SPb. Math. Society Preprint 2004-08 16 Oct 2004

Polynomial–time computation of the degree of a dominant morphism in zero–characteristic II

Alexander L. Chistov

St. Petersburg Institute for Informatics and Automation of the Academy of Sciences of Russia
14th line 39, St. Petersburg 199178, Russia,
e-mail: labta@iias.spb.su or sliss@iias.spb.su

2004

Abstract

Consider a projective algebraic variety W which is an irreducible component of a set of all common zeroes of a family of homogeneous polynomials of degrees less than d in $n+1$ variables in zero–characteristic. Consider a dominant rational morphism from W to W' given by homogeneous polynomials of degree d'. We suggest algorithms to construct objects in general position related to the morphism $W \to W'$ generalizing the algorithms from the first part of the paper to the case $\dim W > \dim W'$. These algorithms are deterministic and polynomial in $(dd')^n$ and the size of input.

Introduction

This paper continues [9]. We investigate general rational morphisms in zero–characteristic from algorithmic point of view now. Everything is prepared for the next paper, where an algorithm for the first Bertini theorem is constructed. The last algorithm will use Theorem 2, its statement see at the end of the Introduction. The second application of the results of the present paper is the substantiation of the method of representing an algebraic variety using representative systems of points of its irreducible components suggested in [7]. Namely, in one of the next papers we shall consider this substantiation as well as an algorithm for constructing the smooth stratification described in [7] in details. Theorem 1, see below, will be the main tool there. Besides that we shall see in the same next, that Theorem 1 allows to compute the intersections and the indices of intersection of algebraic varieties given by method from [7] (although the irreducible components of the intersections not always can be given using the method from [7] with a good upper bound for degrees of corresponding polynomials).

In the present paper we need to elaborate a new method for solving systems of polynomial equations and inequalities with squares of absolute values, see [3], in the case when these systems contain generic linear forms (or even generic homogeneous polynomials of given degrees; but in this paper we need only the case of linear forms). Namely, let it is a system in $(n+1)\nu$ variables $X_{i,j}$, $0 \leq j \leq n$, $1 \leq i \leq \nu$, where $\nu = O(1)$ (is considered as a constant, $\nu < n$), and also in linear forms $U_\alpha = \sum_{0 \leq j \leq n} u_{\alpha,j} X_{i,j}$, $1 \leq \alpha \leq \mu$, where all $u_{\alpha,j}$ are variables and $\mu = O(n)$. Suppose also that this system depends on other $O(n)$ variables. So the total number of variables may be $\geq Cn^2$ for a constant C. Hence applying the general method for solving such systems, see [3], we get an estimate polynomial in d^{n^2} for the working time of the algorithm for constructing a solution, where d is a bound for degrees of all equations and inequalities. In brief, the suggested method allows to replace the generic linear forms by the ones in ν variables $X_{i,j_1}, \ldots, X_{i,j_\nu}$, where $0 \leq j_1 < \ldots < j_\nu \leq n$ are chosen, and obtain a new system. So there are only $\mu\nu = O(n)$ variables u_{α,j_β} in the new systems corresponding to the different choices of $j_1, \ldots, j_\nu$. This method reduces constructing a solution of the initial system to constructing the ones of $\binom{n+1}{\nu}$ new systems in $O(n)$ variables.

The results of the present paper are based actually on this method (although we do not describe it formally separately from the considered problems). It is applied in Section 1 to strengthen Theorem 1 [8], Section 4 and Section 6. Also we would like to underline here the importance of [5] and its consequence Theorem 2 [9] in the present paper. Besides that, now we use [8] (the correction of the statement of Lemma 2 [8] see in [9]), [9] and other our previous papers.

