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Wavelet Analysis in Solving the Cauchy Problem for the Wave Equation in Three-Dimensional Space

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Abstract. We propose a new method of solving the Cauchy problem for the wave equation in three-dimensional space. The method is based on continuous wavelet-analysis. We show that the exact non-stationary solution of the wave equation with finite energy found in [1] at any fixed moment of time should be regarded as a mother wavelet. This solution was named in [1] "Gaussian wave packet". It is a new three-dimensional axially symmetric wavelet which is given by a simple explicit formula as well as its Fourier transform. This wavelet has an infinite number of vanishing moments. It is a smooth function, i.e. it has derivatives of any order with respect to spatial coordinates and time. We show that using the wavelet decomposition of the initial data we can find the exact formula for the solution of the Cauchy problem as a linear superposition of "Gaussian packets".

1 Introduction

Our aim here is to give the exact formula which should express the solution of the Cauchy problem for the wave equation as a linear superposition of the localized solution of this equation found in [1]. The representation of the function from some class as a superposition of some elementary functions is convenient for solving many problems of the wave propagation. The best known example of such a representation is the Fourier transform when the plane waves are taken as elementary functions. However, the Fourier transform is not very convenient when studying the nonstationary or localized signals.

There is also a relatively new method, called wavelet analysis, which make it possible to present the solution as superposition of the elementary localized functions (see, say, [2]-[4]). The elementary function is any function which is well localized both in the space- and in spatial frequency domain and which satisfies some admissibility condition. This elementary function is called the basic- or mother wavelet and will be designated $\varphi(\boldsymbol{r})$. The popularity of the wavelet analysis in different applications grows very quickly. It is used in signal processing, in the analysis of images, in astronomy, in fluid dynamics, in medical physics etc. (see [2]). The mother wavelets which are used in such problems are usually not connected with differential equations. There is also a wavelet, associated with the solution of the Schroedinger equation (see, say [2]) , called the Schroedinger wavelet.

We propose here to regard the exact localized solution of the wave equation found in [1] as the mother wavelet. We show that this solution at a fixed moment of time represents a three dimensional axially symmetric mother wavelet. It has such advantages as the vanishing moments of any order and smoothness. Furthermore, it is an exact solution of the wave equation, and it can be useful for representing the solution of the Cauchy problem with the initial data of finite energy.

2 Wavelet Decompositions of Functions of Three Variables

We give here the main formulas of wavelet analysis in $L^2(\mathbb{R}^3)$ which will be used below in our work. They can be found for example in [3], [5].

Let there be some function which is fit to be a mother wavelet and which is axially symmetric. Using this function we construct a family of functions

$$\varphi^{(\boldsymbol{b},a,\alpha,\beta)} = \frac{1}{a^{3/2}}\varphi\left(\mathbf{M}^{-1}(\alpha,\beta)\frac{\boldsymbol{r}-\boldsymbol{b}}{a}\right). \tag{1}$$

Each function of this family is a mother wavelet, shifted to the vector $\boldsymbol{b}\in\mathbb{R}^3$ and dilated with the parameter $a>0$. The axis of the mother wavelet is rotated by the 3×3 matrix M parameterized in terms of two angles $0\le\alpha\le\pi$, $0\le\beta<2\pi$. This matrix is a product of two matrices: the matrix of rotation round the x-axis $\mathbf{M}_x(\alpha)$ and the matrix of rotation round the z-axis $\mathbf{M}_z(\beta)$

$$\mathbf{M}(\alpha,\beta) = \mathbf{M}_z(\beta)\mathbf{M}_y(\alpha), \tag{2a}$$

$$\mathbf{M}_y(\alpha) = \begin{pmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{pmatrix}, \quad \mathbf{M}_z(\beta) = \begin{pmatrix}\cos\beta & -\sin\beta & 0\\ \sin\beta & \cos\beta & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{2b}$$

We determine the continuos wavelet transform of any square integrable function $f(\boldsymbol{r})$ with respect to the wavelet φ in the following way

$$T_f^{(\boldsymbol{b},a,\alpha,\beta)} = \int\limits_{\mathbb{R}^3} f(\boldsymbol{r})\overline{\varphi^{(\boldsymbol{b},a,\alpha,\beta)}(\boldsymbol{r})}d^3\boldsymbol{r}. \tag{3}$$

