GRADED LIE ALGEBRAS AND DYNAMICAL SYSTEMS

A. M. VERSHIK

Introduction

We consider a class of infinite-dimensional Lie algebras which is associated to dynamical systems with invariant measures. There are two constructions of the algebras – one based on the associative cross product algebra which considered as Lie algebra and then extended with nontrivial scalar two-cocycle; the second description is the specification of the construction of the graded Lie algebras with continuum root system in spirit of the papers of Saveliev-Vershik [SV1, SV2, V] which is a generalization of the definition of classical Cartan finite-dimensional algebras as well as Kac–Moody algebras. In the last paragraph we present the third construction for the special case of dynamical systems with discrete spectrum. The first example of such algebras was so called sine-algebras which was discovered independently in [SV1] and [FFZ] and had been studied later in [GKL] from point of view Kac–Moody Lie algebras. In the last paragraph of this paper we also suggest a new examples of such type algebras appeared from arithmetics: adding of 1 in the additive group Z_p as a transformation of the group of p-adic integers. The set of positive simple roots in this case is Z_p; Cartan subalgebra is the algebra of continuous functions on the group Z_p and Weyl group of this Lie algebra contains the infinite symmetric group. Remarkably this algebra is the inductive limit of Kac–Moody affine algebras of type $A^1_{p^n}$.

1. Lie algebra generated by automorphism

1.1. **Associative algebra $\mathcal{A}(\mathcal{X}, T)$.** Let $(\mathcal{X}, \mu)$ be a separable compactum with Borel probability measure μ which is positive for any open set $A \subset \mathcal{X}$ and T is a measure preserving homeomorphism of $\mathcal{X}$.

It is old and well-known construction of W^*-algebra (von Neumann) and C^*-algebra (Gel'fand) generated by $(\mathcal{X}, \mu, T)$, see for example [D, ZM]. Algebraically this is an associative algebra $\mathcal{A}(\mathcal{X}, T)$ which is semidirect product of $C(\mathcal{X})$ and $C(\mathbb{Z})$ with the action of $\mathbb{Z}$ on $\mathcal{X}$ and consequently on $C(\mathcal{X})$.

St. Petersburg Division, Steklov Mathematical Institute, RAS,
27, Fontanka, St. Petersburg, 191011, Russia.

Partially supported by grant RFBR 99-01-00098.

This paper contains the subject of my talk on the Conference in Twente in December 2000.

As a linear space this is direct sum

$$\mathcal{A}(\mathcal{X},T)=\bigoplus_{n\in\mathbb{Z}}C(\mathcal{X})\otimes U^n$$

where $C(\mathcal{X})$ is Banach space of all continuous functions on $\mathcal{X}$, and $U=U_T$ is linear operator $(U_Tf)(x)=f(Tx)$, $f\in C(\mathcal{X})$. The multiplication of the monomials is defined by formula

$$(\varphi\otimes U^n)\cdot(\psi\otimes U^m)=(\varphi\cdot U^n\psi)\otimes U^{n+m},\qquad n,m\in\mathbb{Z},\quad \varphi,\psi\in C(\mathcal{X}).$$

Involution on $\mathcal{A}(\mathcal{X},T)$ is the following:

$$(\varphi\otimes U^n)^*=(U^{-n}\bar{\varphi})\otimes U^{-n}.$$

Completion of $\mathcal{A}(\mathcal{X},T)$ with respect to the appropriate C^*-norm gives a corresponding C^*-algebra.

It is possible to include to this construction a 2-cocycle of the action of $\mathbb{Z}$ with values in $C(\mathcal{X})$ to obtain another C^*-algebra which are unsplittable extensions (see [ZM, VSh]), but we restrict ourself to the case of the trivial cocycle.

If we use a measure μ as a state on $\mathcal{A}(\mathcal{X},T)$ and construct $*$-representation corresponding to this state then W^*-closure of image of $\mathcal{A}(\mathcal{X},T)$ gives us W^*-algebra generated by triple $(\mathcal{X},\mu,T)$.

There are two classical representations of the algebra $\mathcal{A}(\mathcal{X},T)$ — Koopmans representation (in $L^2_\mu(\mathcal{X})$) and von Neumann one in $L^2_{\mu\times m}(\mathcal{X}\times\mathbb{Z})$ (m is Haar measure on $\mathbb{Z}$). This area called “algebraic theory of dynamical systems” and there are many papers on this. Extremally popular is so called rotation algebras (also called as ”quantum torus”) which is associative C^*-algebra generated by irrational rotation of the unit circle.

