On Goodness-of-Fit in Accelerated Life Testing

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Abstract. Goodness-of-fit test for the generalized Sedyakin's model is proposed when accelerated experiments are done under step-stresses. Various alternatives are considered. Power of the test against the approaching alternatives is investigated.

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1 Introduction

In accelerated life testing (ALT) units are tested at higher-then-usual levels of stress to induce early failures. The results are extrapolated to estimate the lifetime distribution at the design stress using models which relate the lifetime to the stress. The most general accelerated life model is the generalized Sedyakin's model (see Bagdonavičius (1978), Nelson (1990), Bagdonavičius and Nikulin (1999). We'll consider a goodness-of-fit test for this model when the data are obtained from accelerated experiments.

To see the possible alternatives to this model consider the most important accelerated life models and their relations.

Suppose that stresses are deterministic time functions:

$$x(\cdot) = (x_1(\cdot), ..., x_m(\cdot))^T \ : [0, \infty) \to B \in R^m.$$

We suppose that $x(\cdot)$ are right-continuous. If $x(\cdot)$ is constant in time, we'll write x instead of $x(\cdot)$ in all formulae.

Denote by $T_{x(\cdot)}$ the time-to-failure, by $S_{x(\cdot)}(t) = P\{T_{x(\cdot)} > t\}$ the survival function, by

$$\alpha_{x(\cdot)}(t) = \lim_{h \downarrow 0} \frac{1}{h} P\{T_{x(\cdot)} \in (t, t+h] \mid T_{x(\cdot)} > t\}$$

the hazard rate and by

$$A_{x(\cdot)}(t) = \int_0^t \alpha_{x(\cdot)}(u) du = -ln\{S_{x(\cdot)}(t)\}$$

the accumulated hazard rate under the stress $x(\cdot)$. We write $x(\cdot) < y(\cdot)$ if $S_{x(\cdot)}(t) > S_{y(\cdot)}(t)$ for all $t > 0$.

We formulate the models in terms of the hazard rate $\alpha_{x(\cdot)}(t)$ which for any moment t caracterizes the risk of failure just after this moment for the units which survived until t.

Definition 1. *The generalized Sedyakin's (GS) model (Bagdonavičius (1978)) holds on a set of stresses E if there exist a positive on $E \times R^+$ function g such that for all $x(\cdot) \in E$*

$$\alpha_{x(\cdot)}(t) = g\left(x(t), S_{x(\cdot)}(t)\right).$$

Thus the hazard rate at the moment t is a function of the value of the stress at this moment and of the probability of survival until this moment.

Equivalently, the model can be written in the form

$$\alpha_{x(\cdot)}(t) = g_1\left(x(t), S_{x(\cdot)}(t)\right)$$

with $g_1(x, s) = g(x, exp\{-s\})$.

Note that GS model does not state the presence of any connections between the survival or hazard rate functions under different constant stresses, because for any hazard rates $\alpha_x(t)$ under constant stresses $x \in E_0$ we can take $g_1(x, s) = \alpha_x(A_x^{-1}(s))$ and thus the GS model holds on E_0. So the GS model shows only the influence of time-variability of stresses on survival.

In terms of hazard rates and accumulated hazards under time varying and constant in time stresses this model has the form

$$\alpha_{x(\cdot)}(t) = \alpha_{x(t)}\left(A_{x(t)}^{-1}(A_{x(\cdot)}(t))\right).$$

in accelerated life testing The most used time-varying stresses in accelerated life testing are the step-stresses: units are placed on test at an initial low stress and if they do not fail in a predetermined time t_1, the stress is increased. If they do not fail in a predetermined time $t_2 > t_1$, the stresses is increased once more, and so on.

Consider a set E_m of step-stresses of the form

$$x(\tau) = \begin{cases} x_1, & 0 \le \tau < t_1, \\ x_2, & t_1 \le \tau < t_2, \\ \dots & \dots \\ x_m, & t_{m-1} \le \tau < t_m. \end{cases} \tag{1}$$

Set $t_0 = 0$.

If the GS model holds on E_m then the hazard rate function $\alpha_{x(\cdot)}(t)$ and the survival function $S_{x(\cdot)}(t)$ verify the equalities:

$$\alpha_{x(\cdot)}(t) = \alpha_{x_i}(t - t_{i-1} + t_{i-1}^*), \quad \text{if} \quad t \in [t_{i-1}, t_i),$$

$$S_{x(\cdot)}(t) = S_{x_i}(t - t_{i-1} + t_{i-1}^*), \quad \text{if} \quad t \in [t_{i-1}, t_i)\ (i = 1, 2, \dots, m), \tag{2}$$

where where t_i^* can be found by solving the equations

$$S_{x_1}(t_1) = S_{x_2}(t_1^*), \dots, S_{x_i}(t_i - t_{i-1} + t_{i-1}^*) = S_{x_{i+1}}(t_i^*)\ (i = 1, \dots, m-1). \tag{3}$$

Thus the survival functions under step-stresses are obtained from the survival functions under constant in time stresses by the *rule of time shift.*

Note also that

$$S_{x(\cdot)}(t_i) = S_{x_{i+1}}(t_i^*).$$

The moment t_i under the stress $x(\cdot)$ is equivalent to the moment t_i^* under the stress x_{i+1}.

Definition 2. *The additive accumulation of damages (AAD) model (Bagdonavičius (1978)) holds on E if there exists a positive function r on E and a positive on $[0, \infty)$ function q such that for all $x(\cdot) \in E$*

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\, q\{S_{x(\cdot)}(t))\}.$$

The AAD model implies that under constant in time stresses survival functions have the form

$$S_x(t) = S_0\{r(x)t\},$$

where the function S_0 does not depend on x, i.e. survival functions under different constant in time stresses differ only in scale parameters.

If the AAD model holds on E_m then the hazard rate function $\alpha_{x(\cdot)}(t)$ and the survival function $S_{x(\cdot)}(t)$ verify the equalities (2) with t_i^* determined by the equalities

$$t_i^* = \frac{1}{r(x_{i+1})} \sum_{j=1}^{i} r(x_j)(t_j - t_{j-1}), \quad \text{if} \quad t \in [t_{i-1}, t_i).$$

Thus the AAD model is a restriction of the GS model when strict connection between survival functions under different constant in time stresses are assumed. The survival functions under step-stresses are obtained from the survival functions under constant in time functions by the same rule of time shift as in the case of GS model.

The GS model seems very natural but it not means that it is universal. Consider some alternatives to it. In 1972 Sir D.Cox proposed the *proportional hazards* (PH) model for constant in time covariates (stresses): for all $x \in E_0$

$$\alpha_x(t) = r\{x\}\ \alpha_0(t),$$

where the function α_0 does not depend on x.

