UNIVERSALITY AND RANDOMNESS FOR THE GRAPHS AND METRIC SPACES

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Contents

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1. Universal and Random Graphs

In this section we will show the existence of a denumerable, non-directed graph which is universal in the category of all countable, non-directed graphs and homogeneous (with respect to its finite subgraphs). Moreover, we will see that such a graph is determined unique up to isomorphism. This definition was done by R. Rado. We will give the criteria of universality using incidence matrix. Then we consider the random graphs or probability measures on the set of graphs. We generalize the theorem by P. Erdös and A. Rényi that the random graph is universal graph with probability 1.

We will describe a non-directed countable graph by a pair $\Gamma = (\Gamma, \Gamma^{(1)})$, where Γ is the (countable) set of vertices, and $\Gamma^{(1)} \subseteq \Gamma \times \Gamma$ the set of edges. Any subset $S \subseteq \Gamma$ determines a subgraph $(S, \Gamma^{(1)}|_{S\times S})$ of Γ, which we will again denote by S.

Given two graphs Γ_i, $i = 1, 2$. A mapping $\iota : \Gamma_1 \to \Gamma_2$ is called a mono-(iso)morphism, if it is one-to-one (bijective, resp.) and preserves the structure of the graph Γ_1:

$$(x, y) \in \Gamma_1^{(1)} \text{ if and only if } (\iota x, \iota y) \in \Gamma_2^{(1)}.$$

In other words, Γ_1 is isomorphic to the subgraph $\iota(\Gamma_1)$ of Γ_2. Every isomorphism of Γ_1 onto itself is called automorphism.

Definition 1.1. A (countable, non-directed) graph Γ is said to be

(1) *universal*, if to any finite (non-directed) graph F there exists a monomorphism $\iota : F \to \Gamma$. In other words, any finite non-directed graph is isomorphic to a subgraph of Γ.
(2) *homogeneous*, if to any finite subgraphs F_1, F_2 of Γ, and isomorphism $\iota : F_1 \to F_2$, there exists an automorphism $\iota' : \Gamma \to \Gamma$ which extends ι.

1.1. **Construction of the universal graph.** In the sequel, we will show that universality and homogeneity is equivalent to the following extension property:

Definition 1.2. A graph Γ is said to have the *extension property*, if the following holds: Given any finite graph F, and a one-point extension F' of F. If there is a monomorphism $\iota : F \to \Gamma$, then ι can be extended to an monomorphism $\iota' : F' \to \Gamma$, i.e. the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\iota \text{ (monomorphism)}} & \Gamma \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} \\ F' & \xrightarrow{\iota' \text{ (monomorphism)}} & \Gamma \end{array}.$$

In other words, the graph Γ satisfies the extension property if and only if to any finite subgraph F of Γ, and any (exterior) one-point extension $F' = F \cup \{a\}$ of F, there exists a point $\alpha \in \Gamma$ such that for $x \in F$

$$(\alpha, x) \in \Gamma^{(1)} \text{ if and only if } (a, x) \in F'^{(1)}.$$

The extension property allows us to extend locally defined isomorphisms to global isomorphisms via a so-called *back and forth* argument:

Lemma 1.3. *Suppose Γ, $\tilde{\Gamma}$ are countable, non-directed graphs which have the extension property, and $F \subseteq \Gamma$ finite. Then any monomorphism $\iota : F \to \tilde{\Gamma}$ can be extended to an isomorphism $\iota' : \Gamma \to \tilde{\Gamma}$.*

Proof. Write $\Gamma = \{x_1, x_2, \ldots\}$ and $\tilde{\Gamma} = \{y_1, y_2, \ldots\}$. Denote $F_1 = F$, $G_1 = \iota(F_1)$, and $\iota_1 = \iota : F_1 \to G_1$. Define $G_2 = G_1 \cup \{y_1\}$. By the extension property, $\iota_1^{-1} : G_1 \to F_1$ extends to a monomorphism $\iota_2^{-1} : G_2 \to F_2$ onto a subset $F_2 \supseteq F_1$. Now define $F_3 = F_2 \cup \{x_1\}$. Again there is a monomorphism $\iota_3 : F_3 \to G_3$ onto a subset $G_3 \supseteq G_2$. Take $G_4 = G_3 \cup \{y_2\}$. Continuing in the same manner as above, we can construct inductively subgraphs

$$F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots \subseteq \Gamma$$

with

$$F_n \supseteq \{x_k : 1 \le k \le \frac{n-1}{2}\},$$

subgraphs

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots \subseteq \tilde{\Gamma}$$

satisfying

$$G_n \supseteq \{y_k : 1 \le k \le \frac{n}{2}\},$$

and isomorphisms $\iota_n : F_n \to G_n$,

$$\iota = \iota_1 \subseteq \iota_2 \subseteq \cdots \subseteq \iota_n \subseteq \cdots$$

The mapping $\iota_\infty = \bigcup_{n=1}^\infty \iota_n$ maps $\bigcup_{n=1}^\infty F_n = \Gamma$ isomorphically onto $\bigcup_{n=1}^\infty G_n = \tilde{\Gamma}$. □

Theorem 1.4. *A countable, non-directed graph is universal and homogeneous if and only if it has the extension property. Two such graphs are isomorphic.*

Proof. Any universal and homogeneous graph Γ has the extension property: By universality there is an embedding $\kappa : F_2 \to M$. Define $i : \kappa(F_1) \to \iota(M_1)$, $i = \iota \circ \kappa^{-1}$. Now, choose an automorphism $J : M \to M$ which extends i. The mapping $J \circ \kappa : F_2 \to M$ is an extension of ι.

Conversely, if Γ has the extension property, then it is obviously universal. Moreover, if $\iota : F \to F'$ is an isomorphism between finite subgraphs of Γ, then Lemma 1.3 (setting $\Gamma = \tilde{\Gamma}$) shows that ι can be extended to a global automorphism $\iota' : \Gamma \to \Gamma$. Hence Γ is homogeneous.

Suppose $\tilde{\Gamma}$ is another countable graph which satisfies the extension property. To show that Γ and $\tilde{\Gamma}$ are isomorphic, choose any $a \in \Gamma$, $\tilde{a} \in \tilde{\Gamma}$, and apply Lemma 1.3 to the monomorphism $\iota : \{a\} \to \tilde{\Gamma}$, which maps a to $\tilde{a}$. □

Remark 1.5. *A non-directed countable graph Γ which has the extension property is universal in the category of all countable non-directed graphs, i.e. every countable, non-directed graph Γ' can be embedded into Γ.*

Consider a countable, non-directed graph $(\Gamma, \Gamma^{(1)})$, $\Gamma = \{x_1, x_2, x_2, \ldots\}$. We define its incidence matrix $m_\Gamma = (\varepsilon_{ij})_{i,j=1}^\infty \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ by

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } (x_i, x_j) \in \Gamma^{(1)} \\ 0 & \text{otherwise} \end{cases}. \tag{1.1}$$

Of course the definition of m_Γ depends on the order of the x_i.

Definition 1.6. An incidence matrix $(\varepsilon_{ij})_{i,j=1}^\infty$ is said to be *universal*, if for every $n \in \mathbb{N}$

$$\{(\varepsilon_{1k}, \varepsilon_{2k}, \ldots, \varepsilon_{nk}) : k > n\} = \{0,1\}^n. \tag{1.2}$$

In other words, every finite binary word $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ occurs infinitely often:

$$(\varepsilon_{1k}, \varepsilon_{2k}, \dots, \varepsilon_{nk}) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \text{ for infinitely many } k > n. \tag{1.3}$$

The set of all universal matrices is denoted by $\mathcal{U}$.

