SPb. Math. Society Preprint 2003-07 23 Dec 2003

Density of polynomials in the de Branges spaces of entire functions

A. D. Baranov

Introduction[1]

Let E be an entire function satisfying the inequality

$$|E(z)| > |E(\overline{z})|, \qquad z \in \mathbb{C}^+, \tag{1}$$

where $\mathbb{C}^+ = \{z : \operatorname{Im} z > 0\}$ is the upper half-plane. The class of such functions (known as the Hermite-Biehler class) we denote by HB. With each function $E \in HB$ we associate a Hilbert space $\mathcal{H}(E)$ which consists of all entire functions F such that F/E and F^*/E belong to the Hardy class $H^2(\mathbb{C}^+)$ (here and later on $F^*(z) = \overline{F(\overline{z})}$). The inner product, which makes $\mathcal{H}(E)$ a Hilbert space, is defined by the formula

$$\langle F, G\rangle_{\mathcal{H}(E)} = \int\limits_{\mathbb{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2} dt.$$

The theory of spaces $\mathcal{H}(E)$ introduced by L. de Branges [1] has important applications in mathematical physics. At the same time, de Branges spaces are of interest from the point of view of theory of entire functions. As a special case of de Branges spaces the Paley – Wiener space PW_a of entire functions of exponential type at most a square summable on the real axis may be considered (this space corresponds to the function $E(z) = \exp(-iaz)$, $a > 0$).

De Branges spaces are also closely connected with the shift-coinvariant subspaces K_Θ of the Hardy class $H^2(\mathbb{C}^+)$ known also as model subspaces

[1]The work is partially supported by RFBR grant No 03-01-00377, by the grant for Leading Scientific Schools No NSH-2266.2003.1, and by the Ministry of Education grant No E02-1.0-66.

[2, 3]. Let Θ be an inner function in the upper half-plane. Put $K_\Theta = H^2(\mathbb{C}^+) \ominus \Theta H^2(\mathbb{C}^+)$. If E is an entire function satisfying (1), then $\Theta_E = E^*/E$ is inner and the mapping $F \mapsto F/E$ is a unitary operator from $\mathcal{H}(E)$ onto K_{Θ_E}.

In the present note we are concerned with the following three related problems. By $\mathcal{P}$ we denote the set of all polynomials.

1. For which entire functions $E \in HB$ the function $f \equiv 1$ belongs to the space $\mathcal{H}(E)$?

2. For which $E \in HB$ the inclusion $\mathcal{P} \subset \mathcal{H}(E)$ holds?

3. For which $E \in HB$ the polynomials are dense in $\mathcal{H}(E)$?

Density of polynomials is a classical problem for weighted functional spaces. At the same time, there are rather deep motivations for considering these problems for the de Branges spaces particularly. The first question is inspired by the recent works by V.P. Havin and J. Mashreghi [4, 5] where the admissible majorants for the model subspaces were studied in the spirit of the Beurling – Malliavin theorem. By an admissible majorant for the space K_Θ we mean a nonnegative function w on the real line such that $w \geq |f|$ for some nonzero function $f \in K_\Theta$. It turns out that the condition $1 \in \mathcal{H}(E)$ is crucial for the existence of the majorant with the fastest rate of decrease. This problem was also investigated in [6].

The third problem is closely connected with the Hamburger moment problem on the line. It was shown in [7] that the answer to this question will lead to the description of all canonical solutions of the indeterminate Hamburger problem. Moreover, in [8] a criterion for the density of polynomials was obtained, which is, however, somewhat implicit and not so easy to apply. At the same time, there exist very simple geometrical conditions on the zeros of E sufficient for the density of polynomials. It was mentioned by N.I. Akhiezer [9] that if $E \in HB$ is a canonical product of genus 0 with zeros lying in a strip $|\mathrm{Re}\, z| \leq h$ then $\mathcal{P} \subset \mathcal{H}(E)$ and $\mathrm{Clos}_{\mathcal{H}(E)}\mathcal{P} = \mathcal{H}(E)$. Another example is due to V.P. Gurarii [10]: if E is an even function with zeros in the angle $-3\pi/4 \leq \arg z \leq -\pi/4$ then the polynomials are also dense in $\mathcal{H}(E)$.

It should be noted that problems 2 and 3 lead to different classes of functions. It is easy to construct an example of the space $\mathcal{H}(E)$ such that $\mathcal{P} \subset \mathcal{H}(E)$, but $\mathcal{P}$ is not dense in $\mathcal{H}(E)$. It is known that if the function $B = \frac{E^* - E}{2i}$ belongs to $\mathcal{H}(E)$ then it is orthogonal to the domain of the operator of multiplication by z (see [1], Theorem 29). Thus, B is orthogonal to all polynomials in $\mathcal{H}(E)$. For a function $E \in HB$ of zero type one has $B \in \mathcal{H}(E)$ if and only if the zeros z_n of E satisfy the condition $\sum_n |\mathrm{Im}\, z_n|^{-1} < \infty$ [11]. P. Koosis [12] has constructed a much more subtle example showing that polynomials may be not dense in $\mathcal{H}(E)$ even in the case when $B \notin \mathcal{H}(E)$.

Acknowledgements. The author thanks V.P. Havin, E. Abakumov and M. Sodin for posing of the problems and helpful discussions. This work was partially done while the author was visiting McGill University, Montreal, in October 2002. The author thanks this institution for hospitality.