Let k be a field of zero–characteristic with algebraic closure $\overline{k}$. Let $X_0, X_1, \ldots$ be independent variables over k. Denote by $\mathbb{P}^n(\overline{k})$, $n \geq 0$, the projective space over the field $\overline{k}$ with coordinates $X_0, \ldots, X_n$. We shall suppose that $\mathbb{P}^n(\overline{k})$ is defined over k. For arbitrary homogeneous polynomials $g_1, \ldots, g_m \in \overline{k}[X_0, \ldots, X_n]$ we shall denote by $\mathcal{Z}(g_1, \ldots, g_m)$ the set of all common zeroes of polynomials $g_1, \ldots, g_m$ in $\mathbb{P}^n(\overline{k})$. The similar notations will be used for the sets of zeroes of ideals and polynomials with other fields of coefficients in affine and projective spaces (this will be seen from the context). For a field k and a family of elements $a' = \{a'_i\}_{i \in I}$ from an extension of k we shall denote by $k(a')$ (respectively

$k[a']$) the field (respectively the ring) generated over k by all the elements a'_i, $i \in I$. For two (or more) families a', a'' the notations $k(a', a'')$, $k[a', a'']$ have the similar sense. Now suppose that a' is a family of independent variables over k and $f \in k[a']$. Suppose that $b' = \{b'_i\}_{i \in I}$ is a family of elements from k with the same indices as a'. Then we shall denote by $f(b')$ the value of f at $a'_i = b'_i$, $i \in I$, and also we use other similar notations.

In the algorithmic part of the paper we shall suppose that the field k is finitely generated over the field of rational numbers $\mathbb{Q}$. The field k is supplied with a real structure if the embedding $k \to R[\sqrt{-1}]$ is given, where R is a real ordered field.

Consider a system of polynomial equations in n variables and a fixed number of inequalities with squares of absolute values with coefficients from the field k. Let the degrees of equations and inequalities be less than d, see the details in [3] and also in Section 1 of [9], Then, see Proposition 1 [3], one can construct a real structure on k and decide within the time polynomial in d^n and the size of input whether this system has a solution in $\mathbb{A}^n(\overline{k})$ relative to this real structure. When such a solution exists one can construct it within the same time. The coordinates of the solution belong to a finite extension of the field k. This finite extension is constructed and is given by a primitive element and its minimal polynomial. We shall use this proposition many times without special reference to it.

Since k is an arbitrary finitely generated field over $\mathbb{Q}$ this result is valid also in the case if one replace the field of coefficients k by a finite extension of the field $k_i = k(\varepsilon_1, \ldots, \varepsilon_i)$ for fixed i, where $\varepsilon_j > 0$ for $j \geq 1$ is an infinitesimal with respect to the field k_{j-1}, see below. Theorem 2 [9], which is based on [5], strengthens this result. Actually in this paper the most part of systems of polynomial equations and inequalities is solved by Theorem 2 [9].

Let $V \subset \mathbb{P}^n(\overline{k})$ be a projective algebraic variety which is a set of all common zeroes of a family of homogeneous polynomials in $X_0, \ldots, X_n$ of degrees less than d with coefficients from a field k. Let W be a defined over k and irreducible over k component of V and W is given by its point which belongs to only one irreducible over k component of V, see Theorem 1 from [8] and also below for details. Let $e_0, \ldots, e_r \in k[X_0, \ldots, X_n]$, be homogeneous polynomials such that $\deg e_i = d' \geq 1$ for every nonzero e_i, $0 \leq i \leq r$. We shall suppose in what follows that $\dim W \cap \mathcal{Z}(e_0, \ldots, e_r) < \dim W$. Let

$$p : W \setminus \mathcal{Z}(e_0, \ldots, e_r) \to \mathbb{P}^r, \quad (X_0 : \ldots : X_n) \mapsto (e_0 : \ldots : e_r). \tag{1}$$

