THE JACOBI ALGEBRAS CLASSIFICATION ON HOMOGENEOUS MANIFOLDS

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1 Introduction

The problem of classification of the geodesic flows symmetries polynomial on velocity (the Jacobi fields) on the tungent bundle of a homogeneous manifold has the qualitative solution. The question is reduced to the straightforward calculation, with the construction of the manifold and explicit description of the algebra being possible. The idea of such procedure itself is not new, it is equations described earlier by the author which are. The key notion is the tensor equation monodromy group. The list of possible Jacobi algebras coincides with a number of conjugate classes of the monodromy representations in the group of affine symmetries of the universal cover space. Remarkably that for the flat connections this conclusion follows right from the algebraic structure of the Jacobi algebra. Here the calculation is fulfilled for some important cases.

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2 Classification theorem

Let M^n be connected homogeneous C^∞-manifold provided by the symmetrical affine connection ∇, Γ — a transitive pseudogroup of local affine transformations. All objects we will face in this work belong C^∞-category. The elements of the group connecting the points p, q will be denoted φ_{pq}. The theorem cited below belongs to K. Nomizu ([1]). We will apply it for the case of the infinite set of equivariant tensor operators introduced by the author and use the additional algebraic structure (graded Lie algebra) which is invariant as well.

Theorem ([1]). *If a tensor equation is given on a simply connected manifold, such that the dimension of the germs space is constant over it, every germ of local solution is expandable over whole manifold.*

The author has proved in ([2]) that every generalized Jacobi equation

$$\delta^2 A + R \circ A = 0$$

has finite dimensional germ space in each point. Here δ and $\circ$ are the symmetrised covariant derivation and contraction, R is curvature tensor of the connection. In view of homogeneity and the Jacobi equation invariance the dimension function is constant. It is enough to consider the universal cover space $(\widetilde{M}, \pi, M)$. The symmetrical affine connection with the same transitive pseudogroup Γ is induced on the cover space. Each local solution is expandable over the cover. Hence, the Jacobi algebra $\mathfrak{Y}(\widetilde{M}, \widetilde{\nabla})$ consists of globally defined tensor fields. But this algebra is wider than that on base. Not every field on the cover space is the lift of one from the base. In order to describe such lifts we will introduce the notion of the *shift* operator. Let us consider the germ space in arbitrary point $p \in \widetilde{M}$. Every germ is expandable, hence each germ defines some germ over another point q. Such correspondence defines the germ spaces isomorphism. More over this operation saves the Lie bracket. Hence this operation gives the isomorphism T_{pq} of the graded Lie algebras $\mathfrak{Y}(U_p)$ and $\mathfrak{Y}(U_q)$ over some neighborhoods of points p, q, which will be called the shift operator. From another side, any element $\varphi_{qp} \in \Gamma$ defines an isomorphism φ^*_{qp} of these algebras and we can consider an automorphism of the space $\mathfrak{Y}(U_p)$

$$T^{\varphi_{qp}}_{pq} = \varphi^*_{qp} \circ T_{pq} : \mathfrak{Y}(U_p) \to \mathfrak{Y}(U_p).$$

For any pair of points projecting in one on the base correspond to some loop on base with this origin. If π is the cover map and $p, q \in \pi^{-1}(x)$, than the local diffeomorphism $\pi_q^{-1} \circ \pi_p$ of the neighborhoods is the connection automorphism. Hence, the operator $T^{\pi_p^{-1} \circ \pi_q}_{pq}$ is well defined. It gives the monomorphism of the fundamental group to the affine group, with the image acting free and totally disconnected on the cover space

$$R : \pi_1(M) \to \mathfrak{A}(\widetilde{M}).$$

Using this monomorphism one can construct the monodromy representation

$$\left\{ \rho([\gamma]) = T^{R^{-1}([\gamma])|_p}_{pR([\gamma])(p)} \in \mathrm{Aut}\mathfrak{Y}(\widetilde{M}) \middle| \, [\gamma] \in \pi_1(M) \right\}$$

Indeed, it is a homomorphism

$$\rho([\beta] \circ [\alpha]) = \rho([\beta]) \circ \rho([\alpha]).$$

In fact, if the loop is covered by different paths $\widehat{pr}$ and $\widehat{qm}$ on $\widetilde{M}$, than each germ extends along them uniformly. It means that the shift operators are conjugate with respect to the change of coordinates determined by the cover

$$T_{pr} = (\pi_r^{-1} \circ \pi_m) \circ T_{qm} \circ (\pi_q^{-1} \circ \pi_p).$$