§1. Main results

In the present note we give some simple geometric conditions generalizing the results of Akhiezer and Gurarii. In particular, we study the following problem: to describe the subsets Ω of $\mathbb{C}^+$ such that for any function $E \in HB$ with zeros in the conjugate set $\overline{\Omega}$ we have the inclusion $\mathcal{P} \subset \mathcal{H}(E)$. In this case the inclusion $\mathcal{P} \subset \mathcal{H}(E)$ holds independently of the distribution of zeros of E in Ω, and we say that the set Ω is a distribution independent (d. i.) set. The result of Akhiezer states that $\{z = x + iy : y > 0, |x| \le h\}$ is a d. i. set for any $h > 0$.

In what follows we assume that E is a function of the form

$$E(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\overline{z}_n}\right), \tag{2}$$

where $z_n \in \mathbb{C}^+$, $\sum_n |z_n|^{-1} < \infty$. For the sake of convenience we denote the zeros of E by $\overline{z}_n$ so that $z_n \in \mathbb{C}^+$. The case when zeros z_n lie in a Stolz angle $\Gamma_\gamma = \{z : y \ge \gamma|x|\}$, $\gamma > 0$, is of special interest. In this case E belongs to the so-called class A discussed in details in B.Ya. Levin's monograph [13]).

The following theorem gives a condition on the growth of the zeros of E which is sufficient for the density of polynomials.

Theorem 1. *Let $z_n = x_n + iy_n$ and $\inf_n y_n > 0$. If*

$$\sum_{|x_n| > R} \frac{|x_n|}{|z_n|^2} \le AR^{-1}, \qquad R > 0, \tag{3}$$

for some $A > 0$*, then* $\mathcal{P} \subset \mathcal{H}(E)$ *and* $\mathrm{Clos}_{\mathcal{H}(E)}\mathcal{P} = \mathcal{H}(E)$.

Theorem 2. *Let* E *be an even entire function of the form (1),* $\inf_n y_n > 0$ *and*

$$\sum_{|x_n|^2 > y_n^2 + R^2} \frac{|x_n|^2}{|z_n|^4} \le AR^{-2}, \qquad R > 0. \tag{4}$$

Then $\mathcal{P} \subset \mathcal{H}(E)$ *and* $\mathrm{Clos}_{\mathcal{H}(E)}\mathcal{P} = \mathcal{H}(E)$.

As a particular case of Theorem 1 we obtain a description of distribution independent sets generalizing the result of Akhiezer.

Theorem 3. *Let* $\Omega \subset \{z = x + iy : y > qx^2\}$ *for some constant* $q > 0$*. Then* Ω *is a d. i. set. Moreover, this condition is in a sense sharp: for any* $\alpha < 2$ *there exists a function* $E \in HB$ *such that* $z_n \in \{y \ge |x|^\alpha\}$ *and* $1 \notin \mathcal{H}(E)$.

Let zeros z_n lie outside some Stolz angle Γ_γ. Then $|x_n| \asymp |z_n|$ and the condition (3) is equivalent to $\sum_{|z_n|>R} |z_n|^{-1} \le CR^{-1}$. Thus, the sequence z_n tends to infinity very fast (say, as a progression ρ^n, $\rho > 1$). To deal with a more slow growth of z_n one should consider more precise estimates. By $n(t)$ we denote the number of the zeros in the disc $\{|z| \le t\}$.

Theorem 4. *Let zeros* z_n *lie in a Stolz angle* Γ_γ*. If*

$$\sum_{|z_n|>R} \frac{|x_n|}{|z_n|^2} = o\left(\frac{1}{R} \int_0^R \frac{n(t)}{t} dt + R^2 \int_R^\infty \frac{n(t)}{t^3} dt \right), \qquad R \to \infty, \tag{5}$$

then $\mathcal{P} \subset \mathcal{H}(E)$.

Condition (5) is applicable to sequences z_n growing "faster than any power" (say, $|z_n| \asymp \exp(\log^\alpha n)$, $\alpha > 1$), but it fails for $|z_n| \asymp n^\alpha$, $\alpha > 1$.

Theorem 5. *Let* $\{z_n\} \subset \{\gamma_1 x \le y \le \gamma_2 x\}$*,* $\gamma_1, \gamma_2 > 0$*, and* $C_1 n^\alpha \le |z_n| \le C_2 n^\alpha$*. Then there exist exponentials* α_1 *and* α_2 *(depending on* $\gamma_1, \gamma_2, C_1, C_2$*) such that* $E(x) \to \infty$*,* $|x| \to \infty$*, and* $\mathcal{P} \subset \mathcal{H}(E)$ *if* $\alpha > \alpha_1$*, whereas* $E(x) \to 0$*,* $|x| \to \infty$*, if* $\alpha < \alpha_2$.

We construct an example showing that in this case the behavior of E depends essentially on the zeros' distribution. Namely, for any $\alpha > 1$ there exist z_n with $\arg z_n = \pi/4$, $|z_n| \asymp n^\alpha$ and $1 \notin \mathcal{H}(E)$. We also consider the case when zeros are distributed regularly along a single ray: in this case we obtain an explicit formula for the limit exponential.