Universal matrices are connected with universal graphs as follows:

Theorem 1.7. *A countable non-directed graph Γ is universal and homogeneous, if its incidence matrix m_Γ is universal.*

Proof. Write $\Gamma = \{x_1, x_2, x_3, \dots\}$, and let m_Γ be its incidence matrix defined by 1.1. By Theorem 1.4, we have to show that Γ satisfies the extension property if and only if m_Γ is universal. But this is obvious, since Definition 1.2 is only a reformulation of the extension property. □

To show the existence of a universal and homogeneous graph, we construct a universal incidence matrix $(\varepsilon_{ij})_{i,j=1}^{\infty}$ as follows: Choose an enumeration $\{w^{(1)}, w^{(2)}, w^{(3)}, \dots\}$ of all finite words with alphabet $\{0,1\}$, such that every finite word occurs infinitely often. Extend these finite words $w^{(i)}$ arbitrarily to infinite words $\tilde{w}^{(i)} \in \{0,1\}^{\mathbb{N}}$. Now define ε_{ij}, $i < j$, as

$$(\varepsilon_{1,n+1}, \varepsilon_{2,n+1}, \dots, \varepsilon_{n,n+1}) = \pi_n \tilde{w}^{(n)}, \qquad n \in \mathbb{N}$$

where π_n is the projection onto the first n coordinates. Further define $\varepsilon_{ii} = 0$, and $\varepsilon_{ij} = \varepsilon_{ji}$ for $i > j$. This already defines an incidence matrix, such that (1.3) is satisfied.

We thus have proved

Theorem 1.8. *There exists a countable, non-directed graph which is universal and homogeneous. By Theorem 1.4 this graph is determined uniquely up to isomorphism.*

Since such a universal and homogeneous graph is unique (up to isometry), we simply speak of *the* universal graph Γ_U.

Suppose $x, y \in \Gamma_U$ are such that $(x,y) \notin \Gamma_U^{(1)}$. By the extension property, there is a point $z \in \Gamma_U$ such that both $(x,z), (z,y) \in \Gamma_U^{(1)}$. Thus the graph metric

$$d(x,y) = \min \# \text{ edges in } \gamma,$$

where the min is taken over all paths which join x and y, can only attain the values

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{for } (x,y) \in \Gamma^{(1)} \\ 2 & \text{otherwise} \end{cases}.$$

1.2. **Action of the group S^∞ and the set of universal matrices.** We denote by $\mathcal{M} \subseteq \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ the set of all incidence matrices, equipped with the topology that inherits from the product topology on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$, and by $\mathcal{U}$ the subset of all universal incidence matrices. The space $\mathcal{M}$ is compact and metrizable, and $\mathcal{U}$ is a closed subset of $\mathcal{M}$.

Let denote by S^∞ the group of all permutations on the naturals $\mathbb{N}$. Equipped with the topology of point-wise convergence, and choosing a proper metric, for example the usual metric for point-wise convergence

$$d(g_1, g_2) = \sum 1/2^n \frac{|g_1(n) - g_2(n)|}{1 + |g_1(n) - g_2(n)|},$$

S^∞ is a complete separable metric space (=polish space), which is not locally compact. Moreover, *this topology makes S^∞ to a non-locally compact, polish group.* All finite permutations form a dense subgroup S_∞ of S^∞.

For any $g \in S^\infty$ and infinite matrix $(a_{ij})_{i,j=1}^\infty$ we define

$$T_g(a_{ij}) = g\,(a_{ij})\,g^{-1} = (a_{g(i),g^{-1}(j)})_{i,j=1}^\infty. \tag{1.4}$$

Clearly T_g leaves the set $\mathcal{M}$ of incidence matrices invariant. Via $g \mapsto T_g$, the group S^∞ (or S_∞) acts on $\mathcal{M}$. Note that for any finite permutation $g \in S_\infty$, $T_g : \mathcal{M} \to \mathcal{M}$ is a homeomorphism, whereas for any infinite permutation it is only a Borel-automorphism.

Consider a graph Γ with incidence matrix m_Γ: Another (countable, non-directed) graph $\tilde{\Gamma}$ is isomorphic to Γ if and only if

$$m_{\tilde{\Gamma}} = g m_\Gamma g^{-1}$$

for some $g \in S^\infty$, or equivalently if the orbits $S^\infty m_\Gamma$ and $S^\infty m_{\tilde{\Gamma}}$ coincide. In this sense we may regard the group $\operatorname{Aut}\Gamma$ of all automorphisms of Γ as a (closed) subgroup of S^∞.

PROBLEM

How does the group $\operatorname{Aut}(\Gamma_U)$ of automorphisms of the universal graph Γ_U look like?

It is known that this group is simple (see [2]) Very interesting question whether this (uncountable) group has unitary representations which do not extend to the whole group S^∞

As we saw $\mathcal{U}$ is S^∞-invariant subset of $\mathcal{M}$. Moreover, since any two universal graphs are isomorphic, S^∞ *acts transitively on the set $\mathcal{U}$ of universal matrices*, i.e. if $m_1, m_2 \in \mathcal{U}$, then there is a $g \in S^\infty$ such that

$$m_2 = g\, m_1\, g^{-1}.$$

By the universality condition, it is clear that *for any universal matrix m, its orbit $S^\infty m$ is dense in $\mathcal{M}$.*

Theorem 1.9. *The set $\mathcal{U}$ of all universal incidence matrices is a dense G_δ-subset of $\mathcal{M}$.*

Proof. For any finite binary word $w = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, the set $M_w = \{(\varepsilon_{ij})_{i,j=1}^\infty \in \mathcal{M} : (1.3) \text{ holds }\}$ equals

$$\bigcap_{m>n} \bigcup_{k \geq m} \{(\varepsilon_{ij})_{i,j=1}^\infty \in \mathcal{M} : (\varepsilon_{1k}, \varepsilon_{2k}, \dots, \varepsilon_{nk}) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)\}$$

is a G_δ set in $\mathcal{M}$. Therefore, as the intersection of countably many M_w, the set $\mathcal{U}$ is also G_δ.

For any universal incidence matrix m, the orbit $S_\infty M$ is dense in $\mathcal{M}$. Since $\mathcal{U}$ is S_∞-invariant, this completes the proof of the theorem. □

1.3. **Random graphs.** Now we will define the probability measures on the set of all countable graphs and in particular give a probabilistic proof of the existence of the universal graph Γ_U. That fact that the simplest such probability measures (independent entries) gives with probability one the universal graph is due to P. Erdös and A. Rényi ([4] see also [2]).

This proof will also imply the existence S^∞-invariant measures which are concentrated on the set $\mathcal{U}$ of universal incidence matrices.