By the definition we can write

$$\rho([\alpha]) = \left(\pi_p^{-1} \circ \pi_q\right)^* \circ T_{pq}, \quad \rho([\beta]) = \left(\pi_p^{-1} \circ \pi_r\right)^* \circ T_{pr},$$

$$\rho([\beta]) \circ \rho([\alpha]) = (\pi_p^{-1} \circ \pi_r) \circ T_{pr} \circ (\pi_p^{-1} \circ \pi_q) \circ T_{pq} =$$

$$(\pi_p^{-1} \circ \pi_r) \circ \left((\pi_r^{-1} \circ \pi_m) \circ T_{qm} \circ (\pi_q^{-1} \circ \pi_p)\right) \circ (\pi_p^{-1} \circ \pi_q) \circ T_{pq} =$$

$$(\pi_p^{-1} \circ \pi_m) \circ T_{qm} \circ T_{pq} = (\pi_p^{-1} \circ \pi_m) \circ T_{pm} = \rho([\beta] \circ [\alpha]).$$

Finally, we can formulate the *classification theorem.*

Theorem. *Any subgroup π of $\mathfrak{A}(\widetilde{M})$ acting free and totally disconnected on homogeneous simply connected manifold $\widetilde{M}$ determines the set of automorphisms $\rho(a)$, $a \in \pi$,*

$$\rho(a) = T_{pa(p)}^{a^{-1}}$$

of the Jacobi algebra $\mathfrak{Y}(\widetilde{M})$. The factor space $\widetilde{M}/\pi$ is homogeneous manifold, has the fundamental group π and the Jacobi algebra

$$\mathfrak{Y}(M) = \bigcap_{a \in \pi} \mathrm{Inv}\rho(a).$$

The explicit description is now based on extension of the PDE equations system solution. Any simplification leads to the excessiveness of the list, i.e. the question on realizability must be investigated in addition. An arbitrariness of the map construction in the theorem is double— the choice of the universal cover π and that of point $p \in \tilde{M}$. All cover spaces are diffeomorphic. The diffeomorphism itself is affine map. Hence, both maps R, R' corresponding to the different covers are conjugate by this affine map. As to the choice of the point p' differ from the point p, in this case the operators ρ and ρ' are conjugate with respect to the shift operator $T_{pp'}$. In fact, for the operator R is affine the following diagram is commutative for any $a \in \pi_1$

$$\begin{array}{ccc}
U_p & \xrightarrow{T_{pp'}} & U_{p'} \\
\Big\downarrow{\scriptstyle R(a)} & & \Big\downarrow{\scriptstyle R(a)} \\
U_{R(a)(p)} & \xrightarrow{T_{R(a)(p)R(a)(p')}} & U_{R(a)(p')}
\end{array}$$

By the definition

$$\rho'(a) = T^{R^{-1}(a)}_{p'R(a)p'} = (T_{pp'} \circ T_{p'p}) \circ T^{R^{-1}(a)}_{p'R(a)p'} \circ (T_{pp'} \circ T_{p'p}) =$$

$$T_{pp'} \circ (T_{p'p} \circ (R^{-1}(a))^*) \circ (T_{p'R(a)p'} \circ T_{pp'}) \circ T_{p'p} =$$
$$T_{pp'} \circ ((R^{-1}(a))^* \circ T_{R(a)p'R(a)p}) \circ T_{pR(a)p'} \circ T_{pp'} =$$
$$T_{pp'} \circ ((R^{-1}(a))^* \circ T_{pR(a)p}) \circ T_{pp'} = T_{pp'}\rho(a)T_{p'p}.$$

So, for any pair $a \in \mathfrak{A}(\widetilde{M})$ and $q \in \widetilde{M}$ the conjugation of the representation ρ by the operator $T^{(a^{-1}Ra)^{-1}}_{pq}$ corresponds to some change of coordinates on the manifold. Hence, the algebra is still the same. It means that the set $\mathfrak{A} \times \widetilde{M}$ acts on the set of monodromy representations $\mathrm{Rep}[\pi \to \mathrm{Aut}\mathfrak{Y}]$. To each realizable algebra corresponds some conjugacy class with respect to this action. Finally, *the Jacobi algebras classification is reduced to the description of the realizable classes*

$$\mathrm{Rep}[\pi \to \mathrm{Aut}\mathfrak{Y}]/\mathfrak{A} \times \widetilde{M}$$

Thus for the explicit classification we need a) to know the group $\mathfrak{A}(\widetilde{M})$; b) to list all subgroup acting free and totally discontinuous; c) to know the algebra $\mathfrak{Y}(\widetilde{M})$; d) to describe all shift operators; e) to find all realizable classes. For the homogeneous spaces the items a)–b) are studied rather well (see, for inst. [6]).