Theorem 6. *Let $\gamma > 0$ and $z_n = n^\alpha(1+\gamma i)$, $\alpha > 1$. Then $E(x) \to \infty$, $x \to \infty$, if $\alpha > 2 - \frac{2\mathrm{arctg}\,\gamma}{\pi}$, and $E(x) \to 0$, $x \to \infty$, if $\alpha < 2 - \frac{2\mathrm{arctg}\,\gamma}{\pi}$.*

In the last section we discuss the relationship between the density of polynomials and the hypercyclicity phenomenon. Recall that a vector f is said to be hypercyclic for a bounded linear operator T in a Fréchet space F if its orbit $\{T^n f\}_{n=0}^\infty$ is dense in F. In this case the operator T is also said to be hypercyclic.

The first example of a hypercyclic operator was obtained by G.D. Birkhoff [14] who showed that the translation operators $T_w : f \mapsto f(\cdot + w)$, $w \in \mathbb{C}$, $w \neq 0$, are hypercyclic in the space of all entire functions with the topology of uniform convergence on compact subsets of the plane. It was shown in [15] that the differentiation operator is also hypercyclic in this space, whereas in [16] this result was extended to all operators commuting with the differentiation. Thus, spaces of entire functions proved to be an important source of hypercyclic operators. K.C. Chan and J.H. Shapiro [17] studied the hypercyclicity of translations in the setting of Hilbert spaces of entire functions of "slow growth". As a corollary of the results of [17] we obtain the following theorem.

Theorem 7. *Let $\mathcal{H}(E)$ be a de Branges space such that $\mathcal{P} \subset \mathcal{H}(E)$ and the differentiation operator is bounded in $\mathcal{H}(E)$. Then the translation operators T_w, $w \neq 0$, are hypercyclic in $\mathcal{H}(E)$.*

Note that the translations are bounded in $\mathcal{H}(E)$ whenever the differentiation operator $\mathcal{D}$ is bounded, since $T_w = \exp(w\mathcal{D})$. The de Branges spaces $\mathcal{H}(E)$ such that the differentiation is bounded on $\mathcal{H}(E)$ were described in the author's work [18]. In particular, it is the case if E is an entire function of the form (2) with zeros in a Stolz angle.

§2. Proof of Theorems 1 and 2

We begin with a condition sufficient for the density of polynomials in $\mathcal{H}(E)$. The following proposition is analogous to Theorem 6 in [10]. Recall that the reproducing kernel of the space $\mathcal{H}(E)$ corresponding to the point $w \in \mathbb{C}$ is of the form

$$K(z,w) = \frac{i}{2\pi} \cdot \frac{\overline{E(w)}E(z) - \overline{E^*(w)}E^*(z)}{z - \overline{w}}.$$

Proposition 8. *Assume that there is a sequence P_n of polynomials, which converges to the function E uniformly on any compact set, and*

$$|P_n(x)| \le C|E(x)|, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{6}$$

Then $\mathcal{P} \subset \mathcal{H}(E)$ and $\mathrm{Clos}_{\mathcal{H}(E)}\mathcal{P} = \mathcal{H}(E)$.

Proof. The inclusion $\mathcal{P} \subset \mathcal{H}(E)$ follows immediately from (6). To prove the density we note first that the set of reproducing kernels $\{K(\cdot, s),\ s \in \mathbb{R}\}$ is dense in $\mathcal{H}(E)$. Thus, it suffices to show that $K(\cdot, s) \in \mathrm{Clos}_{\mathcal{H}(E)}\mathcal{P}$. Fix $s \in \mathbb{R}$ and put

$$K_n(z, s) = \frac{i}{2\pi} \cdot \frac{\overline{P_n(s)}P_n(z) - P_n(s)P_n^*(z)}{z - s}.$$

Clearly, $K_n(\cdot, s)$ is a polynomial, and the sequence $K_n(\cdot, s)$ converges to $K(\cdot, s)$ uniformly on any compact set. Let us show that $K_n(\cdot, s)$ converge to $K(\cdot, s)$ in the norm of the space $\mathcal{H}(E)$. Take $A > 0$ and split the norm $\|K_n(\cdot, s) - K(\cdot, s)\|^2_{\mathcal{H}(E)}$ into two parts:

$$\int\limits_{|t-s|\le A} \left| \frac{K_n(t, s) - K(t, s)}{2\pi E(t)(t - s)} \right|^2 dt + \int\limits_{|t-s|>A} \left| \frac{K_n(t, s) - K(t, s)}{2\pi E(t)(t - s)} \right|^2 dt = I_1 + I_2.$$

We estimate the integral I_2. By inequality (6) we have

$$\left| \frac{K_n(t, s) - K(t, s)}{E(t)} \right| \le 2|P_n(s)| \cdot \left| \frac{P_n(t)}{E(t)} \right| + 2|E(s)| \le (2C^2 + 2)|E(s)|.$$

Hence, $I_2 \le C_1|E(s)|^2A^{-1}$ for all $n \in \mathbb{N}$. Thus, choosing a sufficiently large A we can make the integral I_2 as small as we wish uniformly with respect to n. Now, when A is fixed, the integral I_1 tends to zero when n tends to infinity. ◯

We introduce certain notations which are used in what follows. Note that

$$|E(x)|^2 = \prod_n \frac{(x - x_n)^2 + y_n^2}{|z_n|^2} = \prod_n \left(1 - \frac{2xx_n - x^2}{|z_n|^2}\right).$$

We split this product into two parts $\Pi_+(x)$ and $\Pi_-(x)$, where

$$\Pi_-(x) = \prod_{x_n > x/2} \left(1 - \frac{2xx_n - x^2}{|z_n|^2}\right)$$

for $x > 0$; if $x < 0$ we take the product over n such that $x_n < x/2$. Thus, each factor in the product Π_- is smaller than 1 whereas the factors in the product Π_+ are greater or equal to 1.