be the regular morphism. Set $W' = \overline{p(W \setminus \mathcal{Z}(e_0, \ldots, e_r))}$ (here and below in the similar situations the bar denotes closure with respect to the Zariski topology). Hence W' is irreducible over k closed subvariety of $\mathbb{P}^r(\overline{k})$. Note that one can consider the rational morphism $W \to \mathbb{P}^r(\overline{k})$ of projective algebraic varieties given by the same rule but to be more accurate in what follows we deal with regular morphisms of quasiprojective algebraic varieties. Denote by

$$p' : W \setminus \mathcal{Z}(e_0, \ldots, e_r) \to W'$$

the dominant morphism induced by p. Suppose that $\dim W = n - s$ and $\dim W' = n - \sigma$ (one can compute s and σ using the algorithm from [4], see below). So $0 \leq s \leq \sigma \leq n$. Let $v_j = \sum_{0 \leq i \leq n} v_{j,i} X_i$, $0 \leq j \leq n + 1$, be generic linear forms in $X_0, \ldots, X_n$, i.e., the transcendency degree of the family of all

coefficients $v_{j,i}$ over k is $(n+2)(n+1)$. By definition , $v = (v_{s+1}, \dots, v_\sigma)$, $\overline{v} = (v_0, \dots, v_{n+1})$ and k_v (respectively $k_{\overline{v}}$) is the field generated over k by all the $v_{j,i}$, $s+1 \leq j \leq \sigma$, $0 \leq i \leq n$ (respectively $0 \leq j \leq n+1$, $0 \leq i \leq n$). Denote by $\widetilde{v}$ the family of coefficients $v_{j,i}$, $0 \leq j \leq n+1$, $0 \leq i \leq n$. So $k_{\overline{v}} = k(\widetilde{v})$. Now p and p' induce the morphisms

$$p_v \, : \, (W \cap \mathcal{Z}(v_{s+1}, \dots, v_\sigma)) \setminus \mathcal{Z}(e_0, \dots, e_r) \to \mathbb{P}^r(\overline{k_v}),$$
$$p'_v \, : \, (W \cap \mathcal{Z}(v_{s+1}, \dots, v_\sigma)) \setminus \mathcal{Z}(e_0, \dots, e_r) \to W'(k_v)$$

defined over the field k_v. For convenience of notation set $q_v = p'_v$. So q_v is a dominant morphism. The dimension $\dim W \cap \mathcal{Z}(v_{s+1}, \dots, v_\sigma) \setminus \mathcal{Z}(e_0, \dots, e_r) = n - \sigma$. We compute, see Theorem 2 below, the degree $\deg p'_v$.

Let us apply Theorem 1 [9] to the morphism p'_v (in place of p') replacing the field k by k_v in the statement of this theorem. Denote by $\mathcal{V} \subset \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k_v})$ the nonempty open in the Zariski topology subset of $(n-\sigma+1)$-tuples of linear forms from assertion (d) of Theorem 1 [9] applied to p'_v. *Set* $\mathcal{U}'_0 = \mathcal{V} \cap \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$. Then obviously $\mathcal{U}'_0$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$. Hence $L' \in \mathcal{U}'_0$ if and only if (30)–(33), see Section 4, holds with v and ground field k_v in place of λ and k respectively. The set $\mathcal{U}'_0$ is invariant with respect to the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$ and, therefore, is defined over k. We shall construct an element from $\mathcal{U}'_0$. More precisely, we shall define below nonempty open in the Zariski topology sets $\mathcal{U}'''_{n+1} \subset \mathcal{U}'''_0 \subset \mathcal{U}''_0 \subset \mathcal{U}'_0$ related to the morphism p, and construct an element from $\mathcal{U}'''_0$ within the polynomial time, see Theorem 2.

