_0^t r\{x(\tau)\} dS^\gamma_{x_0}(\tau)\}^{1/\gamma}, & \text{if} \quad \gamma \neq 0, \\ exp\{\frac{1}{r(x_0)} \int_0^t r\{x(\tau)\} d\ln S_{x_0}(\tau)\}, & \text{if} \quad \gamma = 0. \end{cases}$$

and for constant over time stresses $x > x_0$ and any $t \in I_x$

$$S_x(t) = \left\{1 - \frac{r(x)}{r(x_0)}(1 - S_{x_0}^{\gamma}(t))\right\}^{\frac{1}{\gamma}}.$$

For $\gamma > 0$ the upper bound $sp_{x(\cdot)}$ of the support $I_{x(\cdot)}$ satisfies the equation

$$r(x_0) + \int_0^{sp_{x(\cdot)}} r\{x(\tau)\} dS_{x_0(\cdot)}^{\gamma}(\tau)\} = 0.$$

Take notice that the condition $x_0 < x(\cdot)$ implies that $sp_x \leq sp_{x_0}$.

For $\gamma < 0$ the upper bound $sp_{x(\cdot)}$ of the support $I_{x(\cdot)}$ satisfies the equation

$$\int_0^{sp_{x(\cdot)}} r\{x(\tau)\} dS^{\gamma}(\tau) = -\infty.$$

If all stresses are constant over time, then $sp_x = sp_{x_0}$ for all x. If $sp_{x_0} = \infty$, then $sp_x = \infty$ for all x.

Take notice that the model (75) is a generalization of the *gamma frailty model* (GFM) (see Vaupel et al. (1979)) *with covariates.*

Indeed, suppose that the frailty variable Z follows a gamma distribution with the scale parameter $\theta > 0$, the shape parameter $k > 0$ and the density

$$p_Z(z) = \frac{z^{k-1}}{\theta^k \Gamma(k)} e^{-z/\theta}, \quad z > 0. \tag{76}$$

The survival function of the resource

$$G(s) = \mathbf{E}e^{-sZ} = (1 + \theta t)^{-k}.$$

Put $\gamma = -1/k < 0$. The formula (2.58) implies that

$$q(u) = -\frac{1}{\lambda\gamma} e^{\gamma u}, \; \gamma < 0. \tag{77}$$

The proportionality constant can be included in α_0 and $q(u)$ can be written in the form

$$q(u) = e^{\gamma u}, \; \gamma < 0.$$

We have the gamma frailty model .

Consider the frailty model with a density $p_Z(z)$ which is the inverse Laplace transformation of the survival function

$$G(t) = (1 - \gamma t)^{1/\gamma} \, \mathbf{1}_{[0,1/\gamma)}(t). \tag{78}$$

The formula (58) implies that

$$q(u) = e^{\gamma u}, \; \gamma > 0.$$

We obtained that under the model (77) the survival function of the resource is

$$G(t) = (1 - \gamma t)^{1/\gamma}, \quad \gamma < 0. \tag{79}$$

For $\gamma > 0$ the support of G is $[0, 1/\gamma)$

For constant in time stresses $x_1, x_2 \in E_0$ and $\gamma < 0$ the equality (79) implies the *generalized proportional odds-rate* (GPOR) *model* (Dabrowska & Doksum (1988)):

$$\frac{S_{x_2}^{-\gamma}(t) - 1}{S_{x_2}^{-\gamma}(t)} = \frac{r(x_2)}{r(x_1)} \frac{S_{x_1}^{-\gamma}(t) - 1}{S_{x_1}^{-\gamma}(t)}.$$

Inverse gaussian frailty model.