Proof of Theorem 1. We show that there exist $l \in \mathbb{N}$ and $C > 0$ such that

$$\Pi_-(x) \geq C(\delta/x)^{2l} \tag{7}$$

for sufficiently large x; here $\delta = \inf_n y_n$. Once the inequality (21) is proved the rest of the theorem follows immediately. Indeed, in this case the polynomials

$$P_k(z) = \prod_{|n| \leq k} \left(1 - \frac{z}{\overline{z}_n}\right).$$

satisfy the conditions of the Proposition 8.

Without loss of generality let $x > 0$. Denote by $N(x)$ the set of n such that $2xx_n - x^2 \geq |z_n|^2/2$ and let $|N(x)|$ be the number of elements in $N(x)$. Then, by (3), we have the estimate

$$\frac{1}{2}|N(x)| \leq \sum_{x_n > x/2} \frac{2xx_n - x^2}{|z_n|^2} \leq 2x \sum_{|x_n| > x/2} \frac{|x_n|}{|z_n|^2} \leq 4A,$$

which implies that there is $l \in \mathbb{N}$ such that $|N(x)| \leq l$ for any $x > 0$. Note also that $|z_n| \leq 4x$ whenever $n \in N(x)$. Hence,

$$\prod_{n \in N(x)} \frac{(x - x_n)^2 + y_n^2}{|z_n|^2} \geq \left(\frac{\delta}{4x}\right)^{2l}.$$

Finally, making use of the elementary estimate $\log(1-t) \geq -2t$, $t \in [0,\, 1/2]$, we get

$$\log \prod_{n \notin N(x),\, x_n > x/2} \left(1 - \frac{2xx_n - x^2}{|z_n|^2}\right) \geq$$

$$\geq -2 \sum_{n \notin N(x),\, x_n > x/2} \frac{2xx_n - x^2}{|z_n|^2} \geq -4x \sum_{x_n > x/2} \frac{|x_n|}{|z_n|^2} \geq 8A.$$

Combining the last two inequalities we obtain the estimate (7) and Theorem 1 is proved. ○

Proof of Theorem 2. In the case of the even entire functions we have the pairs of symmetric zeros $z_n = x_n + iy_n$ and $\tilde{z}_n = -x_n + iy_n$. Hence,

$$|E(x)|^2 = \prod_n \left(1 + \frac{x^4 + 2x^2(2x_n^2 - |z_n|^2)}{|z_n|^2}\right).$$

Now, one should split this product into two parts with factors greater than 1 and smaller than 1 respectively. The inequality (4) implies an estimate from below for the "small" product as in the proof of Theorem 1. We omit the details. ○

§3. Distribution independent sets

Proof of Theorem 3. Let $z_n \in \{y > qx^2\}$. Then

$$\sum_{|x_n|>R} \frac{|x_n|}{|z_n|^2} \le \sum_{|x_n|>R} \frac{|x_n|}{y_n^2} \le \sum_{|x_n|>R} \frac{1}{qx_n y_n} \le \frac{1}{qR} \sum_{|x_n|>R} \frac{1}{y_n}.$$

Since $\sum_n y_n^{-1} < \infty$ we get the estimate (3) (moreover, here $\sum_{|x_n|>R} |x_n|/|z_n|^2 = o(R)$, $R \to \infty$).

Now we show that for any $\alpha < 2$ there exists a function $E \in HB$ such that z_n are in the set $\{y \ge |x|^\alpha\}$ and $1 \notin \mathcal{H}(E)$. Without loss of generality let $\alpha > 1$. Put $z_n = x_n + ix_n^\alpha$ and

$$E(z) = \prod_n \left(1 - \frac{z}{\overline{z}_n}\right)^{k_n}.$$

We choose sequences x_n and k_n tending to infinity very fast. Put $x_1 = 3$, $k_1 = 1$. Assume that $x_1, k_1, \dots, x_{n-1}, k_{n-1}$ are already chosen. Then take any x_n such that $x_n > x_{n-1}^\alpha$ and

$$x_n^{1-\frac{\alpha}{2}} > 2^n \log x_n \sum_{j=1}^{n-1} k_j, \tag{8}$$

and put $k_n = [x_n^{\frac{3\alpha}{2}-1}]$ (by $[s]$ we denote the integer part of the number s). It is easy to see that the sequence x_n increases and tends to infinity.