Let K be a field containing algebraically independent over k elements $u_{j,w}$, $j \in \{0, \sigma+1, \dots, n\}$, $0 \leq w \leq r$, and such that K and $k_{\overline{v}}$ are linearly disjoint over k. Denote by u the family of elements $u_{j,w}$, $j \in \{0, \sigma+1, \dots, n\}$, $0 \leq w \leq r$. Let $l'_{j,w} \in \overline{k}$, $j \in \{0, \sigma+1, \dots, n\}$, $0 \leq w \leq r$. Let $L'_j = l'_{j,0}X_0 + \ldots + l'_{j,r}X_r$ (respectively $U_j = u_{j,0}X_0 + \ldots + u_{j,r}X_r$), $j \in \{0, \sigma+1, \dots, n\}$, be linear forms from $\overline{k}[X_0, \dots, X_r]$ (respectively $K[X_0, \dots, X_n]$). Set $e'_j = L'_j(e_0, \dots, e_r)$ (respectively $u_j = U_j(e_0, \dots, e_r)$), $j \in \{0, \sigma+1, \dots, n\}$. Put $L' = (L'_0, L'_{\sigma+1}, \dots, L'_n)$ (respectively $U = (U_0, U_{\sigma+1}, \dots, U_n)$). In what follows we shall define some objects corresponding to L'. The similar objects corresponding to U are defined by replacing the field k by K.

Let L' be an arbitrary as above (but may be $L' \notin \mathcal{U}'_0$). In what follows unless we state otherwise we shall suppose that the linear forms linear forms $L'_0, L'_{\sigma+1}, \dots, L'_n$ are linearly independent. We shall consider the morphisms

$$p_{L'} \, : \, W \setminus \mathcal{Z}(e'_0, e'_{\sigma+1} \dots, e'_n) \to \mathbb{P}^{n-\sigma}(\overline{k}), \tag{2}$$
$$p_{v,L'} \, : \, W \cap \mathcal{Z}(v_{s+1}, \dots, v_\sigma) \setminus \mathcal{Z}(e'_0, e'_{\sigma+1} \dots, e'_n) \to \mathbb{P}^{n-\sigma}(\overline{k_v}), \tag{3}$$

defined over the fields k and k_v respectively and given by the same formula

$$(X_0 : \ldots : X_n) \mapsto (e'_0 : e'_{\sigma+1} : \ldots : e'_n).$$

For every $\sigma \leq i \leq n$ denote by

- $W_i(L')$ the union of all irreducible over $\overline{k}$ components E of $W \cap \mathcal{Z}(e'_{\sigma+1}, \dots, e'_i)$ such that $\dim E = n - i + \sigma - s$ and the restriction $p_{L'}|_E \, : \, E \setminus \mathcal{Z}(e'_0, e'_{\sigma+1} \dots, e'_n) \to \mathcal{Z}(X_1, \dots, X_{i-\sigma})$ is dominant (here $\mathcal{Z}(X_1, \dots, X_{i-\sigma}) \subset \mathbb{P}^{n-\sigma}(\overline{k})$ is a linear subspace).

Hence $E \not\subset \mathcal{Z}(e'_0, t'_{\sigma+1} \ldots, e'_n)$. The morphism $p_{L'}|_{W_i(L')} : W_i(L') \setminus \mathcal{Z}(e'_0, e'_{\sigma+1}, \ldots, e'_n) \to \mathcal{Z}(X_1, \ldots, X_{i-\sigma})$ is dominant if $W_i(L') \neq \emptyset$ (one can not exclude here that $W_i(L')$ is empty for some i). In particular $W_\sigma(L') = W$ if and only if the morphism $p_{L'}$ is dominant. Note that $W_i(L')$ depends only on $L'_{\sigma+1}, \ldots, L'_i$. So we shall denote also $W_i(L') = W(L'_{\sigma+1}, \ldots, L'_i)$. The morphism $p_{L'}|_{W_i(L')}$ is dominant or $W_i(L') = \emptyset$, for every $\sigma + 1 \leq i \leq n$. Therefore,

$$\dim W(L'_{\sigma+1}, \ldots, L'_{i-1}) \cap \mathcal{Z}(e'_i) \leq n - i + \sigma - s, \tag{4}$$

and

$$\dim W(L'_{\sigma+1}, \ldots, L'_n)) \cap \mathcal{Z}(e'_0) \leq \sigma - s - 1. \tag{5}$$