We first give an intuitive description. Starting with a single point (one-point graph), we successively define random graphs Γ_n, $n = 1, 2, 3, \ldots$ in the following manner: To a given n-point graph $\Gamma_n = \{x_1, x_2, \ldots, x_n\}$ we add a $n+1$-th point x_{n+1} and independently set edges in $\Gamma_{n+1} = \Gamma_n \cup \{x_{n+1}\}$ between the new point x_n and the old points $x \in \Gamma_n$ according to

$$\begin{aligned}\mathrm{Prob}[(x_{n+1}, x_m) \in \Gamma_{n+1}] &= p \\ \mathrm{Prob}[(x_{n+1}, x_m) \notin \Gamma_{n+1}] &= 1 - p,\end{aligned}$$

where $0 < p < 1$ is a fixed number. Then, with probability one, the so obtained graph is the universal graph Γ_U:

Theorem 1.10 (P. Erdos, A. Renyi)**.** *Suppose* $\{\xi_{ij} : i < j\}$ *is an array of independent identically distributed random variables with*

$$\begin{aligned}P[\xi_{ij} = 1] &= p, \\ P[\xi_{ij} = 0] &= 1 - p,\end{aligned}$$

where $0 < p < 1$*. Set* $\xi_{ii} = 0$ *and* $\xi_{ij} = \xi_{ji}$ *for* $i > j$*. Then, with probability one,* $(\xi_{ij})_{i,j=1}^{\infty}$ *is a universal incidence matrix.*

Proof. We only have to show that, with probability one, every binary word $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{0, 1\}^n$ occurs infinitely often:

$$P[(\xi_{11}, \xi_{2k}, \ldots, \xi_{nk}) = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \text{ for infinitely many } k > n] = 1.$$

But this is evident since all the ξ_{ij}, $i < j$, are independent, and since for $k > n$

$$0 < P[(\xi_{1k}, \xi_{2k}, \ldots, \xi_{nk}) = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)] < 1,$$

which follows from $0 < p < 1$. □

Remark 1.11. The measures which were defined are invariant and ergodic with respect to the NW-shift $\sigma_{NW} : \mathcal{M} \to \mathcal{M}$ defined by

$$\sigma_{NW}(a_{i,j})_{i,j=1}^{\infty} = (a_{i+1,j+1})_{i,j=1}^{\infty}. \tag{1.5}$$

There are lot of measures (besides those above) which are invariant and ergodic with respect to the group S_∞ which acts on the set of symmetric $0-1$ matrices and concentrated on the subset of the universal matrices (see [13]). So we have a strong version of so called Kolmogorov's effect: there are uncountable many invariant ergodic measures for the transitive action of nonlocally compact group S_∞ .

Moreover we can say that the set of the probability measures on the space of symmetric $0 - 1$ matrices which have as support the set of universal matrices is everywhere dense G_δ-set in the weak topology of the space of measures.

In another words for the generic probability measure on the set of countable graphs a universal graph has probability 1.

2. The Urysohn Metric Space and Random Metric Spaces

In his last works [10], Paul S. Urysohn (1898–1924) gave a concrete construction of a universal separable metric space which is now known as “Urysohn Space”. Different to other universal separable metric spaces (for example $C([0,1])$ with the sup-norm, as was proven by Banach and Mazur), the Urysohn space is more special, since it is homogeneous in the sense of Definition 2.1 and the space $C([0,1])$ is not homogeneous. Moreover it turns out that, up to isometric equivalence, there is only one universal and homogeneous space – Urysohn space $\mathfrak{U}$.

In this section we give an explicit construction of the Urysohn space $\mathfrak{U}$ which is in the spirit of our construction of the universal graph in Section 1, i.e. it is based on the property that isometries from finite metric spaces F into $\mathfrak{U}$ can be extended to arbitrary one-point extensions of F (see also [12]). Beside universality, this extension property also implies homogeneity and the uniqueness of the Urysohn space. Another analogue to the case (Gurarij spaces (see [8]), or the Poulsen Simplex (see [7]) will be considered elsewhere. For different constructions of the Urysohn space, see [6] or [10].

Definition 2.1. A polish space (U, d) is said to be

(1) *universal*, if to any finite metric space (F, d_F) there exists an isometry $\iota : (F, d_F) \to (U, d)$.
(2) *homogeneous* , if to any finite subsets F_1, F_2 of U, and bijective isometry $\iota : F_1 \to F_2$, there exists an automorphism (=bijective isometry) $\iota' : U \to U$ which extends ι.
(3) a *Urysohn space*, if it is universal and homogeneous.

Note that any separable metric space (M, d) can be embedded isometrically into a separable Banach space $(B, \|\,.\,\|)$: For example, choose any point $x_0 \in M$ and consider the mapping

$$\iota : M \to C_b(M), \qquad x \mapsto \iota(x) = d(x,\,.\,) - d(x_0,\,.\,),$$

where $C_b(M)$ is the space of all bounded continuous functions on M, equipped with the sup-norm $\|\,.\,\|_\infty$. The triangular inequality

$$|d(x,z) - d(y,z)| \le d(x,y), \qquad x, y, z \in K,$$

implies that $\|\iota(x) - \iota(y)\|_\infty = d(x,y)$, and setting $z = y$ in the formula above shows that $\|\iota(x) - \iota(y)\|_\infty = d(x,y)$. Thus ι is an isometric embedding of M into $C_b(M)$. Set $B =$ closed linear span of ιM, $\|\,.\,\| = \|\,.\,\|_\infty$. This is a Banach space and by the separability of M, B is also separable.

This shows that any universal (separable) Banach space (for example $C[0,1]$ with the sup-norm, see [3] or other textbooks on topology) is also a universal separable metric space. But every known model of a universal Banach space is not homogeneous in the sense of Definition 2.1.

2.1. **Extending Isometries.**

In the sequel we will show that the Urysohn property can be characterized by certain extension properties of isometries. This enables us to see that Urysohn spaces are uniquely determined up to isomorphism.

Definition 2.2. Let (M_1, d_1), (M_2, d_2) be metric spaces and $\varepsilon > 0$. A mapping $\iota : M_1 \to M_2$ is said to be an *ε-isometry*, if

$$\|d_1(x,y) - d_2(x,y)\| \le \varepsilon, \qquad x, y \in M_1.$$

Definition 2.3. Suppose $\varepsilon \geq 0$. A metric space (M,d) is said to have the *ε-extension property*, if the following holds: Given any finite metric space (F, d_F), and a one-point extension $(F', d_{F'})$ of (F, d_F). If there is an isometric mapping $\iota : F \to M$, then ι can be extended to an ε-isometry $\iota' : F' \to M$, i.e. the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\iota \quad (\text{isometry})} & M \\ {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow \\ F' & \xrightarrow{\iota' \quad (\varepsilon-\text{isometry})} & M \end{array}.$$

If $\varepsilon = 0$, we simply speak of the *extension property.*

In other words, (M,d) satisfies the ε-extension property if and only if to any finite subspace $F \subseteq M$, and any (exterior) one-point metric extension $(F' = F \cup \{a\}, d_{F'})$ of F, there exists a point $a_\varepsilon \in M$ such that

$$|d(a_\varepsilon, x) - d_{F'}(a, x)| \leq \varepsilon, \qquad x \in F.$$

The extension property allows us to extend locally defined isometries to global isometries via a so-called *back and forth* argument:

Lemma 2.4. *Suppose (M,d), $(\tilde{M}, \tilde{d})$ are polish spaces which have the extension property, $F \subseteq M$ finite. Then any isometric mapping $\iota : F \to \tilde{M}$ can be extended to a bijective isometry $\iota' : M \to \tilde{M}$.*

Proof. Choose countable dense subsets $\{x_1, x_2, \ldots\} \subseteq M$ and $\{y_1, y_2, \ldots\} \subseteq \tilde{M}$.

