MEANS GENERATING THE CONE SECTIONS AND THE THIRD DEGREE POLYNOMIALS

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Introduction

Historically the first "rich" class of means appeared from the Lagrange mean value theorem. This class consists of so called f-differential means

$$D_f(u,v) = \left(f'^{-1}\right)\left(\frac{f(u)-f(v)}{u-v}\right)$$

and after Stolarsky's generalization of logarithmic mean D_{t^α} [1], named after his name, was intensive investigated.

After that new class of quasi-arithmetic means was introduced

$$M_f(u,v) = f^{-1}\left(\frac{f(u)+f(v)}{2}\right).$$

And as the first comparison theorem of classes we can count G. Aumann's theorem which states that all solutions of the differential-functional equation

$$D_f = M_F$$

are arcs of cone sections. This result was generalized by Sh. Haruki (see, for inst., [2]). It gave the impact to the investigation of functional equations of type

$$\frac{f(x)-f(y)}{x-y} = \varphi(\xi(x,y)),$$

so called Stemet type equations [3].

Recently many rich classes of means are described: Tobey's homoigeneous means [4] and their generalizations [5], Toader's means [6], α,β-means [7], functional Stolarsky means [8].

In the works of J. Sándor [9] and M. Kuczma [10] was noted that all monotone convex (vertex) solutions of equation

$$\frac{f(x)-f(y)}{x-y} = \varphi(h(x),h(y)),$$

where $\varphi(u,v) = u+v; uv; 1/(u+v)$, are arcs of cone sections. Assuming $h, \varphi \in C$, $\varphi(u,u)$ being inversable, the above equation is reducable to the form

$$\frac{f(x)-f(y)}{x-y} = K(f'(x), f'(y)), \tag{1}$$

where K is necessarily generalized mean. For $K = \frac{u+v}{2}$ it is the Lagrange theorem of 2^{nd} degree polinomials characterisation.

In this form Kuczma's result can be formulated in the following way: all solutions of equation

$$\frac{f(x)-f(y)}{x-y} = \frac{f'(x)+f'(y)}{2}; \quad \sqrt{f'(x)f'(y)}; \quad \frac{2f'(x)f'(y)}{f'(x)+f'(y)}$$

are vertical paraboles, arcs of increasing vertical hyperboles and horizontal paraboles correspondently.

The aim of present article is to describe all means $K(u,v)$ generating as solutions arcs of the cone sections.

The equation (1) itself is found out have relations with different disciplines such as economics [11], operational research [12,13] and analysis [14], low-dimesional geometry [15].

§1 Classification theorem

We consider the class of functions $f(t) \in C^1$, convex (vertex) on its domain of definition, $K(u,v)$— smooth symmetrical generalized mean. Let $f(t)$ be a solution of equation (1), the transformations of function

$$f(t+\alpha)+\beta; \quad \frac{1}{k}f(kt),$$

where $\alpha, \beta, k \in I\!R$, are solutions as well. So we can assume without lose of generality that

$$f(t) : [0,1] \to I\!R, \quad f(0) = 0,$$

Lemma. *If for given $K(u,v)$ and C the function $\theta(u) = K(u,C) \in C^1$, then the monotone convex (vertex) solution $f(t)$ of the equation (1) if exists is unique up to the homotety.*

P r o o f. As the function $K(u,v)$ is symmetrical mean $K(C,C) = C$ and $K_u(C,C) = 1/2$, i.e. for some interval the function $\theta(u)$ is monotone (therefore inversable) and the only solution of equation $\theta(u) = u$ is C. For given initial data we can write $f'(x) = s(f(x)/x)$, where $s(u) = \theta^{-1}(u) \in$

C^1. The statement of the lemma follows from the fact: for any $s(t)$ such that $s(c) = c$ is the only solution and $s'(C) = 2$ the solution of the above homogeneous ODE exists, belongs to C^2, has the form

$$f(t) = \frac{(\alpha t) F^{-1}(\ln(\alpha t))}{\alpha},$$

where $\alpha \in I\!R$, $F(z) = \int_z^a \frac{d\mu}{s(\mu)-\mu}$, and is determined up to the $f''(0)$. Geometrically it means the uniqueness up to the homotety. Note, $f'(0) = C$. Let's now consider the found solution $f(t)$ and construct the mean K_f by the formula

$$K_f(u, v) = \frac{f(f'^{-1}(u)) - f(f'^{-1}(v))}{f'^{-1}(u) - f'^{-1}(u)}$$

If $K_f = K$ the function $f(t)$ is the solution of the equation (1). $\square$

Let's now derive some relations between K_f and K_g for f and g related somehow.