Denote for brevity $L'_{n+1} = L'_0$, $e'_{n+1} = e'_0$, $W_{n+1}(L') = W(L'_{\sigma+1}, \ldots, L'_{n+1}) = \emptyset$. Therefore,

$$W_{i-1}(L') \cap \mathcal{Z}(e'_i) = W_i(L') \cup W_i^{(1)}(L') \cup W_i^{(2)}(L') \cup W_i^{(3)}(L'), \quad \sigma+1 \leq i \leq n+1,$$

where:

- $W_i^{(1)}(L')$ is the union of all irreducible over $\overline{k}$ components E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that E is an irreducible component of $W \cap \mathcal{Z}(e'_0, e'_{\sigma+1} \ldots, e'_n)$;
- $W_i^{(2)}(L')$ is the union of all the irreducible over $\overline{k}$ components E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that E is contained in $W_j^{(1)}(L')$ for some $s \leq j < i$;
- $W_i^{(3)}(L')$ is the union of all irreducible over $\overline{k}$ components E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that $E \not\subset \mathcal{Z}(e'_0, e'_{\sigma+1} \ldots, e'_n)$ and $\dim \overline{p_{L'}(E \setminus \mathcal{Z}(e'_0, e'_{\sigma+1} \ldots, e'_n))} < n - i + \sigma - s$. Hence $W_i^{(3)}(L') = \emptyset$ for $i = n, n+1$.

Set also $W^{(\beta)}(L') = W_\sigma^{(\beta)}(L') = \emptyset$, $W_0^{(\beta)}(L') = W_{n+1}^{(\beta)}(L')$, $\beta = 1, 2, 3$.

Notice that if $L' \in \mathcal{U}'_0$ then $W' \cap \mathcal{Z}(L'_0, L'_{\sigma+1}, \ldots, L'_n) = \emptyset$ in $\mathbb{P}^n(\overline{k})$. By Lemma 8 if $W' \cap \mathcal{Z}(L'_0, L'_{\sigma+1}, \ldots, L'_n) = \emptyset$ then $W \cap \mathcal{Z}(e_0, \ldots, e_r) = W \cap \mathcal{Z}(e'_0, e'_{\sigma+1}, \ldots, e'_n)$ (actually in this lemma we have a more strong assertion). Therefore, the dimension of every irreducible component of $W \cap \mathcal{Z}(e_0, \ldots, e_r)$ is at least $\sigma - s - 1$. Hence for arbitrary L' (may be, $W' \cap \mathcal{Z}(L'_0, L'_{\sigma+1}, \ldots, L'_n) \neq \emptyset$) every irreducible component of $W_{n+1}^{(1)}(L')$ is an irreducible component of dimension $\sigma - s - 1$ of $W \cap \mathcal{Z}(e_0, \ldots, e_r)$. If $L' \in \mathcal{U}'_0$ then for every $\sigma + 1 \leq i \leq n$ every irreducible component of $W_i^{(1)}(L')$ is an irreducible component of dimension $n-\sigma-s-i$ of $W \cap \mathcal{Z}(e_0, \ldots, e_r)$. If $L' \in \mathcal{U}'_0$ and $\bigcup_{\sigma+1 \leq j \leq i-1} W_j^{(3)}(L')$ does not contain any irreducible component E of $W \cap \mathcal{Z}(e_0, \ldots, e_r)$ such that $\dim E = n - \sigma + s - i$ (in particular, if $\bigcup_{\sigma+1 \leq j \leq i-1} W_j^{(3)}(L') = \emptyset$, or if $L' \in \mathcal{U}''_0$, see below) then, conversely, every irreducible component of dimension $n-\sigma-s-i$ of the variety $W \cap \mathcal{Z}(e_0, \ldots, e_r)$ is an irreducible component $W_i^{(1)}(L')$.