We want to point out that there is another remarkable algebraic object which is associated with dynamical systems—some Lie algebras which are similar to the classical Cartan Lie algebras and to affine Kac-Moody algebras. Some nontrivial central extension included in the definition plays very imporatant role in the whole theory. Upto now only the Lie algebras corresponing to rotation algebras were considered; it was discovered independently in [SV1] and [FFZ] (see also [GKL] were the shift on d-dimensional torus was considered) and called by physisits ”sine algebras” - all those dynamical systems have a *discrete spectrum*,

In whole generality the Lie algebras generated by an arbitrary dynamical system with invariant measure was briefly defined in [SV2] and more systematically in [V]. It is interesting that we started not form the generaltheory of dynamical systems as in the first definition below but from the notions presented in our series of papers with M. Saveliev [SV1, SV2] were we had defined so called “$\mathbb{Z}$-graded Lie algebras with continuous root systems”. Those algebras which we will discuss here were one of the type of the examples and a special case of $\mathbb{Z}$-graded Lie algebras with general root systems. Below we will describe explicitly the modern and detailed version of the construction of Lie algerbas generated by an arbitrary discrete dynamical systems with invariant measure and then will give the link between various definitions. One can hope that this type of algebras can give a new type of invariants of the dynamical systems as well as new examples of calssical and quantum integrable systems.

1.2. **Lie algebras** $\widetilde{\mathfrak{A}}(\mathcal{X}, T)$**.** Most interesting case is the case when T is minimal (= each orbit of T is dense in $\mathcal{X}$) and ergodic with respect to measure μ (= there are no nonconstant T-invariant measurable functions). We assume this in further considerations. It is known that if T is minimal (=each T-orbit is dense in $\mathcal{X}$) then C^*-algebra is simple (= has no proper two-sided ideals), see [ZM]. Algebra $\mathcal{A}(\mathcal{X}, T)$ with brackets

$$[a, b] = ab - ba$$

will be denoted as Lie $\mathcal{A}(\mathcal{X}, T)$; it is still $\mathbb{Z}$-graded Lie algebra and the brackets of monomials are

$$[\varphi \otimes U^n, \psi \otimes U^m] = (\varphi \cdot U^n \psi - \psi \cdot U^m \varphi) \otimes U^{n+m}.$$

This algebra has a center.

Lemma 1. *The center of* Lie $\mathcal{A}(\mathcal{X}, T)$ *is the set of constants functions in zero component subspace:* $\mathfrak{Z} = c \otimes U^0$, $c \in \mathbb{C}$. *The complement linear subspace*

$$\mathcal{A}_0(\mathcal{X}, T) = \bigoplus_{n<0} C(\mathcal{X}) \otimes U^n \oplus C_0(\mathcal{X}) \otimes U^0 \oplus \bigoplus_{n>0} C(\mathcal{X}) \otimes U^n,$$

where $C_0(\mathcal{X}) = \left\{ \varphi \in C_(\mathcal{X}) : \int_{\mathcal{X}} \varphi(x) d\mu = 0 \right\}$, *is Lie subalgebra which is isomorphic to quotient* $\mathcal{A}(\mathcal{X}, T) \big/ \mathfrak{Z}$ *over center* $\mathfrak{Z}$.

Remark. The center is not ideal of associative algebra consequently there is no "associative" analogue of this lemma and $\mathcal{A}_0(\mathcal{X}, T)$ is not a subalgebra of $\mathcal{A}(\mathcal{X}, T)$.

Now we define a 2-cocycle on $\mathcal{A}_0(\mathcal{X}, T)$ with the scalar values and one-dimensional central expansion of it.

Lemma 2. *The following formula defines 2-cocycle on* $\mathcal{A}_0(\mathcal{X}, T)$*:*

$$\alpha(\varphi \otimes U^n, \psi \otimes U^m) = n \int_{\mathcal{X}} \varphi \cdot U^n \psi \, d\mu \cdot \delta_{n+m},$$

so

$$\alpha(\varphi \otimes U^n, \psi \otimes U^m) = \begin{cases} n \int_{\mathcal{X}} \varphi \cdot U^n \psi \, d\mu & \text{if } m = -n, \\ 0 & \text{if } m \neq -n. \end{cases}$$