Let $f(t)$ be a solution of (1) and $g(t) = f(t) + kt$. We can write

$$K_g(g'(t), g'(s)) = \frac{g(t) - g(s)}{t - s} = \frac{f(t) - f(s)}{t - s} + k =$$
$$= K_f(f'(t), f'(s)) + k = K_f(g'(t) - k, g'(s) - k) + k.$$

Therefore

$$K_g(u, v) = K_f(u - k, v - k) + k.$$

Geometrically, described transformation is the affine map of plane $(x, y) \mapsto (x, y + kx)$.

Let's consider now $(x, y) \mapsto (x, ky)$, i.e. $g(t) = kf(t)$.

$$K_g(g'(t), g'(s)) = kK_f(f'(t), f'(s)) = k_f(g'(t)/k, g'(s)/k),$$

i.e.

$$K_g(u, v) = kK_f\left(\frac{u}{k}, \frac{v}{k}\right).$$

Analogously, for $g(t) = f(t/k)$ we have

$$K_g(u, v) = \frac{1}{k} K_f(ku, kv).$$

For transformation $(x, y) \mapsto (y, x)$, i.e. $g(t) = f^{-1}(t)$, we can easy derive the relation

$$K_g(u, v) = \frac{1}{K_f\left(\frac{1}{u}, \frac{1}{v}\right)}.$$

The compositions of all described maps generate all affine transformations of plane. For such one with the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \mu \end{pmatrix}; \quad \alpha\mu - \beta\gamma \neq 0.$$

we consider the function $r(t) = \frac{\mu t+\gamma}{\beta t+\alpha}$. For any $f(t)$ and its A-image $g(t)$

$$K_g(u,v) = r(K_f(r^{-1}(u), r^{-1}(v))),$$

so called r-conjugancy of K (r^*K).

Let's denote

$$K_a = \frac{u+v}{2}; \quad K_g = \sqrt{uv}; \quad K_c = \frac{\sqrt{(u^2+1)(v^2+1)} + uv - 1}{u+v}$$

and

$$\Lambda = \{r^*K | K = K_a, K_g, K_c\}.$$

Theorem. *For given $K \in \Lambda$ all monotone convex (vertex) solutions of the equation (1) are arcs of paraboles, hyperboles or ellipses.*

P r o o f. Let's consider K_a, K_g, K_c at first.

From the Lagrange theorem it follows that all solutions of equation (1) for K_a are vertical paraboles.

From the Kuczma's theorem it follows that all solutions of equation (1) for K_g are arcs of increasing vertical hyperboles.

Let's now consider the arc of cycle $f(t) = \sqrt{1-t^2}$, $t \geq 0$. Streightforward calculation gives $K_f = K_c$, $u, v \leq 0$. For three others arcs of cycle the means are the same, as $-K_c(-u,-v) = K_c(u,v)$ for $uv \geq 0$. Any ellipse on plane can be devided on four arcs of monotonisity. There is an affine transformation mapping these arcs to the arcs of monotonisity of canonical cycle. Therefore the arcs of ellipse generates the means conjugated to K_c. Note, for points of cycle belonging to the different arcs of it the relation is not K_g and no mean at all.

To finish the proof note that affine images of described curves are arbitrary cone sections and means generated by them are from Λ. $\square$

§2. 3^d degree polynomials characterisation

It is naturally to ask about the functional-differential equation characterising the polynomials of degree three. This question is answered for the arbitrary degree by G. Gross [16]. But the form of it is not symmetrical. We will describe one-parameter family of means generating the 3d degree polynomials as solution of equation (1).

By Lemma it is enough to describe K_p, where p is polynomial up to shifts and homotety. Let's consider the polynomial

$$p(t) = a_3t^3 + a_2t^2 + a_1t + a_0.$$

Substitute $t = s - a_2/(3a_1)$ and adding some constant reduce it to the form

$$p(s) = a_3s^3 + cs,$$

where $c = a_1 - a_2^2/(3a_3)$. For $c = 0$

$$K_p(u, v) = H(u, v) = \frac{u + \sqrt{uv} + v}{3}$$

Heron's mean. For arbitrary c

$$K_p(u, v) = H(u - c, v - c) + c = \frac{\sqrt{(u - c)(v - c)} + u + v + c}{3}.$$

Therefore we proved the following result

Theorem. *All monotone convex (vertex) solutions of equation*

$$\frac{f(t) - f(s)}{t - s} = \frac{\sqrt{(f'(t) - c)(f'(s) - c)} + f'(t) + f'(s) + c}{3}$$

are arcs of polynomials

$$p(t) = a_3t^3 + a_2t^2 + a_1t + a_0,$$

for which $c = a_1 - a_2^2/(3a_3)$.

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