Consider the GPH1 model with parametrization

$$q(u) = \frac{1}{1+\gamma u}, \quad \gamma > 0. \tag{80}$$

We obtain the model

$$\alpha_{x(\cdot)}(t) = r(x(t))\, \frac{\alpha_0(t)}{1+\gamma A_{x(\cdot)}(t)}\,. \tag{81}$$

Take notice that this model is the *inverse gaussian frailty model with covariates.*

Indeed, suppose that the frailty variable Z has the inverse gaussian distribution with the density

$$p_Z(z) = \left(\frac{\sigma}{\pi}\right)^{1/2} e^{\sqrt{4\sigma\theta}} z^{-3/2} e^{-\theta z - \frac{\sigma}{z}}, \quad z > 0. \tag{82}$$

The formulas (68) and (58) imply that

$$q(u) = \frac{2\sigma}{u + (4\sigma\theta)^{1/2}}. \tag{83}$$

The proportionality constant can be included in α_0 and $q(u)$ can be written in the form (80).

Consider GM models with G specified. These models are alternative to the PH model.

Generalized logistic regression model.

If the distribution of the resource is *loglogistic*,i.e.

$$G(t) = \frac{1}{1+t}\mathbf{1}_{\{t\geq 0\}}, \tag{284}$$

then $q(t) = e^{-t}$ and the GM model can be formulated in the following way:

$$\frac{\alpha_{x(\cdot)}(t)}{S_{x(\cdot)}(t)} = r\{x(t)\}\frac{\alpha_0(t)}{S_0(t)}. \tag{85}$$

If $x(\cdot) \in E_1$ is the step-stress of the form (3) then we have

$$\frac{\alpha_{x(\cdot)}(t)}{S_{x(\cdot)}(t)} = \begin{cases} \frac{\alpha_{x_0}(t)}{S_{x_0}(t)}, & 0 \leq t < t_1, \\ \rho \frac{\alpha_{x_0}(t)}{S_{x_0}(t)}, & t \geq t_1 \end{cases} \tag{86}$$

or

$$S_{x(\cdot)}(t) = \begin{cases} S_{x_0}(t), & 0 \leq t < t_1, \\ S_{x_0}(t_1)\left[1 + \rho\left(\frac{S_{x_0}(t_1)}{S_{x_0}(t)} - 1\right)\right]^{-1}, & t \geq t_1. \end{cases}$$

If stresses are constant in time then we obtain the model

$$\frac{1}{S_x(t)} - 1 = r(x)\,(\frac{1}{S_0(t)} - 1).$$

It is the analogue of the logistic regression model which is used for analysis of dichotomous data when the probability of "success" in dependence of some factors is analised. The obtained model is near to the Cox model when t is small .

Generalized probit model.

If the resource is *lognormal*, then

$$G(t) = 1 - \Phi(\log t), \quad t > 0,$$

where Φ is the distribution function of the standard normal law. If covariates are constant in time then in terms of survival functions the GM model can be written as follows:

$$\Phi^{-1}\left(S_x(t)\right) = \ log\left(r(x)\right) + \Phi^{-1}\left(S_0(t)\right).$$

It is the *generalized probit model* see Dabrowska & Doksum (1988).

7. Parametrization of the function r in AAD and GPH models

Following Viertl (1988), consider parametrization of the function r in the AAD and GPH models. If the AAD model holds on E_0, then for all $x_1, x_2 \in E_0$

$$S_{x_2}(t) = S_{x_1}(\rho(x_1, x_2)t), \tag{87}$$

where the function $\rho(x_1, x_2) = r(x_2)/r(x_1)$ shows the degree of scale variation. It is evident that $\rho(x, x) = 1$.