Proof. We need to check that $\alpha([x, y], z) + \alpha([y, z], x) + \alpha([z, x], y) = 0$. Let $k + l + n = 0$. Then

$$\alpha\big([\varphi \otimes U^k, \psi \otimes U^l], \gamma \otimes U^n\big) + \dots = \alpha\big((\varphi \cdot U^k \psi - \psi \cdot U^l \varphi) \otimes U^{k+l}, \gamma \otimes U^{-k-l}\big) + \dots$$

$$= (k + l) \int_{\mathcal{X}} \big(\varphi \cdot U^k \psi U^{-n} \cdot U^{-n} \gamma - \psi \cdot U^l \varphi \cdot U^{-n} \gamma\big) d\mu + \dots = 0$$

(Dots mean cyclic permutation of indices, we used here the invariance of measure μ under T.) □

Remark. Cocycle α is not cohomologous to zero because it is easy to check that $\alpha(x,y)$ can not be represented as $f([x,y])$, with any linear functional f. of $C(\mathcal{X})$.

Let us identify scalars c which are extensions of $\mathcal{A}_0(\mathcal{X},T)$ with scalars $c \in C(\mathcal{X}) \otimes U^0 \subset \mathcal{A}(\mathcal{X},T)$. So we can consider again linear space $\mathcal{A}(\mathcal{X},T)$ as one dimensional nontrivial extension of Lie algebra $\mathcal{A}_0(\mathcal{X},T)$. Denote a new Lie algebra by $\widetilde{\mathfrak{A}}(\mathcal{X},T)$. So, Lie algebra $\widetilde{\mathfrak{A}}(\mathcal{X},T)$ as linear space is the same as $\mathcal{A}(\mathcal{X},T)$ but the brackets in $\widetilde{\mathfrak{A}}(\mathcal{X},T)$ differ from the brackets in $\mathcal{A}(\mathcal{X},T)$:

(1)
$$[\varphi \otimes U^n, \psi \otimes U^m] = (\varphi \cdot U^n\psi - \psi \cdot U^m\varphi) \otimes U^{n+m} + \int_{\mathcal{X}} \varphi \cdot U^n\psi \, d\mu \cdot \delta_{n+m}$$

It means that the center of $\widetilde{\mathfrak{A}}(\mathcal{X},T)$ is again scalars $\mathbb{C} \cdot 1 \subset C(\mathcal{X}) \otimes U^0 \subset \operatorname{Lie}\mathcal{A}(\mathcal{X},T)$, but now subspace $\mathcal{A}_0(\mathcal{X},T)$ is not Lie subalgebra and the central extension is not trivial.

Lie algebra $\widetilde{\mathfrak{A}}(\mathcal{X},T)$ is $\mathbb{Z}$-graded Lie algebra. We will give a new definition of it in a framework of Lie algebras with continuous root systems. We will call the subspace of $\widetilde{\mathfrak{A}}(\mathcal{X},T)$ which consists of $\operatorname{Span}\{\gamma \otimes U^{-1}\} \oplus \operatorname{Span}\{\varphi \otimes U^0\} \oplus \operatorname{Span}\{\psi \otimes U^1\}$, $\varphi, \psi, \gamma \in C(\mathcal{X})$, a "local subalgebra". Here are the brackets for local part of $\widetilde{\mathfrak{A}}(\mathcal{X},T)$ are

(2)
$$\begin{aligned}
&[\varphi_1 \otimes U^0, \varphi_2 \otimes U^0] = 0,\\
&[\varphi \otimes U^0, \psi \otimes U^{\pm 1}] = \pm\big((\varphi - U\varphi)\cdot\psi\big) \otimes U^{\pm 1} = \big((I-U)\varphi \cdot \psi\big) \otimes U^{\pm 1},\\
&[\varphi \otimes U^{+1}, \psi \otimes U^{-1}] = (\varphi \cdot U\psi - \psi \cdot U\varphi) \otimes U^0 + \int_{\mathcal{X}} (\varphi \cdot U\psi) d\mu \cdot c.
\end{aligned}$$

The middle term of local algebra $(\{\varphi \otimes U^0; \varphi \in C(\mathcal{X})\})$ is by definition Cartan subalgebra.

This gives the first—"dynamical"—description of the Lie algebra $\widetilde{\mathfrak{A}}(\mathcal{X},T)$.