For constant stresses the survival functions are from the class of functions $\{S_0^{r(x)}(t)\}$, where

$$S_0(t) = \exp\{-\int_0^t \alpha_0(u)du\}.$$

In the statistical literature the following *formal* generalization of the PH model is often used.

Definition 3. *The proportional hazards model holds on the set of time-varying and constant stresses E if for all $x(\cdot) \in E$*

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\ \alpha_0(t). \tag{4}$$

Under the step-stresses the *time-shift rule does not take place*:

$$\alpha_{x(\cdot)}(t) = \alpha_{x_i}(t), \quad \text{if} \quad t \in [t_{i-1}, t_i),\ (i = 1, 2, \ldots, m). \tag{5}$$

The intensity under the stress x_i in the interval $[t_{i-1}, t_i)$ does not depend on stresses which were used until the moment t_{i-1}. This is not natural when items are aging. Thus application of the PH model when stresses are not constant in time must be done with great care.

One of possible causes when the time shift property is not satisfied, can be the presence of unobservable time-varying stresses. For example, suppose that not only the stress $x(\cdot)$ but also some non-observable stress $y(\cdot)$ influences the reliability of items and on the set of stresses of the form $z(\cdot) = (x(\cdot), y(\cdot))$ the AAD model with exponential time-to-failure distributions under constant in time stresses holds:

$$\alpha_{z(\cdot)}(t) = r_1\{x(t)\}\ r_2\{y(t)\}, \quad \alpha_z(t) = r_1\{x\}\ r_2\{y)\}.$$

Then given $y(\cdot)$ the PH model holds on the set of stresses of the form $x(\cdot)$ (with $\alpha_0(t) = r_2\{y(t)\}$) and times-to-failure under constant x are not necessary exponential.

Now we shall "repair" the PH model with time-varying stresses to obtain a restriction of the GS model which coincides with the PH model for constant stresses and has the time shift property for step-stresses.

Definition 4. *The first generalized proportional hazards (GPH1) model holds on E if for all $x(\cdot) \in E$*

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\ \alpha_0\left(A_0^{-1}\left(\frac{A_{x(\cdot)}(t)}{r\{x(t)\}}\right)\right), \tag{6}$$

were the function α_0 does not depend on x and

$$A_0(t) = \int_0^t \alpha_0(u)du.$$

Under constant in time stresses the model (6) implies the PH model:

$$\alpha_x(t) = r(x)\ \alpha_0(t).$$

If $x(\cdot)$ is the step-stress (1) then the hazard rate function $\alpha_{x(\cdot)}(t)$ and the survival function $S_{x(\cdot)}(t)$ verify the equalities (2) with t_i^* defined by the equalities

$$t_1^* = A_0^{-1}\left(\frac{r(x_1)}{r(x_2)}A_0(t_1)\right), \quad t_i^* = A_0^{-1}\left(\frac{r(x_i)}{r(x_{i+1})}A_0(t_i - t_{i-1} + t_{i-1}^*)\right).$$

Consider more general then PH models which do not verify the time shift property.

Definition 5. *The second generalized proportional hazards (GPH2) model (Bagdonavičius & Nikulin (1998)) holds on E if for all $x(\cdot) \in E$*

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\, q\{S_{x(\cdot)}(t)\}\, \alpha_0(t).$$

The particular cases of the GPH1 model are the PH model ($q(u) \equiv 1$) and the AAD model ($\alpha_0(t) \equiv \alpha_0 = const$).

Suppose that not only the stress $x(\cdot)$ but also some non-observable stress $y(\cdot)$ influences the reliability of items and on the set of stresses of the form $z(\cdot) = (x(\cdot), y(\cdot))$ the AAD model holds with not necessary exponential time-to-failure distributions under constant in time stresses:

$$\alpha_{z(\cdot)}(t) = r_1\{x(t)\}\, r_2\{y(t)\}\, q\{S_{z(\cdot)}(t)\}.$$

Then given $y(\cdot)$ the GPH2 model holds on the set of stresses of the form $x(\cdot)$ (with $\alpha_0(t) = r_2(y(t))$). Thus for time-varying covariates the GPH2 model is much more natural then the PH model. If the GPH2 model holds on a set of general step-stresses E_m, then for any $t \in [t_{i-1}, t_i)$

$$S_{x(\cdot)}(t) = G\left\{ (H(S_{x_i}(t)) - H(S_{x_i}(t_{i-1}))) + \sum_{j=1}^{i-1} \frac{r(x_j)}{r(x_i)} (H(S_{x_i}(t_j)) - H(S_{x_i}(t_{j-1}))) \right\},$$

where

$$H(u) = \int_u^1 \frac{dv}{vq(v)}, \quad , G = H^{-1}.$$

Definition 6. *The third generalized proportional hazards (GPH3) model holds on E if for all $x(\cdot) \in E$*

$$\alpha_{x(\cdot)}(t) = u\{x(t), S_{x(\cdot)}(t)\}\, \alpha_0(t).$$

The particular cases of the GPH3 model are the GS model ($\alpha_0(t) \equiv \alpha_0 = const$) and GPH2 model ($u(x,s) = r(x)\, q(s)$).

The numerous examples of real data show that taking two constant in time stresses, say x_1 and x_2, the ratio $\alpha_{x_2}(t)/\alpha_{x_1}(t)$ (which is constant under the PH model), can be *increasing or decreasing* in time and even a *cross-effect* of hazard rates can be observed. Such data can be modeled by sub-models of the GPH2 or more general GPH3 model.

An alternative to the GS model under step-stresses can be obtained by taking into account the influence of switch-off's of stresses on reliability of items. Switch-off's can imply failures of items. Suppose that an item is observed under the stress (1) and after the switch-off at the moment t_i from the stress x_i to the stress x_{i+1} the survival function has a jump:

$$S_{x(\cdot)}(t_i) = S_{x(\cdot)}(t_i-)\, \delta_i;$$

here δ_i is the probability for an item not to fail because of the switch-off at the moment t_i. In this case the GS model for step-stresses can be modified as follows:

$$S_{x(\cdot)}(t) = S_{x_i}(t - t_{i-1} + t^{**}_{i-1})), \tag{7}$$

where

$$t_1^{**} = S_{x_2}^{-1}\{S_{x_1}(t_1)\, \delta_1\}, \quad t_i^{**} = S_{x_{i+1}}^{-1}\{S_{x_i}(t_i - t_{i-1} + t^{**}_{i-1})\, \delta_i\}. \tag{8}$$

Thus the time shift is modified by the jumps.

