Some isometry groups of Urysohn space

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Abstract

We construct various isometry groups of Urysohn space (the unique complete separable metric space which is universal and homogeneous), including abelian groups which act transitively, and free groups which are dense in the full isometry group.

1 Introduction

In a posthumously-published paper, P. S. Urysohn [6] constructed a remarkable complete separable metric space $\mathbb{U}$ which is both *homogeneous* (any isometry between finite subsets of $\mathbb{U}$ can be extended to an isometry of $\mathbb{U}$) and *universal* (every complete separable metric space can be embedded in $\mathbb{U}$). This space is unique up to isometry.

The second author [7], [8] showed that $\mathbb{U}$ is both the generic complete metric space with distinguished countable dense subset (in the sense of Baire category) and the random such space (with respect to any of a wide class of measures).

In this paper, we investigate the isometry group $\mathrm{Aut}(\mathbb{U})$ of $\mathbb{U}$, and construct a few interesting subgroups of this group.

Our main tool is an analogous countable metric space $Q\mathbb{U}$, the unique universal countable homogeneous metric space with rational distances which was also have been considered in [6]. The existence of and uniqueness of $Q\mathbb{U}$ follows from the arguments used to establish the existence and uniqueness of $\mathbb{U}$ (see [8]). Alternatively, this can be deduced from the fact that the class of finite metric spaces with rational distances has the amalgamation property, from which Fraïssé's Theorem [4] gives the result. Now $\mathbb{U}$ is the completion of $Q\mathbb{U}$ (see [7]). In particular, any isometry of $Q\mathbb{U}$ extends uniquely to $\mathbb{U}$. Our notation suggests that $Q\mathbb{U}$ is "rational Urysohn space".

Let $\mathrm{Aut}(Q\mathbb{U})$ and $\mathrm{Aut}(\mathbb{U})$ be the isometry groups of $Q\mathbb{U}$ and $\mathbb{U}$. We show that $\mathrm{Aut}(Q\mathbb{U})$ is dense in $\mathrm{Aut}(\mathbb{U})$ (in the natural topology, induced by the product topology on $\mathbb{U}^{\mathbb{U}}$). We also show that $Q\mathbb{U}$ has an isometry which permutes all its points in a single cycle (indeed, it has $2^{\aleph_0}$ conjugacy classes of such isometries). The closure of the cyclic group generated by such an isometry is an abelian group which acts transitively on $\mathbb{U}$, so that $\mathbb{U}$ carries an abelian group structure (indeed, many such structures). Moreover, the free group of countable rank acts as a group of isometries of $Q\mathbb{U}$ which is dense in the full isometry group (and hence also is dense in $\mathrm{Aut}(\mathbb{U})$).

The space $Q\mathbb{U}$ is characterised by the following property: If A, B are finite metric spaces with rational distances (we say *rational metric spaces*, for short) with $A \subseteq B$, then any embedding of A in $Q\mathbb{U}$ can be extended to an embedding of B. It is enough to assume this in the case where $|B| = |A| + 1$, in which case it take the more convenient form:

($*$) If A is a finite subset of $Q\mathbb{U}$ and g a function from A to the rationals satisfying

- $g(a) \geq 0$ for all $a \in A$,
- $|g(a) - g(b)| \leq d(a,b) \leq g(a) + g(b)$ for all $a, b \in A$,

then there is a point $z \in Q\mathbb{U}$ such that $d(z,a) = g(a)$ for all $a \in A$.

Furthermore, $Q\mathbb{U}$ is homogeneous (any isometry between finite subsets of $Q\mathbb{U}$ extends to an isometry of $Q\mathbb{U}$), and every countable rational metric space can be embedded isometrically in $Q\mathbb{U}$.

There is an evident parallel between the theory of universal metric space and universal graph as well as theory of other universal objects.

2 $\mathrm{Aut}(Q\mathbb{U})$ is dense in $\mathrm{Aut}(\mathbb{U})$

The topology on the group $\mathrm{Aut}(\mathbb{U})$ of isometries of $\mathbb{U}$ is that induced by the product topology on $\mathbb{U}^{\mathbb{U}}$. In particular, $g_n \to g$ if and only if, for any finite sequence $(u_1, \ldots, u_m)$ of points and any $\epsilon > 0$, there exists n_0 such that $d(g_n(u_i), g(u_i)) < \epsilon$ for $1 \le i \le m$ and $n \ge n_0$.