1.3. **Lie algebras with root system** $(\mathcal{X},T)$**.** Definition of Lie algebra will be followed to Kac–Moody pattern but with important changes. First of all we define a *local algebra.* Let $\varphi \in C(\mathcal{X})$; we consider three types of uncountably many generators: $X_{-1}(\varphi)$, $X_0(\varphi)$, $X_{+1}(\varphi)$ where φ runs over $C(\mathcal{X})$. The list of relations is as follows:

$$\begin{aligned}
&[X_0(\varphi), X_0(\psi)] = 0,\\
&[X_0(\varphi), X_{\pm 1}(\psi)] = \pm X_{\pm 1}(K\varphi \cdot \psi), \qquad (3)\\
&[X_{+1}(\varphi), X_{-1}(\psi)] = X_0(\varphi \cdot \psi),
\end{aligned}$$

where product $(\cdot)$ is the product in associative algebra $C(\mathcal{X})$ and K is a linear operator in $C(\mathcal{X})$ which is called Cartan operator:

$$(K\varphi)(x) = 2\varphi(x) - \varphi(Tx) - \varphi(T^{-1}x). \tag{4}$$

It is evident that Jacobi identity is true (if it makes sense) in the local algebra

$$\mathfrak{A}_{-1} \oplus \mathfrak{A}_0 \oplus \mathfrak{A}_{+1},$$

where $\mathfrak{A}_i = \operatorname{Span}\{X_i(\varphi), \varphi \in C(\mathcal{X})\}$, $i = -1, 0, +1$. ($\mathfrak{A}_0 \simeq C(\mathcal{X})$ is Cartan subalgebra.) The further steps are the same as in Kac–Moody theory [K].

We define free Lie algebra which is generated by the local algebra and factorize it over the maximal ideal which has zero intersection with $\mathfrak{A}_0$. The resulting Lie algebra is denoted by $\mathfrak{A}(\mathcal{X}, T)$. The fact is that this algebra is the same as $\widetilde{\mathfrak{A}}(\mathcal{X}, T)$ of subsection 2.

We omit the verification that $\mathfrak{A}(\mathcal{X}, T)$ is the graded Lie algebra with the graded structure (as a linear space) as follow

$$\bigoplus_{n\in\mathbb{Z}} \mathfrak{A}_n$$

and each $\mathfrak{A}_n \simeq C(\mathcal{X})$ (see [V]).

So we can denote the elements of $\mathfrak{A}_n$ as $X_n(\varphi)$, $\varphi \in C(\mathcal{X})$.

Theorem 1. *The following formulas give the canonical isomorphism τ between $\widetilde{\mathfrak{A}}(\mathcal{X}, T)$ and dense part of $\mathfrak{A}(\mathcal{X}, T)$:*

$$\tau(\varphi \otimes U^n) = \begin{cases} X_{-n}(U^{-n}\varphi), & (n > 0) \\ X_0(\varphi - U^{-1}\varphi), & n = 0, \quad \int_{\mathcal{X}} \varphi \, d\mu = 0 \\ X_n(\varphi) & (n > 0) \end{cases}$$
$$\tau(\mathbf{1} \otimes U^0) = X_0(\mathbf{1}).$$

Proof. The kernel of τ is $\mathbf{0}$. Let us check that $\tau([a, b]) = [\tau a, \tau b]$. It is enough to test monomials only.

$$\begin{aligned}\big[\tau(\varphi \otimes U^0), \tau(\psi \otimes U)\big] &= \big[\varphi - U^{-1}\varphi) \otimes U^0, \psi \otimes U\big] \\ &= \psi\big(\varphi - U^{-1}\varphi - U(\varphi - U^{-1}\varphi)\big) \otimes U = (\psi \cdot K\varphi) \otimes U \\ &= X_{+1}(K\varphi \cdot \psi) = \tau\big([\varphi \otimes U^0, \psi \otimes U]\big);\end{aligned}$$

$$\begin{aligned}&\big[X_0(\varphi), X_{-1}(\psi)\big] = \big[(\varphi - U^{-1}\varphi) \otimes U^0, U^{-1}\psi \otimes U^{-1}\big] \\ &= \big(U^{-1}\psi(\varphi - U^{-1}\varphi - U^{-1}\varphi + U^{-2}\varphi)\big) \otimes U^{-1} = \big(U^{-1}(\psi(U\varphi - 2\varphi + U^{-1}\varphi))\big) \otimes U^{-1} \\ &= -U^{-1}(\psi \cdot K\varphi) \otimes U^{-1} = -X_{-1}(K\varphi \cdot \psi);\end{aligned}$$

$$\begin{aligned}\big[X_{+1}(\psi), X_{-1}(\gamma)\big] &= \big[\psi \otimes U, U^{-1}\gamma \otimes U^{-1}\big] \\ &= (\psi\gamma - U^{-1}\gamma \cdot U^{-1}\psi) \otimes U^0 = \big(\psi\gamma - U^{-1}(\psi\gamma)\big) \otimes U^0 = X_0(\psi\gamma);\end{aligned}$$

$$\big[X_{+1}(\mathbf{1}), X_{-1}(\mathbf{1})\big] = cX_0(\mathbf{1}) = c\mathbf{1}.$$