Goodness-of-fit tests for the PH model are numerous (see, for example, Wei (1984), Lin (1991), Lin, Wei and Ying (1993)). Tests for the AAD model are considered in Bagdonavičius (1978, 1990). Tests for the GPH models are considered in Bagdonavičius and Nikulin (2000). A test for the GS model in the case of a special experiment with simple step-stress is considered in Bagdonavičius and Nikoulina (1997).

We consider a test for the GS model from the data of accelerated experiments with multistage step-stresses. Power of the test against several important alternatives is investigated.

2 Test statistic for the GS model

Suppose that a group of n_0 units is tested under the step-stress (1) and m groups of $n_1, \cdots, n_m$ items are tested under constant in time stresses $x_1 \cdots, x_m$ $(x_1 < \cdots < x_m)$, respectively. The units are observed time t_m given for the experiment.

The idea of goodness-of-fit is based on comparing two estimators $\hat{A}^{(1)}_{x(\cdot)}$ and $\hat{A}^{(2)}_{x(\cdot)}$ of the accumulated hazard rate $A_{x(\cdot)}$. One estimator can be obtained from the experiment under the step-stress (1) and another from the experiments under the stresses $x_1, \cdots, x_m$ by using the equalities (2) and (3).

Denote by $N_i(t)$ and $Y_i(t)$ the numbers of observed failures in the interval $[0, t]$ and the number of items at risk just prior the moment t, respectively, for the group of items tested under the stress x_i and $N(t)$, $Y(t)$ the analogous numbers for the group of items tested under the stress $x(\cdot)$.

Set

$$\alpha_i = \alpha_{x_i}, \quad \alpha = \alpha_{x(\cdot)}, \quad A_i = A_{x_i}, \quad A = A_{x(\cdot)} \quad (i = 1, ..., m).$$

If the GS model holds on $E = \{x_1, \cdots, x_m, x(\cdot)\}$, then the accumulated hazards A can be written in terms of accumulated hazards A_i (cf. (2) and (3)):

$$A(t) = A_{i+1}(t - t_i + t_i^*), \ t \in [t_i, t_{i+1}) \ (i = 0, ..., m-1), \tag{9}$$

where

$$t_0^* = 0, \quad t_1^* = A_2^{-1}(A_1(t_1)), \cdots,$$
$$t_{i+1}^* = A_{i+2}^{-1}(A_{i+1}(t_{i+1} - t_i + t_i^*)) \quad (i = 0, ..., m-2). \tag{10}$$

The first estimator $\hat{A}^{(1)}$ of the accumulated hazard A can be the well known Nelson -Aalen estimator obtained from the experiment under the step-stress (1):

$$\hat{A}^{(1)}(t) = \int_0^t \frac{dN(v)}{Y(v)}.$$

The second is suggested by the GS model (formulae (9) and (10)) and is obtained from the experiments under constant stresses:

$$\hat{A}^{(2)}(t) = \hat{A}_{i+1}(t - t_i + \hat{t}_i^*), \ t \in [t_i, t_{i+1}), \ (i = 0, ..., m-1),$$

where

$$\hat{t}_0^* = 0, \quad \hat{t}_1^* = \hat{A}_2^{-1}(\hat{A}_1(t_1)), \ \cdots, \ \hat{t}_{i+1}^* = \hat{A}_{i+2}^{-1}(\hat{A}_{i+1}(t_{i+1} - t_i + \hat{t}_i^*)), \quad (i = 0, ..., m-2),$$
$$\hat{A}_i^{-1}(s) = \inf\{u : \hat{A}_i(u) \geq s\},$$
$$\hat{A}_i(t) = \int_0^t \frac{dN_i(v)}{Y_i(v)} \quad (i = 1, ..., m).$$

The test is based on the statistic

$$T_n = \int_0^{t_m} K(v)\, d\{\hat{A}^{(1)}(t) - \hat{A}^{(2)}(t)\}, \tag{11}$$

where K is the weight function.

We'll consider the weight functions of the type: for $v \in [t_i, t_i + \Delta t_i)$

$$K(v) = \frac{1}{\sqrt{n}} \frac{Y(v) Y_{i+1}(v - t_i + \hat{t}_i^*)}{Y(v) + Y_{i+1}(v - t_i + \hat{t}_i^*)}\, g\left(\frac{Y(v) + Y_{i+1}(v - t_i + \hat{t}_i^*)}{n}\right),$$

where g is nonnegative bounded continuous function with bounded variation on $[0, 1]$ and $n = \sum_{i=0}^m n_i$.

Take notice that the properties of this statistic are different from the properties of the logrank-type statistics used testing hypothesis of the equality of survival functions. The properties of T_n would be similar if t_i^* would be known. The problem is that the points t_i^* are unknown and are estimated. Thus when seeking the limit distribution of the statistic T_n we must take in mind that the estimators $\hat{A}_i$ are approaching A_i with the same rate as the estimators $\hat{t}_i^*$ are approaching t_i^*.

The condition $x_1 < \cdots < x_m$ implies that

$$\mathbf{P}\{T_n \quad \text{is defined}\} \to 1 \quad \text{as} \quad n_i \to \infty.$$

3 Asymptotic distribution of the test statistic

To find the asymptotic distribution of the test statistic, consider at first the asymptotic distribution of the estimators $\hat{t}_i^*$. Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution.

Assumptions A.
a) The hazard rates α_i are positive and continuous on $(0, \infty)$;
b) $A_i(t) < \infty$ for all $t < 0$;
c) $n \to \infty, \quad n_i/n \to l_i, \quad l_i \in (0,1)$.

Lemma. *Suppose that Assumptions A hold. Then*

$$\sqrt{n}(\hat{t}_j^* - t_j^*) \xrightarrow{\mathcal{D}} a_j \sum_{l=1}^{j} d_{jl}\{U_l(t_{l-1}^* + \Delta t_{l-1}) - U_{l+1}(t_l^*)\}, \tag{12}$$

where

$$d_{jl} = \prod_{s=l}^{j-1} c_s, \quad l = 1, ..., j-1; \quad d_{jj} = 1,$$

$$a_j = \frac{1}{\alpha_{j+1}(t_j^*)}, \quad c_s = \frac{\alpha_{s+1}(t_s^* + \Delta t_s)}{\alpha_{s+1}(t_s^*)},$$

$U_1, \cdots, U_m$ *and* U *are independent Gaussian martingales with* $U_i(0) = U(0) = 0$ *and*

$$\mathbf{Cov}\,(U_i(s_1), U_i(s_2)) = \frac{1}{l_i} \frac{1 - S_i(s_1 \wedge s_2)}{S_i(s_1 \wedge s_2)} := \sigma_i^2(s_1 \wedge s_2),$$

$$\mathbf{Cov}\,(U(s_1), U(s_2)) = \frac{1}{l_0} \frac{1 - S(s_1 \wedge s_2)}{S(s_1 \wedge s_2)} := \sigma^2(s_1 \wedge s_2)$$

with $S_i = \exp\{-A_i\}$, $S = \exp\{-A\}$.