Denote $F_1 = F$, $G_1 = \iota(F_1)$, and $\iota_1 = \iota : F_1 \to G_1$. Define $G_2 = G_1 \cup \{y_1\}$. By the extension property, $\iota_1^{-1} : G_1 \to F_1$ extends to an isometry $\iota_2^{-1} : G_2 \to F_2$ onto a subset $F_2 \supseteq F_1$. Now define $F_3 = F_2 \cup \{x_1\}$. Again there is an isometry $\iota_3 : F_3 \to G_3$ onto a subset $G_3 \supseteq G_2$. Take $G_4 = G_3 \cup \{y_2\}$. Continuing in the same manner as above, we can construct inductively subsets

$$F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots \subseteq M$$

with

$$F_n \supseteq \{x_k : 1 \leq k \leq \frac{n-1}{2}\},$$

subsets

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots \subseteq \tilde{M}$$

satisfying

$$G_n \supseteq \{y_k : 1 \leq k \leq \frac{n}{2}\},$$

and isometric bijections $\iota_n : F_n \to G_n$,

$$\iota = \iota_1 \subseteq \iota_2 \subseteq \cdots \subseteq \iota_n \subseteq \cdots$$

The mapping $\iota_\infty = \bigcup_{n=1}^\infty \iota_n$ maps $F_\infty = \bigcup_{n=1}^\infty F_n$ isometrically onto $G_\infty = \bigcup_{n=1}^\infty G_n$. Since F_∞ and G_∞ are dense, and since the metrics on M and $\tilde{M}$ are complete we may extend ι_∞ to an bijective isometry $\iota' : M \to \tilde{M}$. $\square$

Theorem 2.5. *A polish space is a Urysohn space if and only if it has the extension property. Two Urysohn spaces are isometrically isomorphic.*

Proof. A Urysohn space (M, d) has the finite extension property: By universality there is an isometry $\kappa : F_2 \to M$. Define $i : \kappa(F_1) \to \iota(M_1)$, $i = \iota \circ \kappa^{-1}$. Now, choose an automorphism $J : M \to M$ which extends i. The mapping $J \circ \kappa : F_2 \to M$ is an extension of ι.

On the other hand, if (M, d) has the extension property, then (M, d) is obviously universal. Moreover, if $\iota : F_1 \to F_2$ is an isometry between finite subspaces of M, then Lemma 2.4 (setting $M = \tilde{M}$) shows the existence of an automorphism $\iota' : M \to M$ which extends ι.

The uniqueness of two Urysohn spaces (M, d),$(\tilde{M}, \tilde{d})$ is now obvious: Choose any two points $x_1 \in M$, $x_2 \in \tilde{M}$ and consider the isometry $\iota : x_1 \mapsto x_2$. Since both spaces posses the extension property, Lemma 2.4 allows us to extend ι to an isomorphism $\iota' : M \to \tilde{M}$. □

Remark 2.6. The existence of a polish space which is Urysohn is shown in Section 2.2. Since such a space is unique up to isometry, we will simply speak of *the* Urysohn space $(\mathfrak{U}, d)$.

Note that if a polish space (M, d) satisfies the extension property, then it is also universal in the category of polish spaces, i.e. every polish space (P, d) can be embedded isometrically into M: Choose a countable dense subset $\{x_1, x_2, \ldots\} \subseteq M'$, and denote $F_n = \{x_1, \ldots, x_n\}$. There exist exists isometric mappings $\iota_n : F_n \to M$,

$$\iota_1 \subseteq \iota_2 \subseteq \cdots \subseteq \iota_n \subseteq \cdots$$

The mapping $\iota_\infty = \bigcup_{n=1}^{\infty} \iota_n$ maps $F_\infty = \bigcup_{n=1}^{\infty} F_n$ isometrically into M. Since F_∞ is dense in P, and since the metric on M is are complete, we may extend ι_∞ to an isometry $\iota : P \to M$.

Lemma 2.7. *Suppose a complete metric space (M, d) satisfies the ε-extension property for all $\varepsilon > 0$. Then (M, d) has the extension property for compact metric spaces: Given any compact metric space (C, d_C), and any one-point (or polish) extension $(C', d_{C'})$ of (C, d_C). If there is an isometric mapping $\iota : C \to M$, then ι can be extended to an isometry $\iota' : C' \to M$, i.e. the following diagram commutes:*

$$\begin{tikzcd} C \arrow[r, "\iota \quad \text{(isometry)}"] \arrow[d, "id"'] & M \arrow[d, "id"'] \\ C' \arrow[r, "\iota' \quad (\varepsilon-\text{isometry})"] & M \end{tikzcd}$$

Proof. First of all, note that for arbitrary $\varepsilon > 0$ the ε-extension property holds also for compact metric spaces: Given any compact metric space (C, d_C), one-point extension $(C' = C \cup \{a\}, d_{C'})$, and isometry $\iota : C \to M$. Because C is compact, we may choose a finite set $F \subseteq C$ such that

$$\min_{y \in F} d(\iota x, \iota y) \leq \varepsilon/2, \qquad x \in C.$$

By the assumption on M, there exists an $\varepsilon/2$-extension of the restriction of ι to F, which maps a to $x_a \in M$, say. Now the function

$$\iota' : C \cup \{a\} \to M, \begin{cases} \iota' = \iota & \text{on } C \\ a \mapsto x_a \end{cases}$$

is an ε-extension of $\iota : C \to M$, as one easily can verify.

To show the extension property, we have to find a point $\alpha \in M$ such that

$$d(\alpha, \iota(x)) = d_{C'}(a, x), \qquad x \in C. \tag{2.1}$$

Choose ε_n, such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. We will define inductively a sequence of points $x_n \in M$, $n = 0, 1, 2, \ldots$, such that

$$|d(x_n, \iota(x)) - d_{C'}(a, x)| \leq \varepsilon_n, \qquad x \in C, \tag{2.2}$$

and such that the sequence (x_n) is Cauchy in M. Then, since M is complete, there exists a limit α which certainly satisfies (2.1).

Since M satisfies the ε-extension property for $\varepsilon = \varepsilon_0$, we can find a point $x_0 \in M$ which satisfies (2.2) with $n = 0$. Define the other x_n, $n \geq 1$, by applying the following induction step:

Given a point x_n which satisfies (2.2). Define an (exterior) one-point extension $(C \cup \{x_n, b\}, d')$ of $(C \cup \{x_0\}, d)$ by:

$$\begin{aligned} d'(x, y) &= d(x, y) \qquad x, y \in C \cup \{x_n\} \\ d'(x, b) &= d_{C'}(x, a) \qquad x \in C \\ d'(x_n, b) &= \varepsilon_n \end{aligned}$$

In fact, by (2.2), d' satisfies the triangular inequality (if in addition $\varepsilon_n \leq \inf_{x \in C} d_{C'}(x, a)$, which we certainly may assume) as one can easily verify. Since M satisfies the ε-extension property for $\varepsilon = \varepsilon_{n+1}$, we can find a point $x_{n+1} \in M$ such that

$$|d(x_{n+1}, \iota(x)) - d_{C'}(a, x)| \leq \varepsilon_{n+1}, \qquad x \in C,$$

and

$$d(x_n, x_{n+1}) \leq d'(x_n, b) + \varepsilon_{n+1} = \varepsilon_n + \varepsilon_{n+1}.$$