The wavelet transform can be inverted exactly and the reconstruction formula reads as

$$f(\boldsymbol{r}) = \frac{1}{C_\varphi}\int\limits_0^\pi d\alpha\sin\alpha\int\limits_0^{2\pi}d\beta\int\limits_0^{+\infty}\frac{da}{a}\int\limits_{\mathbb{R}^3}\frac{d^3\boldsymbol{b}}{a^3}\,T_f^{(\boldsymbol{b},a,\alpha,\beta)}\varphi^{(\boldsymbol{b},a,\alpha,\beta)}(\boldsymbol{r}). \tag{4}$$

Here the constant C_φ is given by the formula

$$C_\varphi \equiv (2\pi)^3\int_{\mathbb{R}^3}\frac{|\widehat{\varphi}(\boldsymbol{k})|^2}{|\boldsymbol{k}|^3}\,d^3\boldsymbol{k} < \infty \tag{5}$$

where $\widehat{\varphi}(\boldsymbol{k})$ designates the Fourier transform of $\varphi(\boldsymbol{r})$ and $|\boldsymbol{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$. The condition of the finiteness of C_φ guarantees the invertibility of the wavelet transform.

The reconstruction formula provides the decomposition of the signal as a linear superposition of the wavelets $\varphi^{(\boldsymbol{b},a,\alpha,\beta)}$ with the coefficients $T_f^{(\boldsymbol{b},a,\alpha,\beta)}$.

The existence of a certain number of vanishing moments

$$\int\limits_{-\infty}^{+\infty} x^n y^m z^j \varphi(\boldsymbol{r})\, d^3\boldsymbol{r} = 0 \tag{6}$$

is considered to be the advantage of the wavelet φ. This property improves the efficiency of φ when detecting singularities in the signal (see [4], [2]).

3 "Gaussian Wave Packet" and its Time Derivative as Mother Wavelets

We are going to show that the exact solution of the wave equation

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0, \quad c = Const \tag{7}$$

which was found in [1] can be used as the mother wavelet with the time t as a fixed parameter.

The above mentioned solution depends on the spatial coordinates x, y, z and the time t as follows

$$\psi(\boldsymbol{r},t) = \frac{1}{z + ct - i\varepsilon} \exp\left[-2\varkappa\varepsilon\sqrt{1 - \frac{i\theta}{\varepsilon}}\right], \quad \boldsymbol{r} = (x,y,z) \tag{8}$$

where

$$\theta = z - ct + \frac{x^2 + y^2}{z + ct - i\varepsilon} \tag{9}$$

and $\varkappa$, ε are the free positive parameters. The branch of the square root in (8) with the positive real part is understood.

This solution (see [1]) is exponentially localized near the point, moving along the z - axis with the speed of light c. The solution $\psi(\boldsymbol{r},t)$ is axially symmetric in respect to the z direction. This function has smooth derivatives of any order with respect to the spatial coordinates and time. All the moments of this function in respect to spatial coordinates (6) are zero.

3.1 Admissibility Condition for the "Gaussian Wave Packet" and its Time Derivative to Be Regarded as Mother Wavelets

The Fourier transform of the solution with respect to the spatial coordinates for any time t (which is considered to be parameter now)

$$\widehat{\psi}(\boldsymbol{k},t) = \int\limits_{\mathbb{R}^3} \psi(\boldsymbol{r},t) \exp\left(-i\boldsymbol{k}\boldsymbol{r}\right) d^3\boldsymbol{r} \tag{10}$$

is localized exponentially [6]

$$\widehat{\psi}(\boldsymbol{k},t)=\frac{2^{3/2}i\pi c}{|\boldsymbol{k}|(k_z+|\boldsymbol{k}|)^{3/2}}\exp\left[-ict|\boldsymbol{k}|-\varepsilon\left(|\boldsymbol{k}|+\frac{2\varkappa^2}{|\boldsymbol{k}|+k_z}\right)\right]. \tag{11}$$

Here $|\boldsymbol{k}|=\sqrt{k_x^2+k_y^2+k_z^2}, -\infty<k_x,k_y,k_z<+\infty$.

The maximum of the Fourier transform is at the point $k_x=k_y=0, k_z=\varkappa$. If $|\boldsymbol{k}|\to\infty$ or $|\boldsymbol{k}|\to 0$, the function $\widehat{\psi}(\boldsymbol{k},t)$ tends exponentially to zero. Then the integral (5) where $\varphi(\boldsymbol{r})=\psi(\boldsymbol{r},t)$ converges and the admissibility condition (5) is satisfied for any fixed t.