Further, for every $z \in W'$ each irreducible component E of $p^{-1}(z)$ is an irreducible component of $p_{L'}^{-1} p_{L'} p^{-1}(z)$ if $L' \in \mathcal{U}'_0$. Hence $\dim E \geq \sigma - s$. The last inequality is the theorem about dimensions of fibers for the morphism p', cf. [12]. Besides that, for every $\sigma + 1 \leq i \leq n + 1$

- if $W' \cap \mathcal{Z}(L'_0, L'_{\sigma+1}, \ldots, L'_n) = \emptyset$ then $W_i^{(1)}(L')$ is the union of all irreducible over $\overline{k}$ components E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that E is an irreducible component of $W \cap \mathcal{Z}(e_0, \ldots, e_r)$;

- if $W' \cap \mathcal{Z}(L'_0, L'_{\sigma+1}, \ldots, L'_n) = \emptyset$ then $W_i^{(3)}(L')$ is the union of all irreducible over $\overline{k}$ components E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that $E \not\subset \mathcal{Z}(e_0, \ldots, e_r)$ and $\dim \overline{p(E \setminus \mathcal{Z}(e_0, \ldots, e_r))} < n - i + \sigma - s$.

In what follows in the Introduction we assume that $W' \cap \mathcal{Z}(L'_0, L'_{\sigma+1}, \ldots, L'_n) = \emptyset$. Let us represent $W_i(L')$ and $W_i^{(\beta)}(L')$, $\beta = 1, 2, 3$ as the unions of pairwise distinct irreducible over $\overline{k}$ algebraic varieties:

$$W_i(L') = \bigcup_{j \in J_i(L')} W_j, \quad \sigma \le i \le n,$$
$$W_i^{(\beta)}(L') = \bigcup_{j \in J_i^{(\beta)}(L')} W_j, \quad \sigma \le i \le n+1,\ \beta = 1, 2, 3,$$

(actually $J_{\sigma+1}^{(2)}(L') = \emptyset$ and $W_{\sigma+1}^{(2)}(L') = \emptyset$). Denote by $W_i^{(1)}$ the union of all the irreducible components of dimension $n - i + \sigma - s$ of $W \cap \mathcal{Z}(e_0, \ldots, e_r)$. Let W_j, $j \in J_i^{(1)}$, be the family of all the irreducible components of $W_i^{(1)}$. We shall assume in what follows without loss of generality that $J_i^{(1)}(L') \subset J_i^{(1)}$.

Denote by $\iota_j(L')$, $j \in J_{i+1}(L')$ (respectively $\iota_j(L')$, $j \in J_{i+1}^{(\beta)}(L')$, $\beta = 1, 2, 3$), $\sigma \le i \le n$, the index of intersection $i(W_i(L'), \mathcal{Z}(e'_{i+1}); W_j)$ of the algebraic varieties $W_i(L')$ and $\mathcal{Z}(e'_{i+1})$ at W_j, see [10]. Hence by the Bézout theorem, see [10],

$$d' \deg W_i(L') = \sum_{j \in J_{i+1}(L')} \iota_j(L') \deg W_j + \sum_{1 \le \beta \le 3} \sum_{j \in J_i^{(\beta)}(L')} \iota_j(L') \deg W_j. \quad (6)$$

Set

$$D_i(L') = \deg W_i(L') = \sum_{j \in J_i(L')} \deg W_j, \quad \sigma \le i \le n+1,$$
$$D_i^{(\beta)}(L') = \deg W_i^{(\beta)}(L') = \sum_{j \in J_i^{(\beta)}(L')} \deg W_j, \quad \sigma \le i \le n+1,\ \beta = 1, 2, 3.$$