By construction, 2.2 is satisfied, and since $\sum \varepsilon_n < \infty$, the latter inequality shows that (x_n) is a Cauchy sequence, which completes the proof. □

As an immediate consequence of Lemma 2.7, any polish space (M, d) which satisfies the extension property (for finite metric spaces) also satisfies the extension property for compact metric spaces. Repeating the proof of Lemma 2.4 verbatim we may conclude:

Lemma 2.8. *Suppose (M, d), $(\tilde{M}, \tilde{d})$ are polish spaces which have the extension property, $C \subseteq M$ compact. Then any isometric mapping $\iota : C \to \tilde{M}$ can be extended to a bijective isometry $\iota' : M \to \tilde{M}$.*

In particular, this shows that the Urysohn space is homogeneous with respect to its compact subspaces. Hence the Urysohn space $\mathcal{U}$ is a "good" universal object in the category of all compact metric spaces: every compact metric space (C, d_C) can be embedded isometrically into $\mathcal{U}$, and this embedding is unique up to isometry, i.e. if $\iota_1 : C \to U$, $\iota_2 : C \to U$ are two such isometries, then there exists an automorphism $J : \mathcal{U} \to \mathcal{U}$ such that

$$\iota_1 = J \circ \iota_2.$$

Lemma 2.8 is not true assuming C to be closed, even countable (see [1], [5]).

The following Theorem which will be important for our construction of the Urysohn space (See subsection 2.2):

Theorem 2.9. *A polish space is Urysohn if and only if it satisfies the ε-extension property for every $\varepsilon > 0$.*

Proof. Combine Lemma 2.7 and Theorem 2.5. □

2.2. **Construction, Universal Matrices.** Before show the existence of a Urysohn space, we need some preparations.

Suppose (M, d) is an infinite polish space. Choose a countable dense subset $D = \{x_1, x_2, x_3, \dots\}$ and define a matrix $r = (r_{ij})_{i,j=1}^{\infty}$ by

$$r_{i,j} = d(x_i, x_j). \tag{2.3}$$

Any matrix obtained in such a way is called a *(infinite) distance matrix*. The set of all infinite distance matrices is denoted by $\mathcal{R}$. We will consider $\mathcal{R}$ as topological space equipped with the topology that inherits from the product topology on $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$. $\mathcal{R}$ is a closed convex cone in $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$, in particular it is a polish space.

Similarly we say that a finite matrix $(r_{i,j})_{i,j=1}^{n}$ is a *(finite) distance matrix*, if there exists a finite metric space $(\{x_1, x_2, \dots, x_n\}, d)$ such that (2.3) holds. The set of all n-dimensional distance matrices, equipped with the product topology, is denoted by $\mathcal{R}_n$. If we define a (continuous) projection $\pi_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$ by

$$\pi_n (r_{ij})_{i,j=1}^{n+1} = (r_{ij})_{i,j=1}^{n},$$

then we can regard $\mathcal{R}$ as the inverse limit of the cones $\mathcal{R}_n$:

$$\mathbb{R}^+ \overset{\pi_1}{\leftarrow} \mathcal{R}_2 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{n-1}}{\leftarrow} \mathcal{R}_n \overset{\pi_n}{\leftarrow} \mathcal{R}_{n+1} \overset{\pi_{n+1}}{\leftarrow} \cdots$$

Given a n-dimensional distance matrix $r_n = (r_{ij})_{i,j=1}^{n}$, which corresponds to $F = (\{x_1, \dots, x_n\}, d)$. A vector $b = (b_1, \dots, b_n)$ is called *admissible*, if the matrix

$$r_n^b = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} & b_1 \\ r_{21} & r_{22} & \cdots & r_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} & b_n \\ b_1 & b_2 & \cdots & b_n & 0 \end{pmatrix}$$

is also a distance matrix. In other words, b is admissible if there exists a one-point extension $(F \cup \{x\}, d')$ of (F, d), such that

$$d'(x, x_i) = b_i, \qquad i = 1, \dots, n.$$

The set of all admissible vectors is then denoted by $\mathrm{Adm}(r_n)$. It determines all one-point extensions of (F, d) up to isomorphism. Note that

$$\mathrm{Adm}(r_n) = \{(b_i)_{i=1}^{n} \in \mathbb{R}^n : b_i - b_j \le r_{ij} \le b_i + b_j\}, \tag{2.4}$$

hence $\mathrm{Adm}(r_n)$ is a closed convex cone $\subseteq \mathbb{R}^n$.

Lemma 2.10. *Suppose $M = (\{x_1, x_2, x_3, \dots\}, d)$ is countable metric space, and consider the subspaces $F_n = (\{x_1, \dots, x_n\}, d)$, with $r_n = (d(x_i, x_j))_{i,j=1}^{n}$ as their corresponding distance matrices. Then*

$$\mathbb{R}^+ \overset{\pi_1}{\leftarrow} \mathrm{Adm}(r_2) \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_{n-1}}{\leftarrow} \mathrm{Adm}(r_n) \overset{\pi_n}{\leftarrow} \mathrm{Adm}(r_{n+1}) \overset{\pi_{n+1}}{\leftarrow} \cdots,$$

where $\pi_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates.

Proof. It is clear that $\pi_{n+1} \mathrm{Adm}(r_{n+1}) \subseteq \mathrm{Adm}(r_n)$. To prove the opposite inclusion, it suffices to show that to any two vectors $a, b \in \mathrm{Adm}(r_n)$ we can construct a two-point extension $M = (F_n \cup \{y_a, y_b\}, d)$ of F_n with $d(x_i, y_a) = a_i$ and

$d(x_i, y_b) = b_i$, $i = 1, \dots, n$. The only distance left to choose is $\alpha = d(y_a, y_b)$. The validity of the triangular inequality for d is equivalent to

$$|a_i - b_i| \le \alpha \le a_i + b_i, \qquad 1 \le i \le n,$$

which is only possible if

$$\max_{1\le i\le n} |a_i - b_i| \le \min_{1\le i\le n} a_i + b_i.$$

But the latter inequality is true since both vectors a and b are admissible: There exists a one-point extension $(F_n \cup \{z_a\}, d_a)$ of F_n with

$$d_a(z_a, x_i) = a_i, \qquad 1 \le i \le n,$$

and a one-point extension $(F_n \cup \{z_b\}, d_b)$ of F_n with

$$d_b(z_b, x_i) = b_i, \qquad 1 \le i \le n.$$

Now, using the triangular inequality for both metrics, we have

$$\begin{aligned} a_i - b_i = d_a(z_a, x_i) - d_b(z_b, x_i) &\le d_a(z_a, x_j) + d(x_i, x_j) - d_b(z_b, x_i) \le \\ &\le d_a(z_a, x_j) + d_b(z_b, x_j) = a_j + b_j. \end{aligned}$$

Since i and j where arbitrary, this implies the desired inequality. □

Remark. The above lemma shows that the cones $\mathrm{Adm}(r_n)$ are consistent under projections. *They are also consistent under permutations:* If g is any permutation on the set $\{1, 2, \dots n\}$ then we have

$$g\,\mathrm{Adm}(r_n) = \mathrm{Adm}(g r_n g^{-1}),$$

where for vectors $x = (x_1, x_2, \dots, x_n)$, its permutation gx is defined by $gx = (x_{g(1)}, x_{g(2)}, \dots, x_{g(n)})$.