□

Remark. 1. We calculated the bracket only for elements of local algebra, but this is enough because it generates all algebra.
2. The τ-image of $\widetilde{\mathfrak{A}}(\mathcal{X},T)$ is not all $\mathfrak{A}(\mathcal{X},T)$ but dense part of $\mathfrak{A}(\mathcal{X},T)$, for example, the set of functions $\varphi - U\varphi$ is dense in $C_0(\mathcal{X})$ only, but $\mathfrak{A}(\mathcal{X},T)$ is the extension of the image of $\widetilde{\mathfrak{A}}(\mathcal{X},T)$ and we can consider $\mathfrak{A}(\mathcal{X},T)$ as some kind of completion of $\widetilde{\mathfrak{A}}(\mathcal{X},T)$. 3. Using isomorphisms τ we can rewrite the definition of cocycle of $\mathfrak{A}(\mathcal{X},T)$ ($\tau(\varphi\otimes U^{-n}) = X_{-n}(U^{-n}\varphi)$, $n>0$) so

$$\alpha\big(X_n(\varphi), X_m(\psi)\big) = \begin{cases} 0, & \text{if } n+m\neq 0,\\ n\int_{\mathcal{X}}\varphi\psi\,d\mu, & n+m=0.\end{cases}$$

Lie algebra $\mathfrak{A}(\mathcal{X},T)$ defined above is our main object.

Proof. If $n>0$ then $\alpha(X_n(\varphi), X_{-n}(\psi)) = n\int_{\mathcal{X}}\varphi\cdot U^n(U^{-n}\psi)d\mu = n\int_{\mathcal{X}}\varphi\psi\,d\mu$. If $n<0$ the $\alpha(X_{-n}(\varphi), X_n(\psi)) = -n\int_{\mathcal{X}}U^{-n}\varphi\cdot U^{-n}\psi\,d\mu = -n\int_{\mathcal{X}}\varphi\psi\,d\mu$ (U is an unitary operator). □

This consists with the initial formula

$$\big[X_{+1}(\varphi), X_{-1}\big((\varphi)^{-1}\big)\big] = X_0(\mathbf{1}) = \mathbf{1}\cdot c$$

for $n=\pm 1$.

Now we can rewrite the brackets for all monomials (not only for local part). Assume $n,m>0$.

(+,+)
$$\big[X_n(\varphi), X_m(\psi)\big] = \big[\varphi\otimes U^n, \psi\otimes U^m\big] = X_{n+m}(\varphi\cdot U^n\psi - \psi\cdot U^m\varphi);$$

(+,−)
$$\begin{aligned}\big[X_n(\varphi), X_{-m}\psi\big] &= \big[\varphi\otimes U^n, U^{-m}\psi\otimes U^{-m}\big]\\ &= \big(\varphi\cdot U^{n-m}\psi - U^{-m}(\varphi\psi)\big)\otimes U^{n-m}\\ &= \begin{cases} X_{n-m}\big(\varphi\cdot U^{n-m}\psi - U^{-m}(\varphi\psi)\big), & n>m>0,\\ X_0\big((1-U^{-m})(1-U^{-1})^{-1}\varphi\psi\big), & n=m,\\ X_{n-m}\big(U^{-n}\varphi(\psi\cdot U^m\varphi - U^{-n}\psi)\big), & 0<n<m;\end{cases}\end{aligned}$$

(−,−)
$$\begin{aligned}\big[X_{-n}(\varphi), X_{-m}\psi\big] &= \big[U^{-n}\varphi\otimes U^{-n}, U^{-m}\psi\otimes U^{-m}\big]\\ &= (U^{-n}\varphi\cdot U^{-n-m}\psi - U^{-m}\psi\cdot U^{-n-m}\varphi)\otimes U^{-n-m}\\ &= X_{-n-m}(\psi\cdot U^m\varphi - \varphi\cdot U^n\psi)\\ &= -X_{-n-m}(\varphi\cdot U^n\psi - \psi\cdot U^m\varphi)\end{aligned}$$

(0,+)
$$\begin{aligned}\big[X_0(\varphi), X_n(\psi)\big] &= \big[(\varphi - U^{-1}\varphi)\otimes U^0, \psi\otimes U^n\big]\\ &= \big(\psi\cdot(\varphi - U^{-1}\varphi + U^{n-1}\varphi - U^n\varphi)\big)\otimes U^n = X_n(K_n\varphi\cdot\psi),\end{aligned}$$

where $K_n = I - U^{-1} + U^{n-1} - U^n = (I - U^{-1})(I - U^n)$;