In the case of more general GPH1 model

$$S_{x_2}(t) = G(\rho(x_1, x_2)\, H(S_{x_1}(t))), \tag{88}$$

where

$$\rho(x_1, x_2) = \frac{H(S_{x_2}(t))}{H(S_{x_1}(t))} = \frac{f^G_{x_2}(t)}{f^G_{x_1}(t)} \tag{89}$$

shows the degree of the resource usage rate variation. Suppose at first that x is unidimensional. The rate of scale (AAD model) or resourse usage rate (GPH1 model) variation can be defined by the *infinitesimal characteristic* (see Viertl (1988) for AAD model):

$$\delta(x) = \lim_{\Delta x \to 0} \frac{\rho(x, x + \Delta x) - \rho(x, x)}{\Delta x} = [log\, r(x)]'.$$

So for all $x \in E_0$ the function $r(x)$ is given by the formula:

$$r(x) = r(x_0) exp \left\{ \int_{x_0}^{x} \delta(v)\, dv \right\},$$

where $x_0 \in E_0$ is some fixed stress.

Suppose that $\delta(x)$ is *proportional to some known function $u(x)$ of the stress*:

$$\delta(x) = \alpha\, u(x), \quad \alpha > 0.$$

In this case

$$r(x) = e^{\beta_0 + \beta_1 z(x)}, \tag{90}$$

where $z(x)$ is some known function, β_0, β_1 are unknown parameters.

Example 1. $\delta(x) = \alpha$,
i.e. the rate of scale changing is constant. Then

$$r(x) = e^{\beta_0 + \beta_1 x},$$

where $\beta_1 > 0$. It is so called *log-linear model.*

Example 2. $\delta(x) = \alpha/x$.
Then

$$r(x) = e^{\beta_0 + \beta_1 log\, x} = \alpha_1 x^{\beta_1},$$

where $\beta_1 > 0$. It is so called *power rule model.*

Example 3. $\delta(x) = \alpha/x^2$.
Then

$$r(x) = e^{\beta_0+\beta_1/x} = \alpha_1 e^{\beta_1/x},$$

where $\beta_1 < 0$. It is so called *Arrhenius model.*

Example 4. $\delta(x) = \alpha/x(1-x)$.
Then

$$r(x) = e^{\beta_0+\beta_1 \ln\frac{x}{1-x}} = \alpha_1 \left(\frac{x}{1-x}\right)^{\beta_1}, \quad 0 < x < 1,$$

where $\beta_1 > 0$. It is the *model of Meeker-Luvalle* (1995).

The Arrhenius model is widely used to model product life when the stress is the temperature, the power rule model - when the stress is voltage, mechanical loading, the log-linear model is applied in endurance and fatigue data analysis, testing various electronic components (see Nelson (1990)). The model of Meeker-Luvalle is used when x is proportion of humidity.

If it is not very clear which of the first three models to choose, one can take more large class of models. For example, all these models are the particular cases of the class of models determined by

$$\delta(x) = \alpha\, x^{\gamma}$$

with unknown γ or, in terms of the function $r(x)$, by

$$r(x) = \begin{cases} e^{\beta_0+\beta_1(x^{\varepsilon}-1)/\varepsilon}, & \text{if} \quad \varepsilon \neq 0; \\ e^{\beta_0+\beta_1 log\, x}, & \text{if} \quad \varepsilon = 0. \end{cases}$$

In this case the parameter ε must be estimated.

The model (90) can be generalized. One can suppose that $\delta(x)$ is a linear combination of some known functions of the stress:

$$\delta(x) = \sum_{i=1}^{k} \alpha_i u_i(x).$$

In such a case

$$r(x) = exp\left\{\beta_0 + \sum_{i=1}^{k} \beta_i z_i(x)\right\},$$

where $z_i(x)$ are some known functions of the stress, $\beta_0, \ldots, \beta_k$ are unknown (possibly not all of them) parameters.

Example 5. $\delta(x) = 1/x + \alpha/x^2$.
Then

$$r(x) = e^{\beta_0+\beta_1 log\, x+\beta_2/x} = \alpha_1 x e^{\beta_2/x},$$

where $\beta_1 = 1$, $\beta_2 < 0$. It is so called *Eyring model,* applied when the stress x is the temperature.