Remark. Let us consider now any loop γ with the origin $x \in M$. Any germ is extendable along the cover path with endpoints p and q, $p, q \in \pi^{-1}(x)$. The projection of T_{pq} by π on a neighborhood of point x determines an automorphism $\varphi_{[\gamma]} : \mathfrak{Y}(U_x) \to \mathfrak{Y}(U_x)$ depending on the homotopy class of the loop only. This operator is the automorphism of the graded Lie algebra. In the case of the algebra $\mathfrak{Y}(U_x) \cong \mathfrak{Y}(\mathbb{R}^1)$ all such automorphisms have the form φ^*, where φ is some affine transformation of $\mathbb{R}^1$ (([2], 1995)). The hypothesis it is true for $\mathbb{R}^n$ was proved by the author in [3] (2002). Namely, any automorphism of the graded Lie algebra $\mathfrak{Y}(\mathbb{R}^n)$, commuting with the symmetrized covariant derivative on its domain of definition, is generated by an affine transformation of $\mathbb{R}^n$. So a field X is determined on M along the loop γ if its cover field $\tilde{X} \in \mathbf{Inv}\varphi_{[\gamma]}$. The strategy of the *flat* Jacobi algebras classification arises from the above conclusions. One needs to describe the maximal trivial subrepresentation of the fundamental group representation in the affine transformation group. The simplest examples show that not all algebras from the list appeared are realizable. More over, implicit description of $\mathfrak{Y}(\mathbb{R}^n)$ is hard combinatorial task. The key fact in the proof of the last theorem is the perfectness of the affine fields Lie algebra. For the non-flat connection one may hardly expect similar result. In searching of generalizations the new concept of spray algebra was introduced. New interesting interconnections and tensor structures were detected. The last appeared in [4].

3 Explicit descriptions and realizability

As it follows from the classification theorem the explicit description is based on the explicit expression of the operator $T_{pq}^{\varphi_{qp}}$. For the *flat* case such description is possible completely.

Complete connection. If the manifold $\widetilde{M}$ is the complete, flat, simply connected one, than it is diffeomorphic to $\mathbb{R}^n$ with canonical connection. In this case $\mathfrak{A}(\widetilde{M}) \cong \mathrm{Aff}(n)$ and the shift operators $T_{pq}^{\varphi_{qp}}$ are the tensor field shifts with respect to the affine transformation $\varphi_{qp}(\bar{x}) + \vec{pq}$. So, action $\rho(a)$, $a \in \mathrm{Aff}(n)$, coincides with the map a. For map a acts free such transformation has not fix points. The description of the Jacobi algebras is reduced to the enumeration of all $\pi < \mathrm{Aff}(n)$ acting free and totally discontinuous on $\mathbb{R}^n$, up to the group $\mathrm{Aff}(n)$ conjugation.

Theorem. *The classification of the Jacobi algebras is reduced to the enumeration of complete flat not simply connected manifolds generated by the actions of some subgroup of the group* $\mathrm{Aff}(n)$ *on* $\mathbb{R}^n$ *and the description of the maximal invariant subalgebra with respect to this action on the algebra of polynomial symmetries.*

Case $\mathbb{R}^2$ with $\boldsymbol{\pi_1} = \mathbb{Z}$. In the case of $\mathbb{Z}$ action it is enough to define an automorphism $\rho(1) \in \mathbf{Aff}(2)$ acting free on the plane. Any such automorphism can be reduced by some affine transformation of coordinates to one of the map listed below

$$1)\quad \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad 2)\quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Case 1). Let $\lambda = 1$. Polynomial fields are invariant with respect to the parallel shift iff their coefficients are independent of one variable x. Thus, the Jacobi algebra is a subalgebra of $\mathfrak{A}(\mathbb{R}^2)$ consisting of fields depending on one variable. In this case the Jacobi equation is simply second partial derivation with respect to the rest variable. It means that the algebra consists of the tensor fields with linear one variable components

$$\mathfrak{Y} \cong \mathbb{R}^4 \oplus \mathbb{R}^8 \oplus \mathbb{R}^{12} \oplus \dots .$$

Let $\lambda = -1$. Among the polynomial fields $C^i_{j_1 \dots j_s}(y)$ the only fields depending on y in degree $p = 0, 1$, such that $(-1)^{i+j_1+\dots+j_s+p} = 1$, will survive after Ox-symmetry. It is the complete algebra description.