First of all let us check the condition $\sum_n k_n|z_n|^{-1} < \infty$ ensuring that the product for E converges:

$$\sum_n k_n|z_n|^{-1} \le \sum_n x_n^{\frac{3\alpha}{2}-1} x_n^{-\alpha} = \sum_n x_n^{\frac{\alpha}{2}-1} \le \sum_n 2^{-n} < \infty.$$

Now we show that $\sup_{[x_n-1,\, x_n+1]} |E(x)| \to 0$, $n \to +\infty$. Thus, $1/E \notin L^2(\mathbb{R})$. Take $x \in [x_n - 1, x_n + 1]$ and split the product for E into three parts:

$$|E(x)|^2 = \left|\frac{x - z_n}{z_n}\right|^{2k_n} \prod_{l=1}^{n-1} \left|\frac{x - z_l}{z_l}\right|^{2k_l} \prod_{l=n+1}^{\infty} \left|\frac{x - z_l}{z_l}\right|^{2k_l}.$$

Clearly, the last product does not exceed 1. Since $|z_l| > 1$ and $y_l \le x_n$, $l \le n_1$, we have

$$\prod_{l=1}^{n-1} \left(\frac{(x - x_l)^2 + y_l^2}{|z_l|^2}\right)^{k_l} \le \prod_{l=1}^{n-1} (2x_n^2)^{k_l} \le \prod_{l=1}^{n-1} (x_n^3)^{k_l}.$$

Taking into account that $|x - x_n| \le 1$ we get

$$\log \frac{(x - x_n)^2 + y_n^2}{|z_n|^2} \le -\frac{x_n^2 - 1}{x_n^2 + x_n^{2\alpha}} \le -\frac{1}{4x_n^{2\alpha-2}}.$$

Hence,

$$\log |E(x)|^2 \le 3 \log x_n \sum_{j=1}^{n-1} k_j - \frac{k_n}{4x_n^{2\alpha-2}},$$

and, by (8), we have

$$\log |E(x)|^2 \le 3 \log x_n \sum_{j=1}^{n-1} k_j - \frac{[x_n^{\frac{3\alpha}{2}-1}]}{4x_n^{2\alpha-2}} \le -\left(\frac{1}{8} - \frac{3}{2^n}\right) x_n^{1-\frac{\alpha}{2}}.$$

The right-hand side of the last inequality tends to $-\infty$ when n tends to infinity. ◯

Remarks. 1. Though, as Theorem 3 shows, the exponential 2 is sharp, the condition $\Omega \subset \{y > qx^2\}$ for some $q > 0$ is not necessary: there exist d. i. sets Ω such that

$$\limsup_{z \in \Omega,\, z \to \infty} \frac{x^2}{y} = \infty.$$

For example, one can take $\Omega = \{n! + i(n-1)!n!,\ n \in \mathbb{N}\}$. At the same time, the arguments analogous to the proof of sharpness in Theorem 3 show that if Ω is a d. i. set then

$$\limsup_{z\in\Omega,\, z\to\infty} \frac{x^2}{y\log|x|} < \infty.$$

It is not clear whether the last condition is sufficient for Ω to be a d. i. set.

2. One can show that for even entire functions the angle $\{y \ge |x|\}$ is the widest distribution independent angle. Namely, for each $\gamma < 1$ there exists a function $E \in HB$ such that z_n are in $\{y \ge \gamma|x|\}$ and $1 \notin \mathcal{H}(E)$.

§4. Functions with zeros in an angle

In this section we find in a sense sharp asymptotic of the growth on the real axis for an entire function with zeros in a Stolz angle.

Proposition 9. *Let $z_n \in \Gamma_\gamma$. Then there exist functions $A_j(x) > 0$, $j = 1, \dots, 4$, such that*

$$0 < m_j \le A_j(x) \le M_j, \qquad x \in \mathbb{R},$$

and for $x \in \mathbb{R}$ we have

$$\log|E(3x)|^2 = A_1 \int\limits_0^{|x|} \frac{n(t)}{t}dt + A_2 x^2 \int\limits_{|x|}^{\infty} \frac{n(t)}{t^3}dt +$$

$$+A_3|x| \sum_{|z_n|\ge|x|,\, x_n x<0} \frac{|x_n|}{|z_n|^2} - A_4|x| \sum_{|z_n|\ge|x|,\, x_n x>0} \frac{|x_n|}{|z_n|^2}. \tag{9}$$

Here the constants m_j and M_j may depend on γ.

Proof. We use the following elementary estimates: $-t/\delta \le \log(1-t) \le -t$, $t \in [0, 1-\delta]$, and $t/K \le \log(1+t) \le t$, $t \in [0, K]$.

Without loss of generality $x > 0$. Then

$$|E(x)|^2 = \prod_{|z_n|<x/3} \frac{|x-z_n|^2}{|z_n|^2} \prod_{|z_n|\ge x/3,\, x_n<x/2} \frac{|x-z_n|^2}{|z_n|^2} \prod_{x_n\ge x/2} \frac{|x-z_n|^2}{|z_n|^2} =$$

$$= \Pi_1(x)\Pi_2(x)\Pi_3(x).$$

Note that since $z_n \in \Gamma_\gamma$ we have $\frac{2xx_n - x^2}{|z_n|^2} \le \frac{1}{1+\gamma^2}$ whenever $x_n \ge x/2$. Hence,

$$-\frac{\gamma^2+1}{\gamma^2} \sum_{x_n \ge x/2} \frac{2xx_n - x^2}{|z_n^2|} \le \log \Pi_3(x) \le -\sum_{x_n \ge x/2} \frac{2xx_n - x^2}{|z_n^2|}. \tag{10}$$

Analogously,

$$C_1 \sum_{|z_n| \ge x/3,\, x_n < x/2} \frac{x^2 - 2xx_n}{|z_n^2|} \le \log \Pi_2(x) \le$$

$$\le C_2 \sum_{|z_n| \ge x/3,\, x_n < x/2} \frac{x^2 - 2xx_n}{|z_n^2|} \tag{11}$$

for some absolute positive constants C_1 and C_2.

