In reality, all this has begun because of A.Grothendieck.

ON BASES AND APPROXIMATION CONDITIONS IN BANACH SPACES
(trying giving a short survey*))

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One or two times we shall go out of the main theme of the talk to formulate very interesting results, directly (for the first sight) not connected with the objects, mentioned in the title. They are so interesteing that, by the way, it will be hard enough to leave them away. On the other hand they, in any case, are closely related to those approximation conditions.

Let us begin with recalling that *a basis* in a Banach space X is a sequence $(x_n)_1^\infty$ with the property that any $x \in X$ has an unique representation of the form $x = \sum_{n=1}^\infty a_n x_n$ in X. The functionals $x_n' : x \mapsto a_n$ are called *the coordinate functionals* of the basis (x_n). It is not evident (and this is a result of Banach), that these functionals are continuous; hence the so called operators of partial sums $P_N : x \mapsto \sum_{n=1}^N a_n x_n$ are continuous too and, therefore, they are continuous linear projections from X onto the linear (closed) span of $(x_n)_1^N$, $N = 1, 2, \dots$. Of course, any space with a basis is separable.

*)This small lecture was given in some places of Sweden, and was written here just before (or even during) those talks. So, please, the reader (if any...) be not very strict to the author because of not every place of the text is looking nice enough, or maybe even is "enough not nice". Especially, maybe, this concerns my dreadful English (but not only?..).

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The notion of the basis was firstly introduced by Schauder in 1928, when he proved that in the space $C[0,1]$ of all continuous functions with the natural *sup*-norm the "integrated" Haar system is a basis in the above sense (Haar, himself, showed in his thesis that his system is "almost basis" in this space: he proved that every continuous function uniquely expands into an unifomly convergent series with respect to the Haar system; but Haar functions are not continuous! – so they do not give a basis for the space $C[0,1]$). An year later, in 1929, Schauder showed that the Haar system, itself, is a basis in each space $L^p[0,1]$[1], $p \in [1,+\infty)$ (let us recall another, rather hard in proving, examples: the trigonometrical system is a basis in every space $L^p[0,1], p \in (1,+\infty)$, but not in $C[0,1]$ or $L^1[0,1]$). Now it is known that all the classical Banach spaces have bases. For example, a basis in the disc-algebra A (the space of all analytical in open unit disk and continuous in the closed disk functions) was constructed by Bochkarjov; in H^1 – the Hardy space – by B.Maurey. In the spaces H^p, for the other exponents $p \in (1,\infty)$, bases were known from early days of the birth of the notion of the basis.

The question of the existence of a basis in every Banach space was posed by Banach in 1932 (his famous work!). Only in 1972 it has become real to look at an example of the space without basis (P. Enflo), and moreover, — without the so called metric approximation property (and even without the approximation property; see below).

Those operators of partial sums (P_N) possess the properties

1) $P_N \to \mathrm{id}_X$ pointwise;

2) $\sup_N ||P_N|| < +\infty$;

3) (P_N) converges to id_X uniformly on every compact set K in X;

4) $\dim P_N(X) = N$ and $P_N(X) \subset P_{N+1}(X)$.

The property 1) is evident; 2) is a corrolary of Banach–Steinhaus theorem and 1); it follows from 1), taking in account 2), the condition 3); 4) is clear.

Just for a couple of minutes, let us forget that the cause of appearing the assertions 1)–3) was the basis, and consider any sequence (P_N) of finite rank operators in X, satisfying the condition 1) (consequently, 2) and 3)). A (separable) Banach space X in which the

[1]These bases in L^p (when $1 < p < +\infty$) are even unconditional, i.e. they remain being bases under any permutation, but we are not going to discuss this theme; see any corresponding literature.

condition 1) (or 3)) is fulfilled is said to be *the space with the bounded approximation property,* BAP in short. Cleary, the existence of a basis in a Banach space implies that this space has the BAP. On the other hand, if (P_N) is a sequence of projections which satisfies 1)–4), then it is easy to construct the corresponding basis, for which (P_N) is the sequence of operators of partial sums (just take, for each $N = 1, 2, \dots,$ nonzero $x_N \in P_N(X) \setminus P_{N-1}(X)$, where $P_0 = 0$). But in general case, when (P_N) is a sequence of finite rank operators satisfying only 1), it becomes a problem to construct a basis in the space X.