$\lambda > 0$. It is enough to note that this map is conjugate to an affine shift by some diffeomorphism of the plane. This conjugating diffeomorphism has the form

$$\varphi(x, y) = (x, ye^{-\mu x}), \mu = \ln(\lambda).$$

A polynomial field shifted by this diffeomorphism must be independent of x in new coordinates. New components of the field of type $(1, k)$ have the form

$$x^r y^s e^{t\mu x} L(\mu), \quad r, s \in \mathbb{N}, t \in \mathbb{Z},$$

where $L(\mu)$ is some linear combination of the initial polynomial field coefficients. All linear expressions with $r, t \neq 0$, which have to be zero, form a homogeneous linear system. The k-th components of the Jacobi algebra is the solution space of this system. Straightforward calculation shows that the space is independent of spectral parameter and each component is three-dimensional. The explicit form of the invariant polynomial fields is

$$A^1_{1\dots 1} = a, \quad A^1_{\bar{i}} = 0;$$

$$A^2_{1\dots 1} = by, \quad A^2_{21\dots 1} = c, \quad A^2_{\bar{i}} = 0,$$

with arbitrary constants $a, b, c \in \mathbb{R}$. The Lie bracket of these fields is trivial. Thus, the Jacobi algebra is the commutative infinite dimensional Lie algebra

$$\mathfrak{Y} \cong \mathbb{R}^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \dots$$

$\lambda < 0$. The algebra corresponding this spectrum is the intersection of the algebras $\lambda = -1$ and $\lambda > 0$.

Case 2). Polynomial field transferred with such map has to coincide with initial one in new coordinates (x, y). The appeared conditions on tensor components are some functional equations system. Straightforward calculations show that the component $A^2_{1\ldots1}$ has the maximal degree. But it is this component which has degree not exceeded 1. The last follows directly from the generalized Jacobi equation [2]. More over this component does not depend on variable y as a sequence of properties of the appeared tensor equations. The general form of the tensor field is

$$A^2_{1\ldots1} = ax + b, \quad A^1_{1\ldots1} = a, \quad A^2_{21\ldots1} = c.$$

Particularly, the affine algebra of such connection is two-dimensional, commutative and consists of the fields $(0,1)$ and $(1,x)$. The structure of the Jacobi algebra

$$\mathfrak{Y} \cong \mathbb{R}^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \ldots$$

is not trivial. The fields corresponding to the coefficients a and c do not commute. The last remark is that matrix with negative determinant generates the Möbius strip.

Theorem. *There exist exactly three type of the Jacobi algebras of the complete flat affine connection on cylinder.*

1) The typical algebra is infinite dimensional commutative Lie algebra

$$\mathfrak{Y}_1 \cong \mathbb{R}^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \ldots.$$

2) The set of connections of co-dimension 1 in the space of complete connections on cylinder has non-commutative infinite dimensional Jacobi algebra

$$\mathfrak{Y}_2 \cong \mathbb{R}^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \ldots.$$

3) The Jacobi algebra on the Euclidean cylinder consists of tensor fields depending on one variable and linear with respect to it

$$\mathfrak{Y}_3 \cong \mathbb{R}^4 \oplus \mathbb{R}^8 \oplus \mathbb{R}^{12} \oplus \ldots.$$

There exist exactly two type of algebras on the Möbius strip. Both of them are infinite dimensional and commutative. One of them are typical.

In this investigation we do not touch the question of the symmetries completeness. For the obtained explicit classification however the answer is clear. *For any flat complete connection there exists a group of the geodesic flow automorphisms of arbitrary much dimension!* This follows directly from the commutativity and the Palais theorem [5].

From the above description it is easy to obtain the classification for the torus. Indeed, the torus can be generate as factor space only by pair of the commutative independent shifts of cover space. Easy to show that such pair are (up to the conjugation) a) the multiple spectrum map and shift along Oy axis; b) the pair of shifts.