Next we find a rough asymptotic for the product Π_1. Since $|z_n| < x/3$ we get $2x/3 \le |x - z_n| \le 2x$ and

$$2^{2n(x/3)} \frac{(x/3)^{2n(x/3)}}{|z_1|^2 |z_2|^2 \dots |z_{n(x/3)}|^2} \le \Pi_1(x) \le 2^{2n(x/3)} \frac{(x/3)^{2n(x/3)}}{|z_1|^2 |z_2|^2 \dots |z_{n(x/3)}|^2}.$$

Hence,

$$\log \Pi_1(x) = 2 \log \frac{(x/3)^{n(x/3)}}{|z_1 z_2 \dots z_{n(x/3)}|} + A(x) n(x/3) =$$

$$= \int_0^{x/3} \frac{n(t)}{t} dt + A(x) n(x/3), \tag{12}$$

where $A(x) \asymp 1$. Combining the estimates (10)-(12) and replacing x by $3x$ we get the formula (9). We also used the fact that

$$n(R) + R^2 \sum_{|z_n| > R} \frac{1}{|z_n|^2} = R^2 \int_R^\infty \frac{n(t)}{t^3} dt. \qquad \bigcirc$$

Thus, only the last term in the formula (9) is responsible for the possible smallness of the function E on the real axis. If this term is asymptotically smaller than the positive summands in (9), as stated in Theorem 4, then E tends to infinity along $\mathbb{R}$ and Theorem 4 is proved.

Proof of Theorem 5. Now let $z_n \in \{\gamma_1 x \le y \le \gamma_2 x\}$ and $C_1 n^\alpha \le |z_n| \le C_2 n^\alpha$. In this case $n(t) \asymp t^{1/\alpha}$ and we have the following asymptotic for the summands in the formula (9) (all the constants involved depend on γ_i and C_i, but do not depend on R and α):

$$\int_0^R \frac{n(t)}{t} dt \asymp \int_0^R t^{-1+1/\alpha} dt \asymp \alpha R^{1/\alpha}, \qquad R \to \infty$$

$$R^2 \int_R^\infty \frac{n(t)}{t^3} dt \asymp R^2 \int_R^\infty t^{-3+1/\alpha} dt \asymp \frac{\alpha}{2\alpha - 1} R^{1/\alpha}$$

and, finally,

$$R \sum_{|z_n|>R} \frac{x_n}{|z_n|^2} \asymp R \sum_{n=[R^{1/\alpha}]}^{\infty} \frac{1}{n^{1/\alpha}} \asymp \frac{R^{1/\alpha}}{\alpha - 1}.$$

Note that $\alpha + \frac{\alpha}{2\alpha-1} = o\left(\frac{1}{\alpha-1}\right)$, $\alpha \to 1$, and $\frac{1}{\alpha-1} = o(\alpha)$, $\alpha \to \infty$. Hence, the coefficient at $R^{1/\alpha}$ in the formula for $\log|E(3R)|$ is positive when α is sufficiently large and negative when α is close to 1. ◯

In the case of regular distribution along a single ray we obtain an explicit formula for the limit exponential.

Proof of Theorem 6. Here we apply the Levin – Pfluger theory of entire functions of completely regular growth (see [13]). Recall that now $z_n = n^\alpha(1 + i\gamma)$, $\gamma > 0$. Then for any θ, $\psi \in (-\pi, \pi]$ there exists the limit

$$\Delta(\theta, \psi) = \lim_{R\to\infty} \frac{n(R, \theta, \psi)}{R^{1/\alpha}},$$

where $n(R, \theta, \psi)$ denotes the number of zeros z_n in the sector $\{|z| < R, \arg z \in [\theta, \psi)\}$. Note that $\Delta(\theta, \psi) = 0$, $\operatorname{arctg}\gamma \notin [\theta, \psi)$, and $\Delta(\theta, \psi) = C(\gamma)$, $\operatorname{arctg}\gamma \in [\theta, \psi)$, where $C(\gamma)$ is some positive constant. Hence, E is a function of completely regular growth and

$$\lim_{R\to\infty} \frac{\log|E(R)|}{R^{1/\alpha}} = H_\alpha = \frac{\pi C(\gamma)}{\sin\frac{\pi}{\alpha}} \cos\frac{\pi - \operatorname{arctg}\gamma}{\alpha}.$$

Thus, $H_\alpha > 0$ ($H_\alpha < 0$) if and only if $\alpha > 2 - \frac{2\operatorname{arctg}\gamma}{\pi}$ ($\alpha < 2 - \frac{2\operatorname{arctg}\gamma}{\pi}$). ◯

We conclude this section with an example illustrating the subtlety of the growth of a function with power growth of zeros.

Theorem 10. *For any $\alpha > 1$ there exist z_n such that* $\arg z_n = \pi/4$, $|z_n| \asymp n^\alpha$ *and* $1 \notin \mathcal{H}(E)$.