Example 6. $\delta(x) = \sum\limits_{i=1}^{k} \alpha_i/x^i$.
Then

$$r(x) = exp\left\{\beta_0 + \beta_1 log\, x + \sum_{i=1}^{k-1} \beta_i/x^i\right\}.$$

It is so called *generalized Eyring model.*

Suppose now that the stress $x = (x_1, \ldots, x_m)$ is multidimensional. Define (see Viertl (1988)) the infinitesimal characteristics $\delta_i(x)$ by the equalities

$$\delta_i(x) = \lim_{\Delta x_i \to 0} \frac{\rho(x, x+\Delta x_i e_i) - \rho(x,x)}{\Delta x_i} = \frac{\partial log\, r(x)}{\partial x_i},$$

where $e_i = (0, \ldots, 1, \ldots, 0)$. The unity is the ith coordinate.

Generalizing the unidimensional case, $\delta_i(x)$ can be parametrized in the following manner

$$\delta_i(x) = \sum_{j=1}^{k_i} \alpha_{ij} u_{ij}(x),$$

where $u_{ij}(x)$ are known functions, α_{ij} are unknown constants. In this case

$$r(x) = exp\left\{\beta_0 + \sum_{i=1}^{m}\sum_{j=1}^{k_i} \beta_{ij} z_{ij}(x)\right\},$$

where $z_{ij}(x)$ are known functions, β_{ij} are unknown constants.

Example 7. $\delta_1(x) = 1/x_1 + (\alpha_{11} + \alpha_{12}x_2)/x_1^2$ and $\delta_2(x) = \alpha_{21} + \alpha_{22}/x_1$.
Then

$$r(x) = exp\left\{\beta_0 + \beta_1 log\, x_1 + \beta_2 x_2 + \beta_3/x_1 + \beta_4 x_2/x_1\right\}.$$

It is so called *generalized Eyring model*. This model is used for certain semiconductor materials, when x_1 is the temperature and x_2 is the voltage.

Example 8. $\delta_i(x) = \alpha_i u_i(x_i)$, where u_i are known functions.
Then

$$r(x) = exp\left\{\beta_0 + \sum_{j=1}^{k} \beta_j z_j(x_j)\right\},$$

where z_j are known functions. It is so called *generalized Arrhenius model*. It also called the log-linear model.

Generalized additive and additive-multiplicative models

Definition 10. *The generalized additive (GA) model (Bagdonavičius & Nikulin (1995)) holds on E if there exist a function a on E and a survival function S_0 such that for all $x(\cdot) \in E$*

$$\frac{\partial f^G_{x(\cdot)}(t)}{\partial t} = \frac{\partial f^G_0(t)}{\partial t} + a(x(t)) \tag{91}$$

with the initial conditions $f^G_0(0) = f^G_{x(\cdot)}(0) = 0$; here $f^G_0(t) = H(S_0(t))$.

So the stress influences additively the rate of resource using. The last equation implies that

$$S_{x(\cdot)}(t) = G\left(H(S_0(t)) + \int_0^t a(x(\tau))d\tau\right). \tag{92}$$

In terms of exponential resource usage the GA model can be written in the form

$$\alpha_{x(\cdot)}(t) = q\{A_{x(\cdot)}(t)\}(\alpha_0(t) + a(x(t))).$$

The particular case of the GA model is the *additive hazards model* (AH) :

$$\alpha_{x(\cdot)}(t) = \alpha_0(t) + a(x(t)). \tag{93}$$

Both the GM and the GA models can be included into the following model.

Definition 11 *The generalized additive-multiplicative (GAM) model (Bagdonavičius & Nikulin (1997)) holds on E if there exist functions a and r (positive) on E and a survival function S_0 such that for all $x(\cdot) \in E$*

$$\frac{\partial f^G_{x(\cdot)}(t)}{\partial t} = r\{x(t)\}\frac{\partial f^G_0(t)}{\partial t} + a(x(t)) \tag{94}$$

with the initial conditions $f_0^G(0) = f_{x(\cdot)}^G(0) = 0$; here $f_0^G(t) = H(S_0(t))$.