Theorem. *There exist exactly two types of the complete plane affine connections on torus*

1) The typical connection has the algebra

$$\mathfrak{Y}_2 \cong \mathbb{R}^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \dots .$$

2) To the Euclidean tori corresponds the algebra of fields with constant components

$$\mathfrak{Y} \cong \mathbb{R}^2 \oplus \mathbb{R}^4 \oplus \mathbb{R}^6 \oplus \dots .$$

In higher dimensions there are more types of the Jordan forms and the spectral parameters domain is multi-dimensional. The dependance of the solution space on spectrum is not known. In the next subsection an instance of such dependance is described for an uncomplete connection. One can assume that this picture is not trivial for complete ones too.

Non-complete connection. In this case $\mathfrak{Y}(\widetilde{M}) \cong \mathfrak{Y}(\mathbb{R}^n)$ and $\mathfrak{A} < \mathrm{Aff}(n)$. The shift operator T_{pq} coincide with that of Euclidean plane if the points p, q belongs to one normal map (in which the Christoffel symbols are zero identically). If the points are far from each other, any "good enough" path can be split by the set of points, each pair of which belongs to one map. The shift operator in this case is the composition of Euclidean shifts and operators generated by the gluing cocycles of normal maps. Hence, the shift operator belongs to the operator group generated by affine maps of Euclidean plane and the gluing cocycles. In the flat case the latter are affine maps (for hyperbolic spaces,— hyperbolic plane isometries and so on). Henceforth, one should describe all homomorphisms from π to $\mathrm{Aff}(n)$ in order to enumerate all possible Jacobi algebras. The question of realizability will be still open. This is exactly the conclusion of the classification theorem of the author [3].

Case $\boldsymbol{\pi_1} = \mathbb{Z}$ in $\mathbb{R}^n$. The case of $\rho(1) \in \mathbf{Aff}(n)$ without fix points can be regarded as previous one, as we can just remove invariant set from the cover space with saving the fundamental group. The last gives an uncomplete space with the same Jacobi algebra. Let us consider now the elements of $\mathbf{Aff}(n)$ having fix point. The action of an automorphism A on the space of the type $(1, k)$ fields can be regarded as coordinates changing. In such form the description of the algebra coincides with that of the normal form of the field in sense of Birkhof ([7]) with the linear part

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

The classification of the last is completely determined by the operator A spectrum and existence of "resonance" arrays. The explicit description for the prime real spectrum can be found in ([8]). The perfect account of the temporary state of the theory one can read in [9]. As an example let us describe the classification on two dimensional cylinder for the prime positive spectrum.

Let coordinates of a point on the plane correspond to the spectrum of A. All points lying on the curve $y = x^{p/q}, x \neq 1$, where p, q are coprime natural numbers, correspond to isomorphic non-commutative Jacobi algebras. The lines $x = 1$ and $y = 1$ do as well. The others points correspond to one-dimensional algebra consisting of the geodesic flow.

In conclusion, let us note that all algebras of this two dimensional classification are realizable.

References

[1] Nomizu K. *On local and global existence of Killing vector fields* // Ann. math. 1960, **72**.

[2] Kalnitsky V.S., *Algebra of generalized Jacobi fields* // J. Math. Sci. (New-York) 91 (1998), no. 6, pp. 3476–3491.

[3] Kalnitsky V.S., The Jacobi algebra of flat manifold //*J. Math. Sci.*, 2003 (in transl.)

[4] Kalnitsky V.S., Spray Algebra, *Proc. Math. Inst. Nat. Acad. Sci. Ukr.,* v. 50, Part III, pp. 1356–1360, 2004.

[5] Palais, R. *A global formulation of the Lie theory of transformation groups.* // Mem. Amer. Math. Soc. #22, 1957

[6] Loos O. Symmetric spaces. Benjamin, NY-Amsterdam, 1969.

[7] Bibikov Yu.N. Local theory of nonlinear analytic ordinary differential equations //Lecture Notes in Maths. Berlin e.a., 1979. Vol. 702.

[8] Bruno A.D. *Analitic form of the differential equation* // Proc. of Moscow Math. Soc. (in Russian) 1971. v. 25.

[9] Basov V.V., *The normal form method in the local qualitative theory of differential equation* (in Russian) St.P. State Univ. Press, 2002.

[10] Kalnitsky V.S., *Automorphisms of a geodesic vector field* // Vestn. St. Petersb. Univ. Math., Alerton Press, N.-Y., v. 28 (1995) (**2**), 19–20.