(0,$-$)

$$\begin{aligned}\big[X_0(\varphi), X_{-n}\psi\big] &= \big[(\varphi - U^{-1}\varphi)\otimes U^0, U^{-n}\psi \otimes U^{-n}\\ &= \big(U^{-n}\psi\cdot(\varphi - U^{-1}\varphi - U^{-n}\varphi + U^{-n-1}\varphi)\big)\otimes U^{-n}\\ &= X_{-n}((U^{-n}\psi)\cdot(U^n\varphi - U^{n-1}\varphi - \varphi + U^{-1}\varphi) = -X_n(K_n\varphi\cdot\psi)\end{aligned}$$

We can now observe that the formulas (+,+) and ($-$,$-$) are the same, as well as (0,+) and (0,$-$).

Theorem 2. *The formulas for the brackets of monomials in the subalgebra $\mathfrak{A}(\mathcal{X}, T)$ are the following:*
1) $\big[X_n(\varphi), X_m(\psi)\big] = \pm X_{n+m}(\varphi\cdot U^{\pm n}\psi - \psi U^{\pm m}\varphi)$,
where the sign is "+" if $n, m > 0$ and "$-$" if $n, m < 0$.
2) $\big[X_0(\varphi), X_{\pm n}(\psi)\big] = \pm X_n(K_n\varphi\cdot\psi)$.

$$3)\ \big[X_n(\varphi), X_m(\psi)\big] = \begin{cases} X_{n+m}\big(U^m\psi(\varphi\cdot U^n\psi - U^m\varphi)\big), & \begin{cases} m<0,\\ n+m>0\end{cases}\\ X_0\big(\frac{1-U^{-m}}{1-U^{-1}}(\varphi\psi)\big) + n\int_{\mathcal{X}}\varphi\psi\, d\mu & \begin{cases} n+m=0,\\ n\neq 0\end{cases}\\ X_{n+m}\big(U^{-n}\varphi(\psi\cdot U^{-m}\varphi - U^{-n}\psi)\big), & \begin{cases} m<0,\\ n+m<0\end{cases}\end{cases}$$

4) $\big[X_0(\varphi), X_0(\psi)\big] = 0$. □

Lie algebra $\mathfrak{A}(\mathcal{X}, T)$ does not associate with associative algebra; cocycle α has nothing to do with associated crossproduct of subsection 1.1. The role of central extension is very important.

We defined Lie algebra $\mathfrak{A}(\mathcal{X}, T)$ ($\simeq \widetilde{\mathfrak{A}}(\mathcal{X}, T)$) in a new terms, compare with (3). This manner gives us the formulas for local part (1–2) which are similar to classical ones (Cartan simple algebras and Kac–Moody algebras). But the formulas for general monomials are more complicated than dynamical (see subsection 1.1) description.

2. General Lie algebras with continuous root systems and new examples of $\mathfrak{A}(\mathcal{X}, T)$

2.1. **General definition.** We recall [SV1, SV2, V] the definition of graded Lie algebras with continuous root system.

Suppose $\mathcal{H}$ is a commutative associative Lie $\mathbb{C}$-algebra with unity (Cartan subalgebra) and $K : \mathcal{H} \hookleftarrow$ is a linear operator (Cartan operator). The local algebra $[K]$ is, as a linear space, a direct sum

$$\mathcal{H}_{-1}\oplus\mathcal{H}_0\oplus\mathcal{H}_{+1}, \qquad \mathcal{H}_i \simeq \mathcal{H}, \quad i = 0, \pm 1$$

with brackets:

$$\begin{aligned} X_i(\varphi) \in \mathcal{H}_i, \quad i = 0, \pm 1, \quad \varphi, \psi \in \mathcal{H} \\ \big[X_0(\varphi), X_0(\psi)\big] &= 0, \\ \big[X_0(\varphi), X_{\pm 1}(\psi)\big] &= \pm X_{\pm 1}(K\varphi \cdot \psi) \\ \big[X_{+1}(\varphi), X_{-1}(\psi)\big] &= X_0(\varphi \cdot \psi) \end{aligned}$$

The local algebra $\mathcal{H}_{-1} \oplus \mathcal{H}_0 \oplus \mathcal{H}_{+1}$ generates graded Lie algebra $\mathfrak{A}(\mathcal{H}, K)$. in the same spirit as in Subsection 1.2 (and as in the theory of LKM-algebras). Then we obtain $\mathfrak{A}(\mathcal{X}, T)$ from Section 1.