Definition 2.11. A distance matrix $r = (r_{ij})_{i,j=1}^{\infty}$ is said to be *universal*, if for every $n \in \mathbb{N}$

$$\text{closure}\ \{(r_{1k}, r_{2k}, \dots, r_{nk}) : k = n, n+1, \dots\} = \mathrm{Adm}((r_{ij})_{i,j=1}^{n}). \tag{2.5}$$

It is called *weakly universal*, if $S_\infty r$ is dense in $\mathcal{R}$. The set of all universal matrices is denoted by $\mathcal{M}$.

Universal matrices are connected with Urysohn spaces:

Theorem 2.12. *Suppose (M, d) is a polish space, $(x_n)_{n=1}^{\infty}$ a dense sequence. The matrix $r = (d(x_i, x_j))_{i,j=1}^{\infty}$ is universal if and only if M is Urysohn.*

Proof. Suppose M is an Urysohn space. Fix $n \in \mathbb{N}$, $r_n = (d(x_i, x_j))_{i,j=1}^{n}$ and choose an admissible vector $(a_1, a_2, \dots, a_n) \in \mathrm{Adm}(r_n)$. By the extension property, there is an $x \in M$ such that

$$d(x, x_i) = a_i, \qquad i = 1, 2, \dots, n.$$

Since the x_k are dense in M, the property (2.5) is evident.

Conversely, suppose the distance matrix $(d(x_i, x_j))$ is universal. Choose a finite subset $F = \{y_1, y_2, \dots, y_n\} \subseteq M$, and an admissible vector $(a_1, a_2, \dots, a_n) \in \mathrm{Adm}(r_n)$, where $r_n = (d(y_i, y_j))_{i,j=1}^{n}$. Fix $\varepsilon > 0$. Since the x_k are dense, and since (2.5) holds, there are points $x_{k_1}, \dots, x_{k_n}$ and $x = x_{k_{n+1}}$ such that

$$|x_{k_i} - y_i| < \varepsilon/2, \qquad i = 1, 2, \dots, n$$

and

$$|d(x, x_{k_i}) - a_i| < \varepsilon/2, \qquad i = 1, 2, \ldots, n.$$

Therefore $|d(x, y_i) - a_i| < \varepsilon$, $i = 1, \ldots, n$, which proves the ε-extension property of M. Since $\varepsilon > 0$ was chosen arbitrarily, we can conclude by Lemma 2.7 that M has the extension property, hence it is a universal and homogeneous space. $\square$

It is not difficult to show that *the distance matrix* $r = (d(x_i, x_j))_{i,j=1}^{\infty}$ *is weakly universal, if and only if* (M, d) *is weakly universal in the following sense: To any finite metric space* (F, d_F)*, and any* $\varepsilon > 0$*, there exists an* ε*-isometry* ι *which maps* F *into* M. Thus every universal matrix is also weakly universal. The converse is not true: there exists weakly universal matrices (spaces), which are not universal matrices (spaces, resp.). For example consider a sequence of metric spaces

$$(F_1, d_1), (F_2, d_2), (F_3, d_3), \ldots,$$

such that their corresponding distance matrices

$$r_1, r_2, r_3, \ldots$$

enumerate all rational distance matrices of all dimensions. Now consider the disjoint sum (M, d) of the spaces (F_i, d_i), i.e. $M = \bigcup_i F_i$ and

$$d(x, y) = \begin{cases} d_i(x, y) & \text{for } x, y \in F_i \\ \frac{\operatorname{diam}(F_i) + \operatorname{diam}(F_j)}{2} & \text{for } x \in F_i, y \in F_j \end{cases},$$

where $\operatorname{diam}(F_i) = \max_{x,y \in F_i} d_i(x, y)$. The space (M, d) is complete, since it consists of isolated points only, separable, by construction weakly universal but not every finite metric space can be embedded into M.

Before we can start with the construction of the Urysohn space, we need another auxiliary lemma, which proof we leave to the reader.

Lemma 2.13. *Suppose* A_n,A_m *are convex cones in* $\mathbb{R}^n$, $\mathbb{R}^m$ *respectively* $(n > m)$*, such that* $\pi A_n = A_m$*, where* $\pi : \mathbb{R}^n \to \mathbb{R}^m$ *is the projection onto the first* m *coordinates. If* $(a_i)_{i=1}^{\infty}$ *is a sequence dense in* A_m*, then we can find vectors* $(a_i')_{i=1}^{\infty}$ *dense in* A_n *such that*

$$\pi a_i' = a_i, \qquad i \in \mathbb{N}.$$

Now we are able to construct a Urysohn space, or equivalently a universal matrix: To do so, we define a metric d on the set of naturals $\mathbb{N}$ as follows. Choose a sequence $(a_n^{(1)})_{n=2}^{\infty}$ dense in $\mathbb{R}_+$ and define

$$d(1, n) = a_n^{(1)}, \qquad i = 2, 3, \ldots$$

Note that d already defines a metric on the two-point set $\{1, 2\}$. Set $r_1 = (0)$, and let r_2 be the distance matrix which corresponds to the two-point space $(\{1, 2\}, d)$. Since the projection $\pi : \mathbb{R}^2 \to \mathbb{R}$, $(x_1, x_2) \to x_1$ maps $\operatorname{Adm}(r_2)$ onto $\operatorname{Adm}(r_1) = \mathbb{R}_+$, Lemma 2.13 shows that there is a sequence of numbers $(a_n^{(2)})_{n=3}^{\infty}$ such that

$$\operatorname{Adm}(r_2) = \text{closure } \{ \begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \end{pmatrix} : n \geq 3 \}.$$

Define

$$d(2, n) = a_n^{(2)}, \qquad n = 3, 4, \ldots$$

This defines already a metric on the three-point set $\{1,2,3\}$. Now let r_3 be the distance matrix corresponding to $(\{1,2,3\},d)$. Again, we can find a sequence of non-negative numbers $(a_n^{(3)})_{n=4}^\infty$ such that

$$\mathrm{Adm}(r_3) = \text{closure } \{ \begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \\ a_n^{(3)} \end{pmatrix} : i \geq 4 \}.$$

Define

$$d(3,n) = a_n^{(2)}, \qquad n \geq 4.$$

Continuing in this manner, we obtain a metric d on $\mathbb{N}$ such that by construction the matrix $(d(i,j))_{i,j=1}^\infty$ is universal. Therefore the completion of $(\mathbb{N}, d)$ is a Urysohn space. We have proved the main theorem of this section:

Theorem 2.14 (Urysohn)**.** *There exists a polish space $(\mathfrak{U}, d)$ which is universal and homogeneous. By Theorem 2.5 this space is determined uniquely up to isomorphism.*

Remark. For $\alpha > 0$, restricting ourselves to bounded distance matrices $r \in [0,\alpha]^{\mathbb{N}\times\mathbb{N}}$, one similarly can construct a homogeneous polish space $\mathfrak{U}_\alpha$ of diameter α which is universal in the following sense: *Every polish space of diameter $\leq \alpha$ can be embedded isometrically into $\mathfrak{U}_\alpha$.* Such a space is also unique up to isometric equivalence.

2.3. **Some Properties of the Urysohn space.** We now consider topological properties of the Urysohn space.