With this topology, $\mathrm{Aut}(Q\mathbb{U})$ is a dense subgroup of $\mathrm{Aut}(\mathbb{U})$. Here is a proof. It suffices to show the following property of $Q\mathbb{U}$:

Proposition 1 *Given $\epsilon > 0$ and $v_1, \ldots, v_n, v'_1, \ldots, v'_{n-1}, v''_n \in Q\mathbb{U}$ such that $(v_1, \ldots, v_{n-1})$ and $(v'_1, \ldots, v'_{n-1})$ are isometric and*

$$|d(v'_i, v''_n) - d(v_i, v_n)| < \epsilon,$$

there exists $v'_n \in Q\mathbb{U}$ such that $(v_1, \ldots, v_n)$ and $(v'_1, \ldots, v'_n)$ are isometric and $d(v'_n, v''_n) < \epsilon$.

Assuming this for a moment, we complete the proof as follows. We are given an isometry g of $\mathbb{U}$ and points $u_1, \ldots, u_m \in \mathbb{U}$. Choose points $v_1, \ldots, v_m \in Q\mathbb{U}$ with $d(v_i, u_i) < \epsilon/4m$. Now using the above proposition, we inductively choose points $v'_1, \ldots, v'_m$ so that $(v_1, \ldots, v_m)$ and $(v'_1, \ldots, v'_m)$ are isometric and $d(v'_i, g(u_i)) < i\epsilon/m$. For suppose that $v'_1, \ldots, v'_{n-1}$ have been chosen. Choose any point $v''_n \in Q\mathbb{U}$ with $d(g(u_n), v''_n) < \epsilon/4m$. Then

$$d(u_i, u_n) - \epsilon/2m < d(v_i, v_n) < d(u_i, u_n) + \epsilon/2m,$$

and

$$d(g(u_i), g(u_n)) - (4i+1)\epsilon/4m < d(v'_i, v''_n) < d(g(u_i), g(u_n)) + (4i+1)\epsilon/4m,$$

so

$$|d(v'_i, v''_n) - d(v_i, v_n)| < (4i+3)\epsilon/4m \le (4n-1)\epsilon/4m.$$

So we may apply the Proposition to choose v'_n with $d(v'_n, v''_n) < (4n-1)\epsilon/4m$. Then $d(v'_n, g(u_n)) < n\epsilon/m$, and we have finished the inductive step. At the conclusion, we have $d(v'_n, g(u_n)) < n\epsilon/m \le \epsilon$ for $1 \le n \le m$.

Now we find an isometry of $Q\mathbb{U}$ mapping v_i to v'_i for $1 \le i \le m$ (by the homogeneity of $Q\mathbb{U}$), and the proof is complete. $\square$

Proof of the Proposition We have to extend the set $\{v_1', \ldots, v_{n-1}', v_n''\}$ by adding a point v_n' with prescribed distances to $v_1', \ldots, v_{n-1}'$ and distance less than ϵ to v_n''. So it is enough to show that these requirements don't conflict, that is, that

$$|d(v_n', x) - d(v_n', y)| \leq d(x, y) \leq d(v_n\, x) + d(v_n\, y)$$

for $x, y \in \{v_1', \ldots, v_{n-1}', v_n''\}$. There are no conflicts if $x, y \neq v_n''$: this follows from the fact that the points $v_1, \ldots, v_n$ exist having the required distances. So we may assume that $x = v_i'$ and $y = v_n''$, in which case the consistency follows from the hypothesis. $\square$

3 $\mathrm{BAut}(\mathbb{U})$ is dense in $\mathrm{Aut}(\mathbb{U})$

For a metric space M, we define $\mathrm{BAut}(M)$ to be the group of all *bounded* isometries of M (those satisfying $d(x, g(x)) \leq k$ for all $x \in M$, where k is a constant). Clearly it is a normal subgroup of $\mathrm{Aut}(M)$, though in general it may be trivial, or it may be the whole of $\mathrm{Aut}(M)$.

We show that $\mathrm{BAut}(Q\mathbb{U})$ is a dense proper subgroup of $\mathrm{Aut}(Q\mathbb{U})$: in other words, any isometry between finite subsets of $Q\mathbb{U}$ can be extended to a bounded isometry of $Q\mathbb{U}$. This is immediate from the following lemma.