Long ago, A.Pelczyǹski showed that *any Banach space with a basis is a complemented subspace of a (universal and isomorphically unique) space with the (universal) basis.* Besides, he proved *that every separable Banach space with the BAP can be imbedded, complementably, in a space with a basis.* But it was, and was long enough, not known whether a complemented subspace of a space with a basis has a basis itself. Finally, Szarek was successed in showing that the answer to the last question is "no", so solving this (and some other problems) in negative.

Of course, any finite dimensional space has a basis, and not one. But, in any case, there are some natural notions and questions, related to them, in finite dimensional theory too. One can define a quantitative notion connected with a basis (in any Banach space, not only in finite dimensional one). This can be done as follows.

Let X be a Banach space, and (x_n) is its basis. Then the natural projections (operators of partial sums), $P_N : x \mapsto \sum_1^N < x'_n, x > x_n$, as was already said, are bounded linear operators in X. *The basic constant of the basis* (x_n) is defined as the supremum of the norms $||P_N||$ over all N; it is denoted by $bc\,(x_n)$. If X has a basis then *the basic constant of the space* X is not anything as the number $bc\,(X) := \inf\, bc\,(x_n)$, where *inf* is taken over all bases in the space X. It is very interesting to consider this number in the case of a finite dimensional space. It is rather not hard to show that for any n-dimensional normed space X we always have $bc\,(X) \leq \sqrt{n}$ (the reason is that any such space is close enough to the Euclidean space l_2^n, namely, is $\sqrt{n}$-close in the sense of "Banach–Mazur distance", but we are not going to discuss what the last phrase means, because that would lead us far enough from the main theme of our considerations, although to a very nice theory; —

it is not our aim).

So, a very old question was (in connection with the last inequality) a question of the assymptotical precision of that estimate. And, by using probabilistic methods, Gluskin and Szarek showed that indeed, there is a $C > 0$ so that for some sequence $(X_n)_{n=1}^{\infty}$ of finite dimensional spaces with $\dim X_n = n$ one has $bc(X_n) \geq C\sqrt{n}$. By the way, they proved that, in a sense, "no subspase of X_n of large enough dimension is complemented in X_n" (that means, in particular, that any subspace of X_n of the half dimension has the property that there is no projection from X_n onto it with norm less then $c_0\sqrt{n}$, for some absolute constant c_0.)

Well, the last mentioned finite–dimensional result leads to a very old classical question if such a situation is possible in the infinite dimensional spaces. We will give here only two striking results in this direction.

Firstly, G.Pisier, answering one of the question of A.Grothendieck, showed that there exists an infinite dimensional Banach space G all of whose finite dimensional subspaces lie in G very bad: there is a constant $c > 0$ such that for any subspace F in G the best projection from G onto F has the norm more than $c\sqrt{\dim F}$(! – it is well known that every n-dimensional subspace of any Banach space is $\sqrt{n}$-complemented).

Secondly, B.Maurey with a coauther showed that there exists an infinite dimensional Banach space B without any complemented subspaces; more precisely, if only a subspace of B is not finite dimensional itself, or is not finite codimensional, there is no linear continuous projection from B onto this subspace (!!).

But now, let us return from the sky filled with striking results to our bases and approximation conditions...