Proof. Take an integer $K > 2$ and let E be the entire function of the form (2) with zeros at the points $K^{\alpha l} e^{-i\pi/4}$ with multiplicities equal to $K^l - K^{l-1}$, $l \in \mathbb{N}$. Now, if $z_1, z_2, \dots, z_n \dots$ are zeros of E repeated according to the multiplicities, then $|z_n| \asymp n^\alpha$ (actually, we have replaced the group of zeros $n^\alpha e^{-i\pi/4}$, $K^{l-1} < n \le K^l$, by a single zero of the corresponding multiplicity).

Let $N \in \mathbb{N}$ and put $t_N = K^{\alpha N - \frac{1}{2}}$. We show that one can choose a sufficiently large K such that

$$\lim_{N\to\infty} |E(3t_N)| = 0. \tag{13}$$

Then $1/E \notin L^2(\mathbb{R})$ since $E'/E \in L^\infty(\mathbb{R})$ and (13) implies that $\log|E|$ is negative on the intervals $(3t_N - \delta, 3t_N + \delta)$ for some $\delta > 0$ and for sufficiently large N.

By Proposition 9

$$\log|E(3t_N)|^2 = B_1 \int_0^{|t_N|} \frac{n(t)}{t} dt + B_2 n(t_N) +$$

$$+ B_3 t_N^2 \sum_{|z_n| > t_N} \frac{1}{|z_n|^2} - B_4 |t_N| \sum_{|z_n| \ge t_N} \frac{|x_n|}{|z_n|^2}, \tag{14}$$

where $B_1(t_N) + B_2(t_N) + B_3(t_N) \le C$ and $B_4(t_N) \ge c$ for some absolute positive constants C and c. Now we estimate the summands in the formula (14). Note that $K^{\alpha(N-1)} < t_N < K^{\alpha N}$ and

$$n(t_N) = \sum_{l=1}^{N-1} K^l (K-1) = K^N - K.$$

Hence,

$$\int_0^{|t_N|} \frac{n(t)}{t} dt = \log \frac{t_N^{n(t_N)}}{|z_1 z_2 \dots z_{n(t_N)}|} =$$

$$= \left[(K^N - K)(\alpha N - \frac{1}{2}) - \alpha \sum_{l=1}^{N-1} l K^l (K-1) \right] \log K < 2\alpha K^N \log K.$$

Also, we have

$$t_N^2 \sum_{|z_n|>t_N} \frac{1}{|z_n|^2} = K^{2\alpha N-1} \sum_{l=N}^{\infty} \frac{K^l(K-1)}{K^{2\alpha l}} < 2K^N$$

Now we estimate the negative part of $\log |E(t_N)|^2$:

$$t_N \sum_{|z_n|\ge t_N} \frac{|x_n|}{|z_n|^2} = K^{\alpha N-\frac{1}{2}} \sum_{l=N}^{\infty} \frac{K^l(K-1)}{\sqrt{2}K^{\alpha l}} \ge \frac{K^{n+\frac{1}{2}}}{\sqrt{2}}.$$

Thus,

$$\log |E(3t_N)|^2 \le C(3K^N + 2\alpha K^N \log K) - \frac{c}{\sqrt{2}} K^{N+\frac{1}{2}},$$

and taking a sufficiently large K we can make $\log |E(3t_N)| \to -\infty$, $N \to \infty$. ◯

§5. Hypercyclic operators in the de Branges spaces

To prove Theorem 7 we need an auxiliary class of spaces introduced in [17]. Let $\gamma(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ be an entire function such that $\gamma_n > 0$ for each $n \ge 0$ and the sequence $n\gamma_n/\gamma_{n-1}$ decreases when n tends to infinity (in this case we say that γ is an admissible comparison function). Consider the space $E^2(\gamma)$ of all entire functions f such that

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

with

$$\|f\|_{2,\gamma} = \sum_{n=0}^{\infty} \gamma_n^{-2} |f_n|^2 < \infty.$$

Then $E^2(\gamma)$ is a Hilbert space. Moreover, it is easy to see that the condition $\sup_n n\gamma_n/\gamma_{n-1} < \infty$ implies that the differentiation operator $\mathcal{D}$ is bounded on $E^2(\gamma)$. K.C. Chan and J.H. Shapiro have showed that translations are hypercyclic in $E^2(\gamma)$; they also have obtained a much more general result.

Theorem (K.C. Chan, J.H. Shapiro [17])**.** *Suppose that X is a Fréchet space of entire functions with the following properties:*

1. $\mathcal{P} \subset X$ and $\mathrm{Clos}_X \mathcal{P} = X$;

2. the topology of X is stronger than the topology of uniform convergence on compact subsets of the plane;

3. T_w is bounded on X;

4. $E^2(\gamma) \subset X$ for some admissible comparison function γ.

Then T_w, $w \neq 0$, is hypercyclic on X.