So the stress influences the rate of resource usage as multiplicatively as additively. The last equation implies that

$$S_{x(\cdot)}(t) = G\left(\int_0^t r\{x(\tau)\}dH(S_0(\tau)) + \int_0^t a(x(\tau))d\tau\right). \tag{95}$$

In terms of exponential resource usage the GAM model can be written in the form :

$$\alpha_{x(\cdot)}(t) = q\{A_{x(\cdot)}(t)\}(r\{x(t)\}\alpha_0(t) + a(x(t))).$$

In the particular case of the exponential resource we obtain the additive-multiplicative hazards (AMH) model, see Lin and Ying (1996) :

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\alpha_0(t) + a(x(t)). \tag{96}$$

The function a in the GAM models is parametrized as the function $\ln r$ in the GM models and the function q as the function q in the GPH models.

9. Changing shape and scale models

Definition of models

Consider now the important model which does not lie in the class of the GAM models but includes the AAD model as the particular case.

Suppose that the constant in time stresses $x \in E_0$ change not only the scale but also the shape of time-to-failure distribution: there exist on E_0 the positive functions $\theta(x)$ and $\nu(x)$ such that for any $x(\cdot) \in E$

$$S_x(t) = S_{x_0}\left\{\left(\frac{t}{\theta(x)}\right)^{\nu(x)}\right\}. \tag{97}$$

The S_{x_0}-resource used until the moment t under x is $f_x(t) = S_{x_0}^{-1}(S_x(t))$, and the resource usage rate is

$$\frac{\partial}{\partial t}f_x(t) = r(x)\, t^{\nu(x)-1},$$

where $r(x) = \nu(x)/\theta(x)^{\nu(x)}$, $H = G^{-1}$. So the model (97) means that the *resource usage rate under the stress x is increasing, if $\nu(x) > 1$, decreasing, if $0 < \nu(x) < 1$, and constant, if $\nu(x) = 1$.*

In the case $\nu(x) = 1$ we have the AAD model.

Consider the following generalization of the model (97) to the case of time varying stresses :

Definition 12. *The changing shape and scale (CHSS) model (Bagdonavičius & Nikulin (1998)) holds on E if there exist positive on E functions r and ν such that for all $x(\cdot) \in E$*

$$\frac{\partial f_{x(\cdot)}(t)}{\partial t} = r\{x(t)\}\, t^{\nu(x(t))-1}. \tag{98}$$

The equality (98) implies that

$$S_{x(\cdot)}(t) = S_{x_0}\left(\int_0^t r\{x(\tau)\}\tau^{\nu(x(\tau))-1}d\tau\right). \tag{99}$$

In terms of the exponential resource usage the model can be written in the form :

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\, q(A_{x(\cdot)}(t))\, t^{\nu(x(t))-1}.$$

Simple step-stresses

If $x(\cdot) \in E_1$, then the formula (99) implies that

$$S_{x(\cdot)}(t) = \begin{cases} G\left\{\left(\frac{t}{\theta(x_1)}\right)^{\nu(x_1)}\right\}, & 0 \le t < t_1, \\ G\left\{\left(\frac{t_1}{\theta(x_1)}\right)^{\nu(x_1)} + \left(\frac{t}{\theta(x_2)}\right)^{\nu(x_2)} - \left(\frac{t_1}{\theta(x_2)}\right)^{\nu(x_2)}\right\}, & t \ge t_1, \end{cases} \tag{100}$$

where $\theta(x) = \left(\frac{\nu(x)}{r(x)}\right)^{1/\nu(x)}$.