Theorem 2.15. *Any continuous map $f : C \to \mathfrak{U}$ from a compact metric space (C, d_C) into the Urysohn space $(\mathfrak{U}, d)$ is contractible (=homotopic to the constant map).*

Proof. Consider the image $f(C)$ as a subspace of $\mathfrak{U}$. This is a compact metric space and we therefore can find a isometric embedding $\iota : f(C) \to B$ into a separable Banach space $(B, \|\,.\,\|)$ (as was shown at the beginning of this section). By the extension property, the isometry $\iota^{-1} : \iota(f(C)) \to C$ can be extended to an isometry $J : B \to \mathfrak{U}$:

$$\begin{array}{ccc} (C, d_C) & \xrightarrow{f} & (\mathfrak{U}, d) \\ {\scriptstyle g=\iota\circ f}\big\downarrow & & {\scriptstyle id}\big\downarrow \\ (B, \|\,.\,\|) & \xrightarrow{J} & (\mathfrak{U}, d) \end{array}$$

Since the mapping $g : C \to B$ is contractible, f is also contractible. ☐

Theorem 2.16. *The Urysohn space $(\mathfrak{U}, d)$ is path-wise connected. Moreover all homotopy groups π_n (and therefore all homology groups) are trivial.*

Proof. That $\mathfrak{U}$ is path-wise connected is an immediate consequence of the extension property: The mapping which maps two different numbers $a, b \in \mathbb{R}$ to two different points $x, y \in \mathfrak{U}$ is an isometry, provided we have chosen the proper norm on $\mathbb{R}$, and can therefore be extended to an isometric embedding of the whole interval $[a,b]$ into $\mathfrak{U}$.

By Theorem 2.15, any continuous mapping $f : S^n \to \mathfrak{U}$ from the n-dimensional sphere into $\mathfrak{U}$ is contractible, hence all homotopy groups of U are trivial. ☐

PROBLEM

Is Urysohn space $\mathfrak{U}$ contractible or not?

Consider the group $\mathrm{Aut}(\mathfrak{U})$ of isometries of the Urysohn space, equipped with the topology of pointwise convergence. In [11], the author shows that $\mathrm{Aut}(\mathfrak{U})$ is a universal topological group with a countable base, i.e. every Hausdorff topological group with a countable base is isomorphic to a subgroup of $\mathrm{Aut}(\mathfrak{U})$. Apart from that, not very much is known about this group.

PROBLEM

To describe algebraic or topological properties of the Isometry Group $\mathrm{Aut}(\mathfrak{U})$ of the Urysohn space $\mathfrak{U}$.

We continue with miscellaneous considerations about the Urysohn space.

Theorem 2.17. *Consider the product $\mathfrak{U}^2 = \mathfrak{U} \times \mathfrak{U}$ equipped with the maximum metric $d_\infty((x,y),(x',y')) = \max\{d(x,x'),d(y,y')\}$. The product $\mathfrak{U}^2$ is separable and universal, but not isometrically isomorphic to $\mathfrak{U}$*

Proof. We show that $(\mathfrak{U}^2, d_\infty)$ does not satisfy the extension property: Choose a subspace $F_1 = \{x_1, x_2, x_3\} \subseteq \mathfrak{U}$ with distance matrix

$$r_{F_1} = \begin{pmatrix} 0 & 1 & 1/4 \\ 1 & 0 & 1 \\ 1/4 & 1 & 0 \end{pmatrix}.$$

and a subspace $F_2 = \{y_1, y_2, y_3\} \subseteq \mathfrak{U}$ with distance matrix

$$r_{F_2} = \begin{pmatrix} 0 & 1/4 & 1 \\ 1/4 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Let $F \subseteq \mathfrak{U}^2$ be the subspace which consists of the points (x_i, y_i), $i = 1,2,3$. All edges of this triangle have length one, i.e. the distance matrix of F is

$$r_F = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The vector $(1/2, 1/2, 1) \in \mathrm{Adm}(r_F)$, but, by equation (2.4), any vector $(b_1, b_2, b_3) \in \mathrm{Adm}(r_{F_j})$, $j = 1,2$, with $b_j \leq 1/2$ must have $b_3 < 1$. Thus there is no point $(a,b) \in \mathfrak{U}^2$ such that

$$(d((a,b),(x_i,y_i)))_{i=1}^3 = (1/2, 1/2, 1).$$

□

Remark 2.18. The same argument as in the proof of Theorem 2.17 also works out for the product $\mathfrak{U} \times [0,1]^2$ (equipped with the maximum metric). Hence $\mathfrak{U} \times [0,1]^2$ and therefore $\mathfrak{U} \times [0,1]$ (also equipped with the maximum metric) is not isometrically isomorphic to $\mathfrak{U}$.

For any $\alpha > 0$, $(\mathfrak{U}, \alpha d)$ also has the extension property and is therefore isometrically isomorphic to $(\mathfrak{U}, d)$: Suppose $\iota : C \to \mathfrak{U}$ is an isometric embedding of a compact space (C, d_C) into $(\mathfrak{U}, \alpha d)$, and suppose (C', d'_C) is a one-point extension of (C, d_C). By

$$\alpha d(\iota x, \iota y) = d_C(x,y), \qquad x, y \in C$$

the same ι embeds $(C, \alpha^{-1}d_C)$ isometrically into $(\mathfrak{U}, d)$. We thus can find an isometry $\iota' : (C', \alpha^{-1}d_{C'}) \to (\mathfrak{U}, d)$ which extends ι. By the same argument as before, ι' is also an isometric embedding of $(C', d_{C'})$ into $(\mathfrak{U}, \alpha d)$.

2.4. **The set $\mathcal{U}$ of universal matrices.** As for incidence matrices in Section 1, the group S^∞ of permutations on $\mathbb{N}$ acts via formula (1.4) also on the cone of all distance matrices. Unlike to the case of graphs may not regard $\mathrm{Aut}(\mathfrak{U})$ as a subgroup of S^∞.

Theorem 2.19. *The set $\mathcal{U}$ of all universal distance matrices is S_∞-invariant, i.e. $g\mathcal{M}g^{-1} = \mathcal{M}$ for every $g \in S_\infty$. Moreover, it is a dense G_δ-subset of $\mathcal{R}$.*

Proof. ¿From the remark which follows Lemma 2.10 it is obvious that a distance matrix $r = (r_{ij})_{i,j=1}^\infty$ is universal if and only if for every finite subset $\{k_1, k_2, \dots, k_n\} \subseteq \mathbb{N}$,

$$\mathrm{Adm}((r_{k_i k_j})_{i,j=1}^n) = \text{closure } \{(r_{k_1 k}, r_{k_2 k}, \dots, r_{k_n k}) : k = n, n+1, \dots\}.$$

Thus $\mathcal{M}$ is S_∞-invariant.