To apply the Chan – Shapiro theorem to the de Branges spaces satisfying conditions of Theorem 7 (that is, $\mathrm{Clos}_{\mathcal{H}(E)}\mathcal{P} = \mathcal{H}(E)$ and $\mathcal{D}$ is bounded in $\mathcal{H}(E)$) we need to verify only the last condition. Let $r_n = \|z^n\|_{\mathcal{H}(E)}$ and take a sequence $\gamma_n > 0$ such that the sequence $n\gamma_n/\gamma_{n-1}$ decreases and $\{r_n\gamma_n\} \in \ell^2(\mathbb{Z}_+)$. We show that $E^2(\gamma) \subset \mathcal{H}(E)$. Let $f \in E^2(\gamma)$, $f(z) = \sum_n f_n z^n$. Then, for any $y > 0$, we have

$$\left\| \frac{f}{E}(t+iy) \right\|_2 \le \sum_n |f_n| \left\| \frac{(t+iy)^n}{E(t+iy)} \right\|_2 \le \sum_n |f_n| r_n$$

since the functions z^n/E are in the Hardy class $H^2(\mathbb{C}^+)$. Hence,

$$\sup_{y>0} \left\| \frac{f}{E}(\cdot + iy) \right\|_2 \le \|f\|_{2,\gamma} \|r_n\gamma_n\|_{\ell^2}$$

and $f/E \in H^2(\mathbb{C}^+)$. Analogously, $f^*/E \in H^2(\mathbb{C}^+)$. Thus, we get the inclusion $E^2(\gamma) \subset \mathcal{H}(E)$, and Theorem 7 is proved.

Remark. As shown in [17], an interesting feature of the spaces $E^2(\gamma)$ is that they provide examples of hypercyclic operators which are compact or even Schatten class perturbations of the identity operator I. Indeed, choosing the sequence γ_n tending to zero rapidly one can ensure that the differentiation operator in $E^2(\gamma)$ belongs to all Schatten – von Neumann classes and so does the operator $T_w - I$.

The same is true also for "larger" de Branges spaces. In [19] a number of examples are constructed showing that the operator $\mathcal{D}$ in $\mathcal{H}(E)$ may be compact or belong to all Schatten – von Neumann classes. In particular, $\mathcal{D}$ is always compact if the zeros of E lie in a Stolz angle.

References

[1] L. De Branges, Hilbert spaces of entire functions. Prentice Hall, Englewood Cliffs (NJ), 1968.

[2] P. Koosis, Introduction to H^p spaces. Cambridge Univ. Press, Cambridge, 1980.

[3] N.K. Nikol'skii, Treatise on the shift operator. Springer-Verlag, Berlin, 1986.

[4] V. P. Havin, J. Mashreghi, Admissible majorants for model subspaces of H^2. Part I: slow winding of the generating inner function. – Can. J. Math., **55**, 6 (2003), 1231-1263.

[5] V. P. Havin, J. Mashreghi, Admissible majorants for model subspaces of H^2. Part II: fast winding of the generating inner function. – Can. J. Math., **55**, 6 (2003), 1264-1301.

[6] H. Woracek, De Branges spaces of entire functions closed under forming difference quotients. – Integr. Equ. Oper. Theory, **37**, 2 (2000), 238-249.

[7] A. Borichev, M. Sodin, Weighted polynomial approximation and the Hamburger moment problem. – Complex analysis and differential equations, Proceedings of the Marcus Wallenberg Symposium in Honor of Matts Essén, Uppsala University, 1998.

[8] A. Borichev, M. Sodin, The Hamburger moment problem and weighted polynomial approximation on descrete subsets of the real line. – J. Anal. Math., **76** (1998), 219-264.

[9] N.I. Akhiezer, On a generalization of the Fourier transformation and the Paley – Wiener theorem. – Dokl. Akad. Nauk SSSR, **96** (1954), 889-892 (Russian).

[10] V.P. Gurarii, Fourier transform in $L^2(-\infty, \infty)$ with a weight. – Matem. sb., **58**, 4 (1962), 437-452 (Russian).

[11] A. D. Baranov, Isometric embeddings of the spaces K_Θ in the upper half-plane. – Probl. Math. Anal., **21** (2000), 30-44 (Russian); English translation: J. Math. Sci. **105** (2001), 2319-2329.

[12] P. Koosis, Measures orthogonales extrémales pour l'approximation ponderée par des polynomes. – C.R. Acad. Sci. Paris, 311 (1990), 503-506.

[13] B. Ya. Levin, Distribution of zeros of entire functions. GITTL, Moscow, 1956; English translation: Amer. Math. Soc., Providence, 1964; revised edition: Amer. Math. Soc., 1980.

[14] G. D. Birkhoff, Démonstration d'un théoreme elementaire sur les fonctions entiéres. – C.R. Acad. Sci. Paris, 189 (1929), 473-475.

[15] G. R. MacLane, Sequences of derivatives and normal families. – J. Anal. Math., **2** (1952), 72-87.

[16] G. Godefroy, J. H. Shapiro, Operators with dense, invariant cyclic vector manifolds. – J. Funct. Anal., **98** (1991), 229-269.

[17] K. C. Chan, J. H. Shapiro, The cyclic behavior of translation operators on Hilbert spaces of entire functions. – Indiana Math. J., **40**, 4 (1991), 1421-1449.

[18] A. D. Baranov, Bernstein's inequality in the de Branges spaces and embedding theorems. – Proc. St. Petersburg Math. Soc., **9** (2001), 23-53.

[19] A. D. Baranov, On estimates of the L^p-norms of the derivatives in the spaces of entire functions. – Zap. Nauchn. Semin. Pomi, **307** (to appear).

Address: Saint Petersburg State University,
Department of Mathematics and Mechanics,
28, Universitetski pr., St. Petersburg,
198504, Russia

E-mail: antonbaranov@netscape.net