Not knowing whether every separable Banach space has a basis, in 1955 A.Grothendieck has introduced more general notions, — AP (approximation property) and BAP (bounded approximation property), — to go around the basis problem and to cover the situation when the spaces under consideration are not separable (and also, in particular, to prove many facts which it was hardly to get without those additional restrictions; well, when it was possible to prove a result with only the AP, he used it, but in many cases it was needed a stronger version of the approximation in Banach spaces, so he used the MAP).

Here are the definitions (in a sense, inspiring by the above discussion on bases and BAP in separable spaces, i.e. by the condition 1) and, to be exact, by conditions 2) and 3): 3) will give us the AP, while 3) with something like 2) will give the BAP and the MAP). So:

They say that a Banach space X has *the approximation property* (AP), if

($*$) for each $\varepsilon > 0$ and every compact set $K \subset X$ there exists a finite rank operator P in X such that for each $x \in K$ one has $||Px - x|| < \varepsilon$.

The space X has the BAP (*bounded approximation property*), if there is a constant $C > 0$ such that ($*$) is fulfilled with the additional condition $||P|| \leq C$; if we can take $C = 1$, then we say that the space X has *the metric approximation property* (MAP).

P.Enflo (we said about this a little before) proved that *there exists a separable reflexive space without the* AP, using, in particular, the following nice result of A.Grothendieck (certainly, this Grothendieck's theorem is not the main tool in the difficult proof of Enflo): *for reflexive Banach spaces the* AP *is just the same as the* MAP.

This last result and some other assertions of A.Grothendieck led him to the following assumption: *if a weakly compact operator, acting in some Banach spaces, can be approximated, uniformly on each compact, by finite rank operators, then it also can be approximated, uniformly on each compact, by finite rank operators, whose norms are bounded by a constant, depended only on the given weakly compact operator.* I was successed in giving a counterexample to this assumption, constructed a separable space and a compact operator in this space, which is "approximable", but not "bounded approximable". By the way, this space has the AP, but has not the BAP.

As the matter of fact, *the first example of the space with the* AP *and without* BAP *was found by T.Figiel and W.B.Johnson,* who also answered another, close to the problem, question of A.Grothendieck (related to the so called nuclearity of operators in Banach spaces). The last facts led to many interesting questions in the geometrical theory of operators in Banach spaces. But they are not in the sphere of our considerations here. We would like to mention here only the following result which generalises the Figiel-Johnson theorem.

Let $I(X, X)$ be a closed subspace of the space $L(X, X)$ of all linear bounded operators in a separable space X. We now know that the identity map id_X is not nessesary can be compactly approximated by a bounded sequence of finite rank operators, even if X has the AP. But maybe such a situation is impossible if instead of finite rank operarors we will consider the operators from the space $I(X,X)$ (if this space I is large enough, but not the whole $L(X,X)$)? However, in general, the situation here is just the same as in the Figiel-Johnson theorem, e.g.: *there exists a separable Banach space with the* AP *such that the identity map of this space can not be approximated, in the topology of compact convergence, by weakly compact operators.* With the formulation of this result (its proof is somewhat different from the one of Figiel-Johnson) I am going to finish the discussion around bases, AP, BAP etc.. And to end the talk, I would like to propose two, more that twenty years old, open questions.

1) Let X be a separable Banach space. Is it true that if the dual X^* has the AP, then this dual has the BAP?

2) Let T be a linear bounded operator from a Banach space X to a Banach space Y such that there exists a continuous factorization of T^{**} of the kind

$$X^{**} \to l_p \to Y^{**},$$

where $1 < p < +\infty, p \neq 2$. Does there exist a similar factorization for the operator $T : X \to Y$?

Note, finally, that if the space X^* is separable, then the answer to the question 1) is positive (A.Grothendieck). As to the second question, the answer is negative for the cases $p = 1$ or $p = \infty$. Trivially, the answer is positive, when $p = 2$.

The second question does, as we could think, not deal with the AP's; but it is only a first sight on it (deeply inside it the AP seats, as well as some different generalizations of the approximation property do)...

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