General step-stresses

If $x(\cdot) \in E_m$, then for all $t \in [t_{i-1}, t_i)$

$$S_{x(\cdot)}(t) = G\left\{\sum_{j=1}^{i-1}\left[\left(\frac{t_j}{\theta(x_j)}\right)^{\nu(x_j)} - \left(\frac{t_{j-1}}{\theta(x_j)}\right)^{\nu(x_j)}\right]\right.$$

$$\left. + \left(\frac{t}{\theta(x_i)}\right)^{\nu(x_i)} - \left(\frac{t_{i-1}}{\theta(x_i)}\right)^{\nu(x_i)}\right\} =$$

$$S_{x_i}\left\{\theta(x_i)\left(\sum_{j=1}^{i-1}\left[\left(\frac{t_j}{\theta(x_j)}\right)^{\nu(x_j)} - \left(\frac{t_{j-1}}{\theta(x_j)}\right)^{\nu(x_j)}\right] + \right.\right.$$

$$\left.\left.\left(\frac{t}{\theta(x_i)}\right)^{\nu(x_i)} - \left(\frac{t_{i-1}}{\theta(x_i)}\right)^{\nu(x_i)}\right)^{1/\nu(x_i)}\right\}. \tag{101}$$

10. Generalizations

Schabe and Viertl (1995) considered an axiomatic approach to model building.

Proposition 13. (Schabe and Viertl (1995)). *Suppose that there exists a functional*

$$a : E \times E \times [0,\infty) \to [0,\infty)$$

such that for any $x_1(\cdot), x_2(\cdot) \in E$ it is differentiable and increasing in t,

$$a(x_1(\cdot), x_2(\cdot), 0) = 0$$

and

$$T_{x_2(\cdot)} \sim a(x_1(\cdot), x_2(\cdot), T_{x_1(\cdot)}),$$

where $\sim$ denotes equality in distribution.

For any differentiable on $[0,\infty)$ c.d.f. F exists a functional $b : E \times [0,\infty) \to [0,\infty)$ such that for all $x(\cdot) \in E$

$$F_{x(\cdot)}(t) = F\left(\int_0^t b(x(\cdot), u)du\right). \tag{102}$$

Proof. Fix $x_0(\cdot) \in E$ and for all $x(\cdot) \in E$ put

$$a_0(x(\cdot), t) = F^{-1}(F_{x_0(\cdot)}(a(x(\cdot), x_0(\cdot), t))).$$

The distribution of the random variable $R = a_0(x(\cdot), T_{x(\cdot)})$ does not depend on $x(\cdot)$ and its c.d.f. is F. Put

$$b(x(\cdot), t) = \frac{\partial}{\partial t} a_0(x(\cdot), t).$$

Then

$$a_0(x(\cdot),t) = \int_0^t b(x(u),u)du,$$

which implies

$$F_{x(\cdot)}(t) = \mathbf{P}\{T_{x(\cdot)} \le t\} = \mathbf{P}\{R \le a_0(x(\cdot),t)\} = F\left(a_0(x(\cdot),t)\right) = F\left(\int_0^t b(x(u),u)du\right).$$

The proof is complete.

Remark 8. Put

$$G(t) = 1 - F(t), \quad S_{x(\cdot)}(t) = 1 - F_{x(\cdot)}(t), \quad H = G^{-1}, \quad f^G_{x(\cdot)}(t) = H(S_{x(\cdot)}(t)).$$

The equality (102) implies that

$$\frac{\partial}{\partial t} f^G_{x(\cdot)}(t) = b(x(\cdot),t). \tag{103}$$

This model means that the rate of the $G-$resourse usage is a functional of the stress and the time.

The above considered models are submodels of this general model:

1) If $b(x(\cdot),t) = r(x(t))$, we have the AAD model.

2) If $b(x(\cdot),t) = r(x(t))\,\alpha_0(t)$, we have the GM (or, equivalently, GPH1) model.

3) If $b(x(\cdot),t) = r(x(t))\,\alpha_0(t)$ and the resource is exponential, i.e. $G(t) = e^{-t}, t \ge 0$, we have the PH model.