Proof. Under Assumptions A for any $t \in (0, t_m)$ the estimators $\hat{A}_i$ and $\hat{A}^{(1)}$ are uniformly consistent on $[0, t]$, and

$$\sqrt{n}(\hat{A}_i - A_i) \xrightarrow{\mathcal{D}} U_i, \quad \sqrt{n}(\hat{A}^{(1)} - A) \xrightarrow{\mathcal{D}} U \tag{13}$$

on $D[0, t]$, the space of cadlag functions on $[0, t]$ with Skorokhod metric.

We prove (12) by recurrence. If $i = 1$ then

$$\sqrt{n}(\hat{t}_1^* - t_1^*) = \sqrt{n}(\hat{A}_2^{-1}(\hat{A}_1(t_1)) - A_2^{-1}(\hat{A}_1(t_1))) + \sqrt{n}(A_2^{-1}(\hat{A}_1(t_1)) - A_2^{-1}(A_1(t_1))). \tag{14}$$

For any $0 < s_1 < s_2 < \infty$ (see Andersen et al (1993))

$$\sqrt{n}(\hat{A}_2^{-1} - A_2^{-1}) \xrightarrow{\mathcal{D}} U_2^* \tag{15}$$

on $D[s_1, s_2]$, where

$$U_2^*(s) = -\frac{e^{-s} U_2(A_2^{-1}(s))}{p_2(A_2^{-1}(s))}$$

and p_i is the density of T_{x_i}. Note that

$$U_2^*(A_1(t_1)) = -\frac{U_2(t_1^*)}{\alpha_2(t_1^*)}. \tag{16}$$

Consistence of the estimator $\hat{A}_1(t_1)$, the convergence (15) and the formula (16) imply that

$$\sqrt{n}\{\hat{A}_2^{-1}(\hat{A}_1(t_1)) - A_2^{-1}(\hat{A}_1(t_1))\} \xrightarrow{\mathcal{D}} -\frac{U_2(t_1^*)}{\alpha_2(t_1^*)} = -a_1 U_2(t_1^*). \tag{17}$$

Using the delta method and the convergence (13), we obtain

$$\sqrt{n}\{A_2^{-1}(\hat{A}_1(t_1)) - A_2^{-1}(A_1(t_1))\} \xrightarrow{\mathcal{D}} \frac{1}{\alpha_2(t_1^*)} U_1(t_1) = a_1 U_1(t_0^* + \Delta t_0), \tag{18}$$

Thus (14), (17) and (18) imply that

$$\sqrt{n}(\hat{t}_1^* - t_1^*) \xrightarrow{\mathcal{D}} a_1 d_{11}\{U_1(t_0^* + \Delta t_0) - U_2(t_1^*)\}.$$

Suppose that (12) holds for $i = j$. Then similarly as in the case $i = 1$ we have

$$\begin{aligned}\sqrt{n}(\hat{t}_{j+1}^* - t_{j+1}^*) = \sqrt{n}\{\hat{A}_{j+2}^{-1}(\hat{A}_{j+1}(\hat{t}_j^* + \Delta t_j)) - A_{j+2}^{-1}(A_{j+1}(t_j^* + \Delta t_j))\} = \\ \sqrt{n}\{\hat{A}_{j+2}^{-1}(\hat{A}_{j+1}(\hat{t}_j^* + \Delta t_j)) - A_{j+2}^{-1}(\hat{A}_{j+1}(\hat{t}_j^* + \Delta t_j))\} + \\ \sqrt{n}\{A_{j+2}^{-1}(\hat{A}_{j+1}(\hat{t}_j^* + \Delta t_j)) - A_{j+2}^{-1}(A_{j+1}(\hat{t}_j^* + \Delta t_j))\} = \\ \sqrt{n}\{A_{j+2}^{-1}(A_{j+1}(\hat{t}_j^* + \Delta t_j)) - A_{j+2}^{-1}(A_{j+1}(t_j^* + \Delta t_j))\} = \\ a_{j+1}\{U_{j+1}(t_j^* + \Delta t_j) - U_{j+2}(t_{j+1}^*)\} + a_{j+1}\frac{c_j}{a_j}\sqrt{n}(\hat{t}_j^* - t_j^*) + \Delta_n,\end{aligned}$$

where $\Delta_n \xrightarrow{P} 0$ as $n \to 0$. The last formula and the assumption of recurrency imply that

$$\begin{aligned}\sqrt{n}(\hat{t}_{j+1}^* - t_{j+1}^*) \xrightarrow{\mathcal{D}} \\ a_{j+1}\{U_{j+1}(t_j^* + \Delta t_j) - U_{j+2}(t_{j+1}^*)\} \\ + a_{j+1}\frac{c_j}{a_j} a_j \left\{\sum_{l=1}^{j-1}\prod_{s=l}^{j-1} c_s\{U_l(t_{l-1}^* + \Delta t_{l-1}) - U_{l+1}(t_l^*)\} + U_j(t_{j-1}^* + \Delta t_{j-1}) - U_{j+1}(t_j^*)\right\} = \\ a_{j+1}\left\{U_{j+1}(t_j^* + \Delta t_j) - U_{j+2}(t_{j+1}^*) + \sum_{l=1}^{j}\prod_{s=l}^{j} c_s\{U_l(t_{l-1}^* + \Delta t_{l-1}) - U_{l+1}(t_l^*)\}\right\} = \\ a_{j+1}\sum_{l=1}^{j+1} d_{j+1,l}\{U_l(t_{l-1}^* + \Delta t_{l-1}) - U_{l+1}(t_l^*)\}.\end{aligned}$$

The proof is complete.

Consider the limit distribution of the statistic T_n. Note that

$$\frac{K(v)}{\sqrt{n}} \xrightarrow{\mathbf{P}} k(v) = \frac{l_0 l_{i+1}}{l_0 + l_{i+1}} S(v)\, g\left((l_0 + l_{i+1})S(v)\right).$$

Set

$$e_j = a_j\{k(t_j)\ \alpha_{j+1}(t_j^*) - k(t_j + \Delta t_j)\ \alpha_{j+1}(t_j^* + \Delta t_j)$$

$$+ \int_{t_j^*}^{t_j^*+\Delta t_j} \alpha_{i+1}(t_i^* + \Delta t_i)(v) d\, k(v + t_j - t_j^*) \quad (j = 1, ..., m-1),$$

and $f_0 = f_m = 0$, $f_i = \sum_{j=i}^{m-1} e_j d_{ji}$, $(i = 1, ..., m-1)$.