Lemma 2 *Let f be an isometry between finite subsets A and B of $Q\mathbb{U}$, satisfying $d(a, f(a)) \leq k$ for all $a \in A$. Then f can be extended to an isometry g of $Q\mathbb{U}$ satisfying $d(x, g(x)) \leq k$ for all $x \in Q\mathbb{U}$.*

Proof Suppose that $f : a_i \mapsto b_i$ for $i = 1, \ldots, n$, with $d(a_i, b_i) \leq k$. It is enough to show that, for any point $u \in Q\mathbb{U}$, there exists $v \in Q\mathbb{U}$ such that $d(b_i, v) = d(a_i, u)$ for all i and $d(u, v) \leq k$. For then we can extend f to any further point; the same result in reverse shows that we can extend f^{-1}, and then we can construct g by a back-and-forth argument.

The point v must satisfy $d(b_i, v) = d(a_i, u)$ and $d(u, v) \leq k$. We must show that these requirements are consistent; then the existence of v follows from the extension property of $Q\mathbb{U}$. Clearly the consistency conditions for the values $d(b_i, v)$ are satisfied. So the only possible conflict can arise from the inequality

$$|d(v, u) - d(v, b_i)| \leq d(u, b_i) \leq d(v, u) + d(v, b_i).$$

We wish to impose an upper bound on $d(v,u)$, so a conflict could arise only if a lower bound arising from the displayed equation were greater than k, that is, $|d(v,b_i) - d(u,b_i)| > k$, or equivalently, $|d(u,a_i) - d(u,b_i)| > k$. But this is not the case, since

$$|d(u,a_i) - d(u,b_i)| \leq d(a_i,b_i) \leq k.$$

4 A cyclic isometry of $Q\mathbb{U}$

The following theorem is true:

Theorem 3 *There is an isometry g of $Q\mathbb{U}$ such that $\langle g\rangle$ is transitive on $Q\mathbb{U}$. The induced isometry of $\mathbb{U}$ has the property that every orbit of $\langle g\rangle$ is dense in $\mathbb{U}$.*

Proof The second statement follows trivially from the first. So it is enough to show that there is an isometry σ of $Q\mathbb{U}$ such that $\langle\sigma\rangle$ is transitive on $Q\mathbb{U}$. The analogous statement for the universal homogeneous integral metric space was proved in [1], and we require this in the proof.

If a metric space has a cyclic automorphism, we can identify its points with the integers so that the automorphism is the shift. Then the metric is completely determined by the function $f(i) = d(i,0)$ on the non-negative integers; for $d(i,j) = f(|j-i|)$. The function should satisfy the constraints

(a) $f(i) \geq 0$, with equality if and only if $i = 0$.

(b) $|f(i) - f(j)| \leq f(i+j) \leq f(i) + f(j)$ for all i, j.

Now the cyclic metric space given by such a function is isometric to $Q\mathbb{U}$ if and only if f has the following property:

(c) given any function h from $\{1,\ldots,k\}$ to the positive rationals satisfying

$$|h(i) - h(j)| \leq f(|i-j|) \leq h(i) + h(j)$$

for $i, j \in \{1,\ldots,k\}$, there exists a natural number N such that $h(i) = f(N-i)$ for all $i \in \{1,\ldots,k\}$.

Under those conditions a distance matrix (see [8]) $\{d(i,j)\}, i,j \in \mathbb{Z}$ is *Toeplitz matrix* (e.g. commutes with the shift on $\mathbb{Z}$). So, we will call a function satisfying (a) and (b) a *Toeplitz function* (from the form of the metric, $d(i,j) = f(|i-j|)$, and say that it is *universal* if it also satisfies (b).

It is worth mentioning that condition (c) is a special case of necessary and sufficient condition for universality of the arbitrary distance matrix over real numbers which was given in [8]; here universality of matix means that completion of the set $\mathbb{Z}$ under the metric defined by given distance matrix is universal Urysohn space.

We denote by RT_n the space of non-negative rational n-tuples satisfying condition (b) for $i,j \in \{1,\ldots,n\}$. Given $f \in RT_n$, we say that the m-tuple $(h(1),\ldots,h(m))$ is *f-admissible* if

$$|h(i) - h(i+k)| \leq f(k) \leq h(i) + h(i+k)$$

for $1 \leq i < i+k \leq m$ and $k \leq n$. We note that if h is f-admissible, then it is admissible with respect to the restriction of f to $\{1,\ldots,n'\}$ for any $n' \leq n$.