4) If $b(x(\cdot),t) = r(x(t))t^{\nu(x(t))-1}$, we have the CHSS model.

Considering the GS model, it was noted that this (and also AAD) model is not appropriate when the stress is periodic with quick change of its values. Greater is the number of stress cycles, shorter is the life of items. So the effect of cycling must be included in the model.

Suppose that a periodic stress is differentiable. Then the number of cycles in the interval $[0,t]$ is

$$n(t) = \int_0^t \mid d\mathbf{1}\{x'(u) > 0\} \mid .$$

Generalizing the GPH1 (or GM) model we suppose that the G-resource used until the moment t has the form

$$f^G_{x(\cdot)}(t) = \int_0^t r_1\{x(u)\}dH(S_0(u)) + \int_0^t r_2\{x(u)\}d \mid \mathbf{1}\{x'(u) > 0\} \mid \tag{104}$$

The second term includes the effect of cycling on resource usage. In terms of survival functions

$$S_{x(\cdot)}(t) = G\left\{\int_0^t r_1\{x(u)\}dH(S_0(u)) + \int_0^t r_2\{x(u)\} \mid d\mathbf{1}\{x'(u) > 0\}\right\}. \tag{105}$$

If amplitude is constant, $r_2\{x(u)\} = c$ can be considered.

The AAD model is generalised by the model

$$S_{x(\cdot)}(t) = G\left\{\int_0^t r_1\{x(u)\}du + \int_0^t r_2\{x(u)\} \mid d\mathbf{1}\{x'(u) > 0\} \mid\right\}. \tag{106}$$

The GS and AAD models are not appropriate if $x(\cdot)$ is a step stress with many switch on's and switch off's which shorten the life of items. In this case the following model can be considered:

$$f^G_{x(\cdot)}(t) = \int_0^t r_1\{x(u)\}dH(S_0(u)) + \int_0^t r_2\{x(u)\}\mathbf{1}(\Delta x(u) > 0)\frac{\mid dx(u) \mid}{\mid \Delta x(u) \mid}$$

$$+ \int_0^t r_3\{x(u)\}\mathbf{1}(\Delta x(u) < 0)\frac{\mid dx(u) \mid}{\mid \Delta x(u) \mid}. \tag{107}$$

The second and the third terms include the effect of switch-on's and switch-off's (or *vice versa*), respectively, on resource usage. If the step-stress has two values, the functions r_2 and r_3 can be constants.

11. The heredity hypothesis

A process of production is *unstable* if reliability of items produced in different time intervals are different. If items produced in some specified time interval are considered and the AAD, GM (GPH1) or GA models hold on E_0, then for all $x_1, x_2 \in E_0$

$$S_{x_2}(t) = G_{x_1}(\rho(x_1, x_2)t), \tag{108}$$

$$S_{x_2}(t) = G(\rho(x_1, x_2)H(S_{x_1}(t))), \tag{109}$$

$$S_{x_2}(t) = G(H(S_{x_1}(t) + b(x_1, x_2))), \tag{110}$$

respectively.

Definition 13. *If one of the models AAD, GM or GA holds, the process of production is unstable and the function $\rho(x_1, x_2)$ (the models AAD or GM) or $b(x_1, x_2)$ (the model GA) is invariant for groups of items produced in different time intervals, then we'll say that the heredity hypothesis is satisfied.*

Suppose that x_1 is a usual stress and $x_2 > x_1$ an accelerated stress.

If one of the models AAD, GM or GA and the heredity principle hold, then sufficiently large data can be accumulated during a long period of observations and good estimators of the functions $\rho(x_1, x_2)$ or $b(x_1, x_2)$ can be obtained. The reliability of *newly produced items* under the "usual" stress x_1 can be estimated from accelerated life data obtained under the accelerated stress x_2, using the estimators $\hat\rho(x_1, x_2)$ or $\hat b(x_1, x_2)$.

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