Theorem 1. *Under Assumptions A*

$$T_n \xrightarrow{\mathcal{D}} \int_0^{t_m} k(v) d\, U(v) + \sum_{i=0}^{m-1} \{f_{i+1} U_{i+1}(t_i^* + \Delta t_i) - f_i U_{i+1}(t_i^*)$$

$$- \int_{t_i^*}^{t_i^*+\Delta t_i} k(v + t_i - t_i^*) d\, U_{i+1}(v)\}. \tag{19}$$

Proof. Write the statistic (11) in the form

$$T_n = \int_0^{t_m} K(v)\, d\{\hat{A}^{(1)}(t) - A(t)\}$$

$$+ \sum_{i=1}^{m-1} \int_{t_i^*}^{\hat{t}_i^*} K(v + t_i - \hat{t}_i^*)\, d\hat{A}_{i+1}(v) - \sum_{i=1}^{m-1} \int_{t_i^*+\Delta t_i}^{\hat{t}_i^*+\Delta t_i} K(v + t_i - \hat{t}_i^*)\, d\hat{A}_{i+1}(v)$$

$$- \sum_{i=1}^{m-1} \int_{t_i^*}^{t_i^*+\Delta t_i} \{K(v + t_i - \hat{t}_i^*) - K(v + t_i - t_i^*)\}\, d\hat{A}_{i+1}(v)$$

$$- \sum_{i=0}^{m-1} \int_{t_i^*}^{t_i^*+\Delta t_i} K(v + t_i - t_i^*)\, d\{\hat{A}_{i+1}(v) - A_{i+1}(v)\}.$$

It is easy to show that under Assumptions A

$$\int_0^{t_m} K(v)\, d\{\hat{A}^{(1)}(t) - A(t)\} = \int_0^{t_m} k(v)\, d\, U(v) + o_p(1),$$

$$\int_{t_i^*}^{\hat{t}_i^*} K(v + t_i - \hat{t}_i^*)\, d\hat{A}_{i+1}(v) = k(t_i)\, \alpha_{i+1}(t_i^*)\, \sqrt{n}(\hat{t}_i^* - t_i^*) + o_p(1),$$

$$\int_{t_i^*+\Delta t_i}^{\hat{t}_i^*+\Delta t_i} K(v + t_i - \hat{t}_i^*)\, d\, \hat{A}_{i+1}(v) = k(t_i + \Delta t_i)\, \alpha_{i+1}(t_i^* + \Delta t_i)\, \sqrt{n}(\hat{t}_i^* - t_i^*) + o_p(1),$$

$$\int_{t_i^*}^{t_i^*+\Delta t_i} \{K(v+t_i-\hat{t}_i^*) - K(v+t_i-t_i^*)\}\, d\, \hat{A}_{i+1}(v) = -\int_{t_i^*}^{t_i^*+\Delta t_i} \alpha_{i+1}(v)\, d\, k(v+t_i-t_i^*)\, \sqrt{n}(\hat{t}_i^* - t_i^*) + o_p(1),$$

$$\int_{t_i^*}^{t_i^*+\Delta t_i} K(v + t_i - t_i^*)\, d\{\hat{A}_{i+1}(v) - A_{i+1}(v)\} = \int_{t_i^*}^{t_i^*+\Delta t_i} k(v + t_i - t_i^*)\, d\, U_{i+1}(v) + o_p(1),$$

where $o_p(1) \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$. So the statistic T_n can be written in the form:

$$T_n = \int_0^{t_m} k(v) dU(v) + \sum_{i=1}^{m-1} k(t_i) \alpha_{i+1}(t_i^*) \sqrt{n}(\hat{t}_i^* - t_i^*) -$$

$$\sum_{i=1}^{m-1} k(t_i + \Delta t_i)\, \alpha_{i+1}(t_i^* + \Delta t_i)\, \sqrt{n}(\hat{t}_i^* - t_i^*) +$$

$$\sum_{i=1}^{m-1}\int_{t_i^*}^{t_i^*+\Delta t_i}\alpha_{i+1}(v)\,d\,k(v+t_i-t_i^*)\sqrt{n}(\hat t_i^*-t_i^*)-\sum_{i=0}^{m-1}\int_{t_i^*}^{t_i^*+\Delta t_i}k(v+t_i-t_i^*)\,dU_{i+1}(v)+o_p(1). \quad (20)$$

The lemma implies that

$$\sum_{i=1}^{m-1}\{k(t_i)\alpha_{i+1}(t_i^*)-k(t_i+\Delta t_i)\alpha_{i+1}(t_i^*+\Delta t_i)+\int_{t_i^*}^{t_i^*+\Delta t_i}\alpha_{i+1}(v)d\,k(v+t_i-t_i^*)\}\sqrt{n}(\hat t_i^*-t_i^*)=$$

$$\sum_{i=1}^{m-1}e_i\sum_{l=1}^{i}d_{il}\{U_l(t_{l-1}^*+\Delta t_{l-1})-U_{l+1}(t_l^*)\}+o_p(1)$$

$$=\sum_{l=1}^{m-1}\left(\sum_{i=l}^{m-1}e_i d_{il}\right)\{U_l(t_{l-1}^*+\Delta t_{l-1})-U_{l+1}(t_l^*)\}+o_p(1)$$

$$=\sum_{l=1}^{m-1}f_l\{U_l(t_{l-1}^*+\Delta t_{l-1})-U_{l+1}(t_l^*)\}+o_p(1)$$

$$=\sum_{i=0}^{m-2}f_{i+1}U_{i+1}(t_i^*+\Delta t_i)-\sum_{i=1}^{m-1}f_iU_{i+1}(t_i^*)+o_p(1). \quad (19)$$

The formulae (20) and (21) imply the result of the theorem.