We need to show that, if h is f-admissible, then there is some prolongation f^* of f such that h is f^*-admissible and (f,h) is an initial segment of a Toeplitz function. This is proved for integral metric spaces in [1]. For the rational case, multiply everything by the least common multiple of the denominators, apply the integral result, and divide by d. $\square$

The proof shows that, in the sense of Baire category, almost all rational Toeplitz functions are universal, so that almost all rational metric spaces which admit cyclic transitive isometry groups are isometric to $Q\mathbb{U}$. It gives further information too:

Corollary 4 *The group* $\mathrm{Aut}(Q\mathbb{U})$ *contains* $2^{\aleph_0}$ *conjugacy classes of isometries which permute the points in a single cycle. Moreover, representatives of these classes remain non-conjugate in* $\mathrm{Aut}(\mathbb{U})$.

Proof It is clear that, if cyclic isometries g and h are conjugate, then the functions f_g and f_h describing them as in the above proof are equal. For, if $h = k^{-1}gk$, then

$$f_h(n) = d(x, h^n(x)) = d(x, k^{-1}g^n k(x)) = d(k(x), g^n k(x)) = f_g(n).$$

But the set of functions describing cyclic isometries of $Q\mathbb{U}$ is residual, hence of cardinality $2^{\aleph_0}$. $\square$

The cyclic isometries constructed in this section have the property that $d(x,g(x))$ is constant for $x \in Q\mathbb{U}$, and hence this holds for all $x \in \mathbb{U}$. In particular, these isometries are bounded.

5 An abelian group of exponent 2

To extend this argument to produce other groups acting regularly (=freely and transitively) on $Q\mathbb{U}$, it is necessary to change the definition of a Toeplitz function so that the metric is defined by translation in the given group. We give here one simple example.

Proposition 5 *The countable abelian group of exponent* 2 *can act regularly as an isometry group of* $Q\mathbb{U}$.

Proof This group G has a chain of subgroups $H_0 \leq H_1 \leq H_2 \leq \cdots$ whose union is G, with $|H_i| = 2^i$. We show that, given any H_i-invariant rational metric on H_i and any $h \in H_{i+1} \setminus H_i$, we can prescribe the distances from h to H_i arbitrarily (subject to the consistency condition) and extend the result to an H_{i+1}-invariant metric on H_{i+1}. The extension of the metric is done by translation in H_{i+1}: note that $H_{i+1} \setminus H_i$ is isometric to H_i, since $d(h+h', h+h'') = d(h', h'')$ for $h', h'' \in H_i$. Now the resulting function is a metric. All that has to be verified is the triangle inequality. Now triangles with all vertices in H_i, or all vertices in $H_{i+1} \setminus H_i$, clearly satisfy the triangle inequality. Any other triangle can be translated to a triangle containing h and two points of H_i, for which the triangle inequality is equivalent to the consistency condition for extending the metric to $H_i \cup \{h\}$. $\square$

As before, for such a group G almost all G-invariant metrics (in the sense of Baire category) are isometric to $Q\mathbb{U}$.

6 Transitive abelian subgroups of the group $Iso(\mathbb{U})$

The constructions of the last two sections have the following consequence:

Proposition 6 *There are transitive abelian groups of isometries of* $\mathbb{U}$ *of infinite exponent, and transitive groups of exponent* 2.

Proof Let G be one of the abelian groups previously constructed, and $\overline{G}$ its closure in $\mathrm{Aut}(\mathbb{U})$. Since the orbits of G are dense, it is clear that $\overline{G}$ is transitive. Moreover, as the closure of an abelian group, it is itself abelian. For, if $h, k \in \overline{G}$, say $h_i \to h$ and $k_i \to k$; then $h_i k_i = k_i h_i \to hk = kh$. Similarly, if G has exponent 2, then so does $\overline{G}$. $\square$

What is the structure of the closure of action of the infinite cyclic group $\mathbb{Z}$? Since there are many choices for action of $\mathbb{Z}$, we must expect that their closures will not all be alike. In particular, there should be some choices of $\mathbb{Z}$ such that $\overline{\mathbb{Z}}$ is torsion-free, and others for which it is not.

7 Regular actions of other groups on $\mathbb{U}$ and on the universal graph R

The universal graph R (see definition f.e. [3]) could be considered as a universal homogeneous metric space in the class of metric spaces with the distances which takes values $\{0, 1, 2\}$; graph R could be isometrically imbed to $\mathbb{U}$. But there is a natural more deeper parallelism between theory of universal metric space and theory of universal graph. There is one-way relation between transitive group actions on $Q\mathbb{U}$ (or, more generally, group actions on $\mathbb{U}$ with a dense orbit) and transitive actions on the random (universal) graph R, as given in the following result.