The derivative $\psi_t(\boldsymbol{r},t)$ of the function $\psi(\boldsymbol{r},t)$ with respect to t satisfies the inequality (5) where $\varphi(\boldsymbol{r})=\psi_t(\boldsymbol{r},t)$ and has the same properties as the function $\psi(\boldsymbol{r},t)$.

3.2 Wavelet $\psi^{(\boldsymbol{b},a,\alpha,\beta)}(\boldsymbol{r},t)$ and the Solution of the Wave Equation

Each function of the family of dilated, shifted and rotated "Gaussian packets" remains the solution of the wave equation provided we introduce the scaling of time proportional to that of space, i.e. the function $\psi^{(\boldsymbol{b},a,\alpha,\beta)}(\boldsymbol{r},t/a)$ is the solution of (7).

4 Wavelet Representation of the Solution of the Wave Equation

We seek the solution $u(\boldsymbol{r},t)$ of the Cauchy problem for the wave equation:

$$u_{tt}-c^2(u_{xx}+u_{yy}+u_{zz})=0, \tag{12a}$$

$$u(\boldsymbol{r},t_0)=U(\boldsymbol{r}), \tag{12b}$$

$$u_t(\boldsymbol{r},t_0)=V(\boldsymbol{r}). \tag{12c}$$

We determine the wavelet transform of the initial data U using "Gaussian wave packet" (8) $\psi(\boldsymbol{r},0)$ as a mother wavelet. Then we find the wavelet transform of V using the t- derivative of (8) at $t=0$ as a mother wavelet. We calculate these transforms by formula (3) and designate them by $T_U^{(\boldsymbol{b},a,\alpha,\beta)}$ and $T_V^{(\boldsymbol{b},a,\alpha,\beta)}$ respectively.

Our main result here is obtaining the formula for the solution of the problem (12a)-(12c) which reads as

$$\begin{aligned}u(\boldsymbol{r},t)=&\int_0^{\pi}d\alpha\sin\alpha\int_0^{2\pi}d\beta\int_0^{+\infty}\frac{da}{a}\int_{\mathbb{R}^3}\frac{d^3\boldsymbol{b}}{a^3}\\ &\left(\left[\frac{1}{2C_\psi}T_U^{(\boldsymbol{b},a,\alpha,\beta)}+\frac{a}{2C_{\psi_t}}T_V^{(\boldsymbol{b},a,\alpha,\beta)}\right]\psi^{(\boldsymbol{b},a,\alpha,\beta)}(\boldsymbol{r},(t-t_0)/a)+\right.\\ &+\left.\left[\frac{1}{2C_\psi}T_U^{(\boldsymbol{b},a,\alpha,\beta)}-\frac{a}{2C_{\psi_t}}T_V^{(\boldsymbol{b},a,\alpha,\beta)}\right]\psi^{(\boldsymbol{b},a,\alpha,\beta)}(\boldsymbol{r},-(t-t_0)/a)\right).\end{aligned} \tag{13}$$

The constant C_ψ is determined by formula (5) where we assume that $\varphi(\boldsymbol{r}) = \psi(\boldsymbol{r}, 0)$ and C_{ψ_t} is obtained by the same formula with the assumption that $\varphi(\boldsymbol{r}) = \psi_t(\boldsymbol{r}, 0)$.

To verify that the function u gives the solution we sought-for it is sufficient to make sure that the equalities (12a) - (12c) are valid. The function u satisfies (12a) as a linear superposition of solutions. It satisfies the initial conditions. Actually, if we substitute $t = t_0$ in (13), we obtain the wavelet reconstruction formula for $U(\boldsymbol{r})$ because the coefficients $T_V^{(\boldsymbol{b},a,\alpha,\beta)}$ are canceled. If we differentiate the equality (13) and substitute $t = t_0$, we obtain the reconstruction formula for V.

5 Conclusions

We have found the representation of the solution of the Cauchy problem (12a)-(12c) using the wavelet decomposition of the initial data. We used the exact solutions of the wave equation (8) and its derivative with respect to t taken at $t = 0$ as the mother wavelets. The representation (13) can be interpreted as the sum of two groups of dilated, rotated and shifted wave packets moving in opposite directions and having the amplitudes $1/2(T_U^{(\boldsymbol{b},a,\alpha,\beta)} + aT_V^{(\boldsymbol{b},a,\alpha,\beta)})$ and $1/2(T_U^{(\boldsymbol{b},a,\alpha,\beta)} - aT_V^{(\boldsymbol{b},a,\alpha,\beta)})$ respectively.

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