Corollary. Under the assumptions of the theorem

$$T_n\xrightarrow{\mathcal{D}}N(0,\sigma_T^2),$$

with

$$\sigma_T^2=\int_0^{t_m}k^2(v)\,d\,\sigma^2(v)+\sum_{i=0}^{m-1}\{f_{i+1}^2\sigma_{i+1}^2(t_i^*+\Delta t_i)+f_i(f_i-2f_{i+1})\sigma_{i+1}^2(t_i^*)-$$

$$2f_{i+1}\int_{t_i^*}^{t_i^*+\Delta t_i}k(v+t_i-t_i^*)\,d\,\sigma_{i+1}^2(v)+\int_{t_i^*}^{t_i^*+\Delta t_i}k^2(v+t_i-t_i^*)\,d\,\sigma_{i+1}^2(v)\}. \quad (22)$$

If $m=2$ then

$$\sigma_T^2=\int_0^{t_m}k^2(v)\,d\,\sigma^2(v)+f_1^2(\sigma_1^2(t_1)+\sigma_2^2(t_1^*))-2f_1\int_0^{t_1}k(v)\,d\,\sigma_1^2(v)+$$

$$\int_0^{t_1}k^2(v)\,d\,\sigma_1^2(v)+\int_{t_1^*}^{\infty}k^2(v+t_1-t_1^*)\,d\,\sigma_2^2(v).$$

Remark. The variance σ_T^2 can be consistently estimated by the statistic

$$\hat\sigma_T^2=\int_0^{t_m}\hat k^2(v)d\hat\sigma^2(v)+\sum_{i=0}^{m-1}\{\hat f_{i+1}^2\hat\sigma_i^2(\hat t_i^*+\Delta t_i)+\hat f_i(\hat f_i-2\hat f_{i+1})\hat\sigma_i^2(\hat t_i^*)-$$

$$2\hat f_{i+1}\int_{\hat t_i^*}^{\hat t_i^*+\Delta t_i}\hat k(v+t_i-\hat t_i^*)d\,\hat\sigma_i^2(v)+\int_{\hat t_i^*}^{\hat t_i^*+\Delta t_i}\hat k^2(v+t_i-\hat t_i^*)\,d\,\hat\sigma_{i+1}^2(v)\},$$

where

$$\hat k(v)=K(v)/\sqrt{n},\quad \hat\sigma^2(v)=\frac{n}{n_0}\left(\frac{1}{\hat S(v)}-1\right),\quad \hat\sigma_i^2(v)=\frac{n}{n_i}\left(\frac{1}{\hat S_i(v)}-1\right),$$

$\hat{S}$ and $\hat{S}_i$ are the empirical survival functions,

$$\hat{f}_0 = \hat{f}_m = 0, \quad \hat{f}_i = \sum_{s=i}^{2m-1} \hat{e}_s \hat{d}_{si} \quad (i = 1, \cdots, m-1),$$

$$\hat{e}_s = \hat{a}_s \left(\hat{k}(t_s) - \hat{k}(t_s + \Delta t_s) - \hat{k}(t_s)\, \hat{\alpha}_{s+1}(\hat{t}_s^*) + \hat{k}(t_s + \Delta t_s)\, \hat{\alpha}_{s+1}(\hat{t}_s^* + \Delta t_s) \right.$$
$$\left. - \int_{\hat{t}_s^*}^{\hat{t}_s^* + \Delta t_s} \hat{k}(v + t_s - \hat{t}_s^*) d\hat{A}_{s+1}(v) \right),$$

$$\hat{d}_{si} = \prod_{l=i}^{s-1} \hat{c}_l, \quad i = 1, ..., s-1, \quad \hat{d}_{ss} = 1,$$

$$\hat{c}_l = \frac{\hat{\alpha}_{l+1}(\hat{t}_l^* + \Delta t_l)}{\hat{\alpha}_{l+1}(\hat{t}_l^*)}, \quad \hat{a}_s = \frac{1}{\hat{\alpha}_{s+1}(\hat{t}_s^*)},$$

and $\hat{\alpha}_{s+1}(\hat{t}_s^*)$, $\hat{\alpha}_{s+1}(\hat{t}_s^* + \Delta t_s)$ are kernel estimators:

$$\hat{\alpha}_{s+1}(\hat{t}_s^*) = \frac{1}{b} \int_0^{t_m} Ker \left(\frac{\hat{t}_i^* - u}{b} \right) d\hat{A}_{i+1}(u),$$

$$\hat{\alpha}_{s+1}(\hat{t}_s^* + \Delta t_s) = \frac{1}{b} \int_0^{t_m} Ker \left(\frac{\hat{t}_i^* + \Delta t_s - u}{b} \right) d\hat{A}_{i+1}(u);$$

here Ker is some kernel function.

4 The test

The hypothesis

$$H_0 : \textit{GS model holds on } E = \{x_1, \cdots, x_m, x(\cdot)\}$$

is rejected with the approximative significance level α, if

$$\left(\frac{T}{\hat{\sigma}_T} \right)^2 > \chi^2_{1-\alpha}(1),$$

where $\chi^2_{1-\alpha}(1)$ is the $(1-\alpha)$-quantile of the chi-square distribution with one degree of freedom.

5 Consistency and the power of the test against approaching alternatives

Find the power of the test against the following alternatives:

$$H_1 : \textit{PH model with specified non-exponential time-to-failure}$$

$$\textit{distributions under constant stresses}$$

Under H_1

$$\hat{A}^{(1)}(v) \xrightarrow{\mathbf{P}} A_*^{(1)}(v) = A_i(v), \; v \in [t_i, t_{i+1}) \; (i = 0, \cdots, m-1),$$

$$\hat{A}^{(2)}(v) \xrightarrow{\mathbf{P}} A^{(2)}(v) = A_i(v - t_i + t_i^*), \; v \in [t_i, t_{i+1}) \; (i = 0, \cdots, m-1),$$

and

$$\frac{1}{\sqrt{n}} K(v) \xrightarrow{\mathbf{P}} k^*(v),$$

where for $v \in [t_i, t_{i+1})$

$$k^*(v) = \frac{l_0 l_{i+1} S_*^{(1)}(v) S_i(v - t_i + t_i^*)}{l_0 S_*^{(1)}(v) + l_{i+1} S_i(v - t_i + t_i^*)} g\left(l_0 S_*^{(1)}(v) + l_{i+1} S_i(v - t_i + t_i^*)\right),$$

and $S_*^{(1)}(v) = exp\{-A_*^{(1)}(v)\}$. Convergence is uniform on $[0, t_m]$.

Proposition 1. *Assume that Assumptions A hold under H_1 and*

$$\Delta^* = \int_0^{t_m} k^*(v)\, d\{A^{(1)}(v) - A(v)\} \neq 0.$$

Then the test is consistent against H_1.