Proposition 7 *Let G be a group acting on Urysohn space $\mathbb{U}$ and X is a countable dense orbit. Then G preserves the structure of the random graph R on X (which is definied below).*

Proof Partition the positive real numbers into two subsets E and N such that, for any $C, \epsilon > 0$, there are consecutive intervals of length at most ϵ to the right of C with one contained in E and the other in N. (For example, take a divergent series (a_n) whose terms tend to zero, and put half-open intervals of length a_n alternately in E and N.)

We define a graph on X by letting $\{x, y\}$ be an edge if $d(x, y) \in E$, and a non-edge if $d(x, y) \in N$. Clearly this graph is G-invariant; we must show that it is isomorphic to the random graph R.

Let U and V be finite disjoint sets of points of X, and let the diameter of $U \cup V$ be h and the minimum distance between two of its points be m.

Choose $C > h/2$ and $\epsilon < m/2$, and find consecutive intervals I_E and I_N as above. Let $U \cup V = \{w_1, \ldots, w_n\}$. For all values $a_i \in I_E \cup I_n, i = 1 \ldots n$, the consistency condition

$$|a_i - a_j| \leq d(w_i, w_j) \leq a_i + a_j$$

is always satisfied. So choose the values such that a_i is in the interior of I_E if $w_i \in U$, and in the interior of I_n if $w_i \in V$. Let z be a point of U with $d(z, w_i) = a_i, i = 1 \ldots n$. Since X is dense, we can find $x \in X$ such that $d(x, z)$ is arbitrarily small; in particular, so that $d(x, w_i)$ is in I_E (resp. I_N) if and only if $d(z, w_i)$ is. Thus x is joined to all verties in U and to none in V. This condition characterises R as a countable graph.

Corollary 8 *If a countable group G can act on Urysohn space $\mathbb{U}$ with dense orbits then this group can act transitively on the universal graph R.*

The converse is not true. A special case of the result of Cameron and Johnson [2] shows that a sufficient condition for a group G to act regularly on the universal graph R is that any element has only finitely many square roots. In a group with odd exponent, each element has a unique square root. So any such group acts regularly on R. But we have the following:

Proposition 9 *The countable abelian group of exponent* 3 *cannot act on* $\mathbb{U}$ *with a dense orbit, and in particular cannot act transitively on $Q\mathbb{U}$.*

Proof Suppose that we have such an action of this group A. Since the stabiliser of a point in the dense orbit is trivial, we can identify the points of the orbit with elements of A (which we write additively).

Choose $x \neq 0$ and let $d(0, x) = \alpha$. Then $\{0, x, -x\}$ is an equilateral triangle with side α. Since $\mathbb{U}$ is universal and A is dense, there is an element y such that $d(x, y), d(-x, y) \approx \frac{1}{2}\alpha$ and $d(0, y) \approx \frac{3}{2}\alpha$. (The approximation is to within a given ϵ chosen smaller than $\frac{1}{6}$. Then the three points $0, y, x - y$ form a triangle with sides approximately $\frac{3}{2}\alpha$, $\frac{1}{2}\alpha$, $\frac{1}{2}\alpha$, contradicting the triangle inequality.

8 Unbounded isometries of $\mathbb{U}$

The subgroup $\mathrm{BAut}(\mathbb{U})$ is not the whole isometry group, because unbounded isometries exist. The simplest way to see this is to mention thatf.e. euclidean space R^n can be imbedded to $\mathbb{U}$ in such a way that the group of motions $Iso(R^n$ is monomorphically imbedded to $Iso(\mathbb{U})$ and the group of rotations of R^n extended to the unbounded isometry. But we will give a direct construction of such isometry. We are grateful to Jaroslav Nešetřil for the following argument.

Proposition 10 *There exist unbounded isometries of $Q\mathbb{U}$ (and hence of $\mathbb{U}$).*

The proof depends on a lemma.

Lemma 11 *Let A be a finite subset of $Q\mathbb{U}$ and let g be a function on A satisfying the consistency conditions $(*)$. Then the diameter of the set*

$$\{z \in Q\mathbb{U} : d(z,a) = g(a) \text{ for all } a \in A\}$$

is twice the minimum value of g.