The spectrum of commutative algebra $\mathcal{H}$ (if it exists) is root system of $\mathfrak{A}(\mathcal{H}, K)$ by definition (see [V]), more exactly the set of simple positive roots. But it could be no spectra (say, $\mathcal{H}$ is the algebra of rational functions) so we have Lie algebras without simple roots but with Cartan operator.

The condition of constant or polynomial growth of the dimension (in an appropriate sense) puts essential restriction on the operator K.

Remark. Let $E \subset \mathcal{H}$ is an invariant under K subalgebra of $\mathcal{H}$. Then $\mathfrak{A}(E, K)$ is Lie subalgebra of $\mathfrak{A}(\mathcal{H}, K)$. In particular, if $E_1 \subset E_2 \subset \dots$, $\cup_i E_i = \mathcal{H}$, is a sequence of K-invariant subalgebras of $\mathcal{H}$ then $\mathfrak{A}(\mathcal{H}, K) = \cup_{i=1}^{\infty} \mathfrak{A}(E_i, K)$.

2.2. **New examples of algebras of type $\mathfrak{A}(\mathcal{X}, T)$.** The first nontrivial example of algebras of type $\mathfrak{A}(\mathcal{X}, T)$ was so called sine-algebra. We will not consider it because it was done before from different point of view (see [SV2, FFZ, GKL, V]). It was defined independently in [SV2] and [FFZ]. We give now general example of similar type.

Let $(\mathcal{X}, T, \mu)$ be an ergodic system with discrete spectrum. It means that operator $U = U_T$ has spectral decomposition

$$Uf = \sum_{\lambda} \lambda f_\lambda \chi_\lambda, \text{ where } f = \sum_{\lambda} f_\lambda \chi_\lambda,$$

sum is over eigenvalues of U $(\lambda \in \mathbb{T}^1)$, and χ_λ is the eigenfunction corresponding to λ. It is well-known (von Neumann theorem) that such system can be realized on the compact abelian group $G = \mathcal{X}$ with Haar measure $m = \mu$ and T is translation on some element $g_0 \in G$, then χ_λ is a character of G $(\chi_\lambda \in G^\wedge)$ and

$$\lambda = \chi_\lambda(g_0) \in \mathbb{T}^1.$$

Sine-algebra corresponds to the case $\mathcal{X} = \mathbb{T}^1$ and T is translation on irrational number $\theta \in S^1 = \mathbb{R}/BZ$. In [GKL] was considered also the case $\mathcal{X} = \mathbb{T}^d \ni \theta$.

The case

$$\mathcal{X} = Z_p, \qquad Tx = x + 1,$$

where Z_p is additive group of p-adic integers, p is a prime, and T is adding of unity, is more interesting from our point of view. The measure $\mu = m$ is Haar (additive) measure on Z_p.

The group of characters $Z_p^\wedge = Q_{p\infty}$ is a group of all roots of unity of the degree p^n, $n \in \mathbb{N}$, and the characters $\mu \in Q_{p\infty}$ (as function on Z_p with values in $\mathbb{T}^1$) are eigenfunctions of operator $U = U_T$.

We give the description of the general case when operator $U = U_T$ has discrete spectrum. The specific property of algebra $\mathfrak{A}(\mathcal{X}, T)$ in this case is existence of the natural *linear basis* in $\mathfrak{A}(\mathcal{X}, T)$.

Suppose G is abelian (additive) compact group and $G^\wedge$ is a countable group of the (multiplicative) characters on G. We fix the element $\lambda \in G$ with dense set of powers: $\mathrm{Cl}\{\lambda^n, n \in \mathbb{Z}\} = G$. Then $T_\lambda g = Tg = g + \lambda$, $U_T = U$, $(Uf)(g) = f(g + \lambda)$, $f \in C(G)$. Each character $\chi \in G^\wedge$ is an eigenfunction of U with eigenvalue $\chi(\lambda) \in \mathbb{T}^1$.

Theorem 3. *Linear basis in the Lie algebra $\mathfrak{A}(G, T)$ is the set $\{Y_{\chi,n} : \chi \in G^\wedge, n \in \mathbb{Z}\}$ with the following brackets*

$$[Y_{\chi,n}, Y_{\chi_1,n_1}] = \big(\chi_1(\lambda)^n - \chi(\lambda)^{n_1}\big) Y_{\chi\chi_1, n+n_1} + \delta_{n+n_1} \tilde{\delta}_{\chi\chi_1} \cdot n \cdot c, \tag{5}$$

where $\delta_n = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$, $\tilde{\delta}_\chi = \begin{cases} 1, & \chi = \mathbf{1} \\ 0, & \chi \neq \mathbf{1} \end{cases}$.