Proof. Write the test statistic in the form

$$T_n = \int_0^\infty K(v)\, d\{\hat{A}^{(1)}(v) - A_*^{(1)}(v)\} - \int_0^\infty K(v)\, d\{\hat{A}^{(2)}(v) - A^{(2)}(v)\} +$$

$$\int_0^\infty K(v)\, d\{A_*^{(1)}(v) - A^{(2)}(v)\} = T_{1n} + T_{2n} + T_{3n}. \tag{23}$$

Analogously as in the case when seeking the limit distribution of the statistic T_n under the hypothesis H_0, we obtain that under H_1

$$T_{1n} + T_{2n} \xrightarrow{\mathcal{D}} N(0, {\sigma_T^*}^2),$$

where ${\sigma_T^*}^2$ has the same form (22) with only difference that k is replaced by k^* and $\sigma^2(t)$ is replaced by

$$(\sigma^{(1)})^2(t) = \frac{1}{l_0}\left(\frac{1}{S^{(1)}(t)} - 1\right).$$

Under H_1 we have

$$\hat{\sigma}_T^2 \xrightarrow{\mathbf{P}} {\sigma_T^*}^2 \tag{24}$$

and

$$\frac{T_{1n} + T_{2n}}{\hat{\sigma}_T} \xrightarrow{\mathcal{D}} N(0, 1). \tag{25}$$

The third member in (23) can be written in the form

$$T_{3n} = \sum_{i=1}^{m-1} \int_{t_i}^{t_{i+1}} K(v)\, \{\alpha_{i+1}(v) - \alpha_{i+1}(v - t_i + t_i^*)\}\, dv. \tag{26}$$

The assumptions of the proposition and the equalities (23)-(26) imply that under H_1

$$\frac{1}{\sqrt{n}} T_{3n} \xrightarrow{\mathbf{P}} \Delta^*, \quad \frac{T_n}{\hat{\sigma}_T} \xrightarrow{\mathbf{P}} \infty.$$

Thus under H_1, see for example, Greenwood and Nikulin (1996),

$$\mathbf{P}\left\{\left(\frac{T}{\hat{\sigma}_T}\right)^2 > \chi^2_{1-\alpha}(1)\right\} \to 1.$$

The proposition is proved.

Remark. *If α_i are increasing (decreasing) then the test is consistent against H_1.* Proof. We'll show now by recurrence that $t_i > t_i^*$ for all i. Really, the inequalities $x_1 < \cdots < x_m$ imply that

$$S_1(t_1^*) > S_2(t_1^*) = S_1(t),$$

which give $t_1 > t_1^*$. If we assume that $t_{i-1} > t_{i-1}^*$ then

$$S_{i+1}(t_i^*) = S_i(t_i - t_{i-1} + t_{i-1}^*) > S_i(t_i - t_{i-1} + t_{i-1}) = S_i(t_i) > S_{i+1}(t_i),$$

which imply $t_i > t_i^*$. If α_i are increasing (decreasing) then $\Delta^* > 0$ ($\Delta^* < 0$) under H_1. The proposition implies the consistency of the test.

Consider the sequence of approaching alternatives

$$H_n: \; PH \; with \; \alpha_i(t) = \left(\frac{t}{\theta_i}\right)^{\frac{\varepsilon_i}{\sqrt{n}}}$$

with fixed $\varepsilon_i > 0$ $(i = 1, \cdots, m)$. Then

$$T_{3n} \xrightarrow{\mathbf{P}} \mu = -\sum_{i=1}^{m-1} \varepsilon_i \int_{t_i}^{t_{i+1}} k^*(v) \; \ln(1 + \frac{t_i^* - t_i}{v}) dv > 0$$

and

$$\frac{T_n}{\hat{\sigma}_T} \xrightarrow{\mathcal{D}} N(a, 1), \quad \left(\frac{T}{\hat{\sigma}_T}\right)^2 \xrightarrow{\mathcal{D}} \chi^2(1, a),$$

where $a = \mu/\sigma_T^*$ and $\chi^2(1, a)$ denotes the chi-square distribution with one degree of freedom and the non-centrality parameter a (or a random variable having such distribution).

The power function of the test is approximated by the function

$$\beta = \lim_{n\to\infty} \mathbf{P}\left\{\left(\frac{T}{\hat{\sigma}_T}\right)^2 > \chi^2_{1-\alpha}(1) \mid H_n\right\} = \mathbf{P}\left\{\chi^2(1, a) > \chi^2_{1-\alpha}(1)\right\}. \tag{27}$$

Find the power of the test against the following alternatives:

H_2 : the model (7) with specified time-to-failure

distributions under constant stresses

Under H_2

$$\hat{A}^{(1)}(v) \xrightarrow{\mathbf{P}} A_{**}^{(1)}(v) = A_i(v - t_i + t_i^{**}), \; v \in [t_i, t_{i+1}) \; (i = 0, \cdots, m-1),$$

$$\hat{A}^{(2)}(v) \xrightarrow{\mathbf{P}} A^{(2)}(v) = A_i(v - t_i + t_i^{*}), \; v \in [t_i, t_{i+1}) \; (i = 0, \cdots, m-1),$$

and

$$\frac{1}{\sqrt{n}} K(v) \xrightarrow{\mathbf{P}} k^{**}(v),$$

where for $v \in [t_i, t_{i+1})$

$$k^{**}(v) = \frac{l_0 l_{i+1} S_{**}^{(1)}(v) S_i(v - t_i + t_i^*)}{l_0 S_{**}^{(1)}(v) + l_{i+1} S_i(v - t_i + t_i^*)} g\left(l_0 S_{**}^{(1)}(v) + l_{i+1} S_i(v - t_i + t_i^*)\right),$$

and $S_{**}^{(1)}(v) = exp\{-A_{**}^{(1)}(v)\}$. Convergence is uniform on $[0, t_m]$.

Proposition 2. *Assume that Assumptions A hold under H_2 and*

$$\Delta^{**} = \int_0^\infty k^{**}(v) \, d\{A^{(1)}(t) - A(t)\} \neq 0.$$

Then the test is consistent against H_2.

Proof. Write the test statistic in the form (23). Analogously as in the case when seeking the limit distribution of the statistic T_n under the hypothesis H_0, we obtain that under H_2

$$T_{1n} + T_{2n} \xrightarrow{\mathcal{D}} N(0, \sigma_T^{**2}),$$

where ${\sigma_T^*}^2$ has the same form (20) with only difference that k is replaced by k^{**} and $\sigma^2(t)$ is replaced by

$$(\sigma^{(1)})^2(t) = \frac{1}{l_1}\left(\frac{1}{S^{(1)}(t)} - 1\right).$$

The third member in (23) can be written in the form

$$T_{3n} = \sum_{i=1}^{m-1} \int_{t_i}^{t_{i+1}} K(v)\,\{\alpha_{i+1}(v - t_i + t_i^{**}) - \alpha_{i+1}(v - t_i + t_i^*)\}dv.$$

The assumptions of the proposition and the the last equality imply that

$$\frac{1}{\sqrt{n}} T_{3n} \xrightarrow{\mathbf{P}} \Delta^{**}, \quad \frac{T_n}{\hat{\sigma}_T} \xrightarrow{\mathbf{P}} \infty.$$

The proposition is proved.