Proof Let z_1 and z_2 be two points realising g. Consider the problem of adding z_2 to the set $A \cup \{z_1\}$. The consistency conditions for z_2 are precisely those for z_1 together with the conditions

$$|d(z_2,z_1) - d(z_2,a)| \leq d(z_1,a) \leq d(z_2,z_1) + d(z_2,a)$$

for all $a \in A$. Since $d(z_1,a) = d(z_2,a) = g(a)$, the only non-trivial restriction is $d(z_1,z_2) \leq 2d(z_1,a) = 2g(a)$, which must hold for all $a \in A$. $\square$

Proof of the Proposition We construct an isometry f of $Q\mathbb{U}$ by the standard back-and-forth method, starting with any enumeration of $Q\mathbb{U}$. At odd-numbered stages we choose the first point not in the range of f and select a suitable pre-image. At stages divisible by 4 we choose the first point not in the domain of f and select a suitable image. This guarantees that the isometry we construct is a bijection from $Q\mathbb{U}$ to itself.

At stage $4n+2$, let $\mathbb{U}$ be the domain of f. Choose an unused point z whose least distance from $\mathbb{U}$ is n. Now the diameter of the set of possible images of z is $2n$; so we can choose a possible image $f(z)$ whose distance from z is at least n. Then the constructed isometry is not bounded. $\square$

We can improve this argument to construct an isometry g such that all powers of g except the identity are unbounded. In fact, even more is true:

Lemma 12 *There are two isometries a, b of $Q\mathbb{U}$ which generate a free group, all of whose non-identity elements are unbounded isometries.*

Proof We begin by enumerating $Q\mathbb{U} = (x_0, x_1, \ldots)$. We follow the argument we used to show that unbounded isometries exist. We construct a and b simultaneously, using the even-numbered stages for a back-and-forth argument to ensure that both are bijections, and the odd-numbered stages to ensure that any word in a and b is unbounded. The first requirement is done as we have seen before.

Enumerate the words $w(a, b)$ in a and b and their inverses. (It suffices to deal with the cyclically reduced words, since all others are conjugates of these.) We show first how to ensure that $w(a, b) \neq 1$. At a given stage, suppose we are considering a word $w(a, b)$. Choose a point x_i such that neither a nor b, nor their inverses, has been defined on x_j for $j \geq i$. Suppose that w ends with the letter a. Since there are infinitely many choices for the image of x_i under a, we may choose an image x_j with $j > i$. Now define the action of the second-last letter of the word on x_j so that the image is x_k with $k > j$. Continuing in this way, we end up with a situation where $w(a, b)x_i = x_m$ with $m > i$. So $w(a, b) \neq 1$.

To ensure that $w(a, b)$ is unbounded, we must do more. Enumerate the words so that each occurs infinitely often in the list. Now, the kth time we revisit the word w, we can ensure (as in our construction of an unbounded isometry) that $d(x_i, w(a, b)x_i) \geq k$. Thus $w(a, b)$ is unbounded.

9 A dense free subgroup of $\mathrm{Aut}(\mathbb{U})$

We can now use a trick due to Tits [5] to show that there is a dense subgroup of $\mathrm{Aut}(\mathbb{U})$ which is a free group of countable rank.

Theorem 13 *There is a subgroup F of $\mathrm{Aut}(Q\mathbb{U})$ which acts faithfully and homogeneously on $Q\mathbb{U}$ and is isomorphic to the free group of countable rank.*

Proof Since the free group F_2 contains a subgroup isomorphic to F_ω, choose a group H with free generators h_i for $i \in \mathbb{N}$, such that $H \cap \mathrm{BAut}(Q\mathbb{U}) = 1$. Enumerate the pairs of isometric n-tuples of elements of $Q\mathbb{U}$, for all n, as (α_0, β_0), (α_1, β_1), $\ldots$. Now, for each i, Lemma 2 shows that we can choose $n_i \in \mathrm{BAut}(Q\mathbb{U})$ such that $n_i h_i(\alpha_i) = \beta_i$. Let F be the group generated by the elements $n_1 h_1, n_2 h_2, \ldots$. Clearly F acts homogeneously on $Q\mathbb{U}$. We claim that F is free with the given generators. Suppose that $w(n_i h_i) = 1$ for some word w. Since $\mathrm{BAut}(Q\mathbb{U})$ is a normal subgroup, we have $nw(h_i) = 1$ for some $n \in \mathrm{BAut}(Q\mathbb{U})$. Since n is bounded and $w(h_i)$ unbounded, this is impossible. In fact this argument shows that all the non-identity elements of F are unbounded isometries.

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