By the above construction, there is at least one universal matrix $r \in \mathcal{R}$. Its orbit $S_\infty r$ dense in $\mathcal{R}$, and consists of universal matrices only. Hence $\mathcal{U}$ is dense in $\mathcal{R}$. For fixed $n, m \in \mathbb{N}$ and $\alpha > 0$, the function

$$\varepsilon_{n,m,\alpha} : r = (r_{ij}) \mapsto \sup_{a \in \mathrm{Adm}(\pi_n r), \|a\| \le \alpha} \ \min_{n < k \le m} \|a - (r_{ik})_{i=1}^n\|,$$

where $\|\cdot\|$ is the maximum norm on $\mathbb{R}^n$, is a continuous map $\mathcal{R} \to \mathbb{R}$. By definition,

$$\mathcal{U} = \bigcap_{k \in N} \bigcap_{n \in \mathbb{N}} \bigcap_{\alpha \in \mathbb{N}} \bigcup_{m > n} [\varepsilon_{n,m,\alpha} < \frac{1}{k}]$$

is therefore a G_δ-set. $\square$

We will now give a probabilistic proof of the existence of the Urysohn space in the spirit of Erdös Theorem 1.10. This procedure will also give an idea of a random metric space. To emphasize the principle idea we only give an intuitive but not formally rigorous presentation:

Starting with a single point we successively construct a random sequence (M_n, d_n) of n-point metric spaces (or n-dimensional distance matrices r_n) ,

$$(M_0, d_0) \subseteq (M_1, d_1) \subseteq (M_2, d_2) \subseteq \dots \subseteq (M_n, d_n) \subseteq \dots,$$

as follows: to a given n-point metric space M_n (with distance matrix $r_n \in \mathcal{R}_n$) we randomly add a $n+1$-th point choosing the distances between the new and the previous points (= admissible vector $\in \mathrm{Adm}(r_n)$) according to a certain (conditional) probability distribution μ_{r_n} (this is a probability measure on $\mathrm{Adm}(r_n)$).

To ensure that, with probability one, the so obtained space is Urysohn (or equivalently, its distance matrix is universal), we choose the distributions μ_r as follows: On the one-dimensional admissible cone $\mathbb{R}^+$ we take any fixed measure $\mu_\emptyset$ which supports the whole half-line (i.e. to any open subset B of $\mathbb{R}^+$, $\mu_\emptyset(B) > 0$). Now we continue inductively via the following procedure:

Suppose we had constructed all the measures μ_{r_n} for all one-dimensional distance matrices $r_n \in \mathcal{R}_n$ such that its support $\mathrm{supp}\, \mu_{r_n}$ is always the whole admissible cone $\mathrm{Adm}(r_n)$. For any $n+1$-dimensional distance matrix $r_{n+1} \in \mathcal{R}_{n+1}$, recall that

$$\mathrm{Adm}(\pi_n r_{n+1}) \overset{\pi_n}{\leftarrow} \mathrm{Adm}(r_{n+1}),$$

where the projection is onto. Now in $\mathrm{Adm}(r_{n+1})$ we take any measure $\mu_{r_{n+1}}$ such that $\pi_n\mu_{r_{n+1}} = \mu_{r_n}$ and

$$\operatorname{supp}\mu_{r_{n+1}} = \mathrm{Adm}(r_{n+1})$$

Such a measure does exist, but we won't give the detailed construction here.

On the cone $\mathcal{R}_1$ we put a probability measure μ_1, $\mu_1 = \mu_\emptyset$. Using the measures μ_r, we define probability measures μ_n on $\mathcal{R}_n$ via

$$\mu_n(\,.\,|\pi_{n-1}r) = \mu_{\pi_{n-1}r}(\tilde{\pi}_{n-1}\,.\,),$$

where $\tilde{\pi}_{n-1}(r_{ij})_{i,j=1}^n = (r_{1,n}, r_{2,n}, \dots, r_{n-1,n})$. Note that $\tilde{\pi}_{n-1}$ maps each matrix $r \in \mathcal{R}_n$ into the the admissible cone $\mathrm{Adm}(\pi_{n-1}r)$, hence this definition makes sense.

The so constructed family μ_n is consistent under projection,

$$\pi_m\mu_n = \mu_m, \qquad m \le n,$$

hence it defines a probability measure μ on $\mathcal{R}$. By construction the probability that the random distance matrix is universal equals one, thus

$$\mu(\mathcal{U}) = 1.$$

Remark 2.20. To assure that, with probability one, the random distance matrix is universal, it is certainly not necessary that the conditional distributions are consistent under projections, i.e.

$$\mu_{\pi_m r} = \pi_m\mu_r, \qquad r \in \mathcal{R}_n, m \le n,$$

as assumed in the discussion above (see [12]). Anyway, note that the measure μ in the above construction is generically not S_∞-invariant.

At the end of this section we gather

- (cf. Theorem 2.14) There exists polish space $(\mathfrak{U}, d)$ which is Urysohn, i.e. universal and homogeneous. This space is unique up to isometric equivalence.
- (cf. Theorem 2.16) The Urysohn space $(\mathfrak{U}, d)$ is path-wise connected. All homotopy groups π_n of $(\mathfrak{U}, d)$, and therefore all homology groups, are trivial.
- Open question: Is $(\mathfrak{U}, d)$ contractible?
- Open question: Find familiar spaces to which $(\mathfrak{U}, d)$ is homeomorphic.
- (cf. Theorem 2.12) A space is Urysohn if and only if its distance matrix of any (= some) everywhere dense countable subset is universal.
- (cf. Theorem 2.19) The set of all universal matrices is dense G_δ subset of $\mathcal{R}$, which is invariant under finite permutations.

2.5. **Random Metric Space.** Here we will give a sketch of the construction of the random metric space. For this we will define a probability measure on the space of distance matrices which is concentrated on the subset of universal matrices. This construction is analogue to the same construction of the random graph but more complicate because it is not possible to use independence of entries (triangle inequality is the obstruction to that). Choose some measure γ on the half line $\mathbf{R}_+$ which has full support (e.g. gaussian measure on the half line) and let entries

$$r_{1,2}, \dots r_{1,n}, \dots$$

are i.i.d with distribution γ. We will define the conditional distribution of the element of distance matrix $r_{m,n}, (n > m)$ under the condition that all the elements with indices (i, k) are fixed where $1 \le i \le m-1, 1 \le j \le n$

Then the conditional distribution of the element $r_{m.n}$ is uniform distribution on the interval $[a, b]$ where

$$a = \max_{i=1,\dots m-1} |r_{i,m} - r_{i,n}|; \mathrm{b} = \min_{i=1\dots m-1} |r_{i,m} + r_{i,n}|$$

Lemma 2.10 shows that $a \leq b$ so interval $[a, b]$ is not empty and we can by induction to define the distributions of all entries of the distance matrix. Remark that distribution of the element $r_{m,n}$ actually depends on the distribution of $r_{i,m}$ and $r_{i,n}$ for $i = 1, \dots m-1$ only. As result we had define a probability measure μ_γ on the space of distance matrices with parameter – measure γ on the halfline.

Theorem 2.21. *For any absolutely continuous measure γ on the half line* $\mathbf{R}$ *with positive density the set of universal distance matrices has measure 1 with respect to the measure μ_γ. In another word. Let r is a random distance matrix with distribution mu_γ in the space of all distance matrices; then a metric space which is completion of the set of naturals* $\mathbf{N}$ *under the distance matrix r is the Urysohn space with probability 1.*

The proof uses the law of large numbers in the special situation but we will not prove the theorem here. As in the case of graphs we emphasize that the last fact is valid for very reach set of measures on the space of distance matrices, – we gave only one simple example which is in a sense similar to the example with independent entries for the graphs. Namely, as in paragraph 1 for the case of graph, the set of probability measures on the space of distance matrices for which subset of universal matrices has a full measure is again everywhere dense G_δ subset of the space of all probability measures with weak topology.

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