Remark. *If α_i are increasing (decreasing) then the test is consistent against H_2.*

Proof. We'll show now by recurrence that $t_i^{**} > t_i^*$. Really, the inequalities $x_1 < \cdots < x_m$ imply that

$$S_2(t_1^{**}) = S_1(t_1)\,\delta_1 < S_1(t_1) = S_2(t_1^*),$$

which give $t_1^{**} > t_1^*$. If we assume that $t_{i-1}^{**} > t_{i-1}^*$ then

$$S_{i+1}(t_i^{**}) = S_i(t_i - t_{i-1} + t_{i-1}^{**})\,\delta_i < S_i(t_i - t_{i-1} + t_{i-1}^{**}) < S_i(t_i - t_{i-1} + t_{i-1}^*) = S_{i+1}(t_i^*),$$

which imply $t_i^{**} > t_i^*$. If α_i are increasing (decreasing) then $\Delta > 0$ ($\Delta < 0$) under H_2. The proposition 2 implies the consistency of the test.

Consider the sequence of approaching alternatives

H_n : *the model (7) with specified time-to-failure distributions*

under constant stresses and $\delta_i = 1 - \frac{\varepsilon_i}{\sqrt{n}}$.

Find the limit of $\sqrt{n}(t_i^{**} - t_i^*)$ by recurrence. If $i = 1$, then

$$\sqrt{n}(t_1^{**} - t_1^*) = \sqrt{n}\left\{S_2^{-1}\left(S_1(t_1)(1 - \frac{\varepsilon_1}{\sqrt{n}})\right) - S_2^{-1}\left(S_1(t_1)\right)\right\} \to -\frac{\varepsilon_1}{\alpha_2(t_1^*)} = -a_1\varepsilon_1$$

Suppose that

$$\sqrt{n}(t_i^{**} - t_i^*) \longrightarrow a_i \sum_{j=1}^{i} d_{ij}\varepsilon_j, \tag{28}$$

where a_i, d_{ij} are defined in the formulation of the lemma. Then

$$\sqrt{n}(t_{i+1}^{**} - t_{i+1}^*) = \sqrt{n}(S_{i+2}^{-1}\left(S_{i+1}(t_{i+1} - t_i + t_i^{**})(1 - \frac{\varepsilon_{i+1}}{\sqrt{n}})\right) - S_{i+2}^{-1}(S_{i+1}(t_{i+1} - t_i + t_i^*))$$

$$= \sqrt{n}\frac{1}{p_{i+2}(t_{i+1}^*)}\left\{p_{i+1}(t_{i+1} - t_i + t_i^*)(t_i^{**} - t_i^*) - S_{i+1}(t_{i+1} - t_i + t_i^*)\frac{\varepsilon_{i+1}}{\sqrt{n}}\right\} + o(1) =$$

$$\frac{1}{p_{i+2}(t_{i+1}^*)}\left\{-p_{i+1}(t_{i+1} - t_i + t_i^*)a_i\sum_{j=1}^{i} d_{ij}\varepsilon_j - S_{i+1}(t_{i+1} - t_i + t_i^*)\varepsilon_{i+1}\right\} + o(1) = a_{i+1}\sum_{j=1}^{i+1} d_{i+1,j}\varepsilon_j + o(1).$$

We note p_i the densities of T_{x_i}. Thus the convergence (28) holds for all i. It implies that

$$T_{3n} \xrightarrow{\mathbf{P}} \mu = -\sum_{i=1}^{m-1} a_i \sum_{j=1}^{i} d_{ij}\,\varepsilon_j \int_{t_i}^{t_{i+1}} k(v)\,d\,\alpha_{i+1}(v - t_i + t_i^*)$$

and

$$\frac{T_n}{\hat\sigma_T} \xrightarrow{\mathcal{D}} N(a,1), \quad \left(\frac{T_n}{\hat\sigma_T}\right)^2 \xrightarrow{\mathcal{D}} \chi^2(1, \mid a \mid),$$

where $a = \mu/\sigma_T^{**}$.

The parameter μ is positive (negative) if the functions α_i are convex (concave).

The power function of the test is approximated by the function (27) with $a = \mu/\sigma_T^{**}$.

REFERENCES

1. P.K. Andersen, O. Borgan, R.D. Gill, N. Keiding (1993). Statistical Models Based on Counting Processes. *Springer,* New York.

2. Bagdonavičius, V., (1978). Testing the hypothesis of the additive accumulation of damages. *Probab. Theory and its Appl.*, **23** (2) , 403-408.

3. Bagdonavičius, V., (1990). Accelerated life models when the stress is not constant, *Kybernetika*, **26**, 289-295.

4. Bagdonavičius, V. and Nikoulina, V. (1997). A goodness-of-fit test for Sedyakin's model. *Romanian Journal of Pure and Applied Mathematics*, 52 (1) , p. 5-14.

5. Bagdonavičius, V. and Nikulin, M. (1999). Generalized Proportional Hazards Model Based on Modified Partial Likelihood. *Lefetime Data Analysis*, **5**, 329-350.

6. Bagdonavičius, V. and Nikulin, M. (1999). Model Building in Accelerated Experiments. In: *Statistical and Probabilistic Models in Reliability,* (Eds. D.Ionescu, N. Limnios), Birkhauser:Boston, 51-73.

7. Bagdonavičius, V. and Nikulin, M. (2000). On goodness-of-fit for linear transformation and frailty models, *Statistics and Probability Letters*,**47**, 177-188.

8. Greenwood P.E., Nikulin M. (1996). *A Guide to Chi-Squared Testing,* J.Wiley and Sons: N.Y.

9. Lin, D.Y. (1991). Goodness-of-fit analysis for the Cox regression model based on a class of parameter estimators. *J. Am. Statist. Assoc.*, **86**, 725-8.

10. Lin, D.Y., Wei, L.J., Ying, Z. (1993). Checking the Cox model with cumulative sums of martingale-based residuals. *Biometrika*, **80**, No.3, 557-72.

11. Nelson, W. (1990). Accelerated testing: statistical models, test plans, and data analyses. *Wiley* , 601 p.

12. Sedyakin, N.M. (1966). On one physical principle in reliability theory, *Techn. Cybernetics*, **3**, 80-87.

13. Wei, L.J. (1984). Testing goodness-of-fit for proportional hazards model with censored observations, *JASA*, **79**, 387, 649-652.