Algebra $\mathfrak{A}(G, T)$ is $\mathbb{Z} \times G^\wedge$-graded algebra; the subalgebra $\{c1 : c \in \mathbb{C}\}$ in Cartan subalgebra $\mathfrak{A}_0 = C(G)$ is the center of $\mathfrak{A}(G, T)$.

Proof. Assume $Y_{\chi,n} = \chi \otimes U^n$ as an element of $\mathfrak{A}_n$, where χ is the character of G as a function $G \to \mathbb{T}^1$. It is easy to check that the brackets (see formula (1) in Section 1) give us formula (5). Note that the center is not direct summand, so we have

$$[Y_{\chi,n}, Y_{\chi^{-1},-n}] = n \cdot c. \quad \square$$

This is the third description of our algebra $\mathfrak{A}(\mathcal{X}, T)$ with linear basis; this description is valid for discrete spectrum only.

The group G is the set of simple roots for $\mathfrak{A}(G, T)$ and "Dynkin" diagram is the set of arrows $G \ni g \to g + \lambda \in G$. In opposite to Kac–Moody case our algebras $\mathfrak{A}(\mathcal{X}, T)$ have no imaginary roots.

2.3. **The case of p-adic integers.** Return back to the case $G = Z_p$, $Tx = x+1$. In this case $\mathfrak{A}(Z_p, T)$ is $\mathbb{Z} \times Q_{p\infty}$-graded algebra. Cartan subalgebra is space $C(Z_p)$. Consider finite dimensional subspaces $L_n \subset C(Z_p)$ of functions depending on the points of the quotient $Z_p \to \mathbb{Z}/p^n$; it is clear that subspace L_n is U_T-invariant, so $\mathcal{L}_n = \bigoplus_{m \in \mathbb{Z}} L_n \otimes U^m$
is a subalgebra of $\mathfrak{A}$.

Theorem 4. *The Lie algebra $\mathcal{L}_n$ is canonically isomorphic to the algebra $\mathcal{A}^{(1)}_{p^n}$. Consequently, $\mathfrak{A}(Z_p, T)$ is (completion) of the inductive limit of Kac–Moody Lie algebras*[1]

$$\mathfrak{A}(Z_p, T) \supset \lim_n \mathrm{ind}\, \mathcal{L}_n. \quad \square$$

Remark. It is possible to define Weyl group W for this algebra. Group W contains the group of permutations of the coordinates in Z_p.

It is very instructive to study the link between theory of Kac–Moody affine algebras and our theory looking on this example.

References

[SV1] M. Saveliev, A. Vershik. *Continuum analogues of contragradient Lie algebras.* CMP **126** (1989), 367–378.

[SV2] M. Saveliev, A. Vershik. *New examples of continuum graded Lie algebras.* Phys. Lett. A **143** (1990), 121–128.

[V] A. Vershik. *Lie algebras generated by dynamical systems.* St. Petersburg Math. J. **4** no. 6 (1993).

[K] V. Kac. *Infinite dimensional Lie algebras.* Camb. Univ. Press, 1991.

[GKL] M. Golenishcheva–Kuznezova, D. Lebedev. *$\mathbb{Z}$-graded trigonometric subalgebras and its representations with vertex operator.* Func. Anal., **27** no. 1 (1993), 12–24.

[VSh] A. Vershik, B. Shoikhet. *Graded Lie algebras whose Cartan subalgebras is the algebra of polynomials in one variable.* Th. and Math. Phys. **123** no. 2 (2000), 345–352.

[D] J. Dixmier. *C^*-algebras and its representations.* Paris, 1969.

[ZM] G. Zeller–Meier. *Produits croisés d'une C^* algebre par un groupe d'automorphismes.* J. Math. pures et appl. **47** (1968), 101–239.

[FFZ] D. Fairlie, P. Fletcher, C. Zachos. *Trigonometric structure constants for new infinite dimensional algebras.* Phys. Lett. B **218** no. 2 (1989), 203–206.

[1] More exactly, inductive limit contains only linear combinations of monomials of type $\varphi \otimes U^n$, where φ are cylindric functions; so it is enough to extend the set of φ onto arbitrary continuous functions which means to make a completion.