We shall denote also $D_\alpha(L') = D(L'_{\sigma+1}, \ldots, L'_\alpha)$, $J_\alpha(L') = J(L'_{\sigma+1}, \ldots, L'_\alpha)$, $W_\alpha(L') = W(L'_{\sigma+1}, \ldots, L'_\alpha)$ for $\sigma \le \alpha \le n+1$; and $W_\alpha^{(\beta)}(L') = W^{(\beta)}(L'_{\sigma+1}, \ldots, L'_\alpha)$, $D_\alpha^{(\beta)}(L') = D^{(\beta)}(L'_{\sigma+1}, \ldots, L'_\alpha)$, $J_\alpha^{(\beta)}(L') = J^{(\beta)}(L'_{\sigma+1}, \ldots, L'_\alpha)$; for $\sigma+1 \le \alpha \le n+1$, $\beta = 1, 2, 3$.

Set

$$D_i = D_i(U),\ J_i = J_i(U), \quad \sigma \le i \le n+1,$$
$$D_i^{(\beta)} = D_i^{(\beta)}(U),\ J_i^{(\beta)} = J_i^{(\beta)}(U), \quad \sigma \le i \le n+1,\ \beta = 1, 2, 3,$$
$$\iota_j = \iota_j(U), \quad j \in J_i^{(\beta)},\ \sigma \le i \le n+1,\ \beta = 1, 2, 3, \quad (7)$$

and $D_0 = D_{n+1}$, $D_0^{(\beta)} = D_{n+1}^{(\beta)}$, $J_0^{(\beta)} = J_{n+1}^{(\beta)}$, $\beta = 1, 2, 3$. The inequality $D_i(L') \le D_i$ holds for every $\sigma \le i \le n$, see Lemma 12. *If* $L' \in \mathcal{U}'_0$ *then by Lemma 11 from [9] the index of intersection* $\iota_j(L') = 1$ *for every* $j \in J_i(L')$, $\sigma + 1 \le i \le n$. In particular, $\iota_j = 1$ for every $j \in J_i$, $\sigma + 1 \le i \le n$. Further, if $L' \in \mathcal{U}'_0$ and $J_i^{(1)}(L') = J_i^{(1)}$ then $\iota_j(L') \ge \iota_j$ for every $j \in J_i^{(1)}$, see

Lemma 11 of the present paper (actually the last inequality is fulfilled under the weaker conditions from this lemma). Besides that, if additionally $i = n+1$ then $\iota_j(L') = \iota_j$, see Lemma 12.

Denote by W''' the closure in the Zariski topology of all points $z \in W'$ such that $\dim p^{-1}(z) > s - \sigma$. Denote by $W^{(4)}$ the closure in the Zariski topology of the union of all irreducible components E of $p^{-1}(W''')$ such that the dimension of closure $\dim \overline{p(E)} < \sigma - s + \dim E$. By the theorem about dimension of fibers, see [12] or above, for every $z \in W'''$ the dimension of every irreducible component (if it exists, $p^{-1}(z)$ may be empty) of $p^{-1}(z)$ is at least $\sigma - s + 1$. Hence the quasiprojective algebraic variety $W^{(4)} \setminus \mathcal{Z}(e_0, \ldots, e_r)$ is exactly the set of all points $x \in W \setminus \mathcal{Z}(e_0, \ldots, e_r)$ such that for every irreducible component E of the fiber $p^{-1}p(x)$ if $x \in E$ then $\dim E > \sigma - s$. Hence if $L' \in \mathcal{U}'_0$ then $W_i^{(3)}(L') \subset W^{(4)}$, $\sigma + 1 \le i \le n-1$.

Let $\sigma \le i \le n$ be an integer. We shall need also conditions (if the dimension here is negative then the variety is empty)

$$\begin{aligned} &\dim(\, W^{(4)} \setminus \mathcal{Z}(e_0, \ldots, e_r)\,) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i) \\ &\le n - i + \sigma - s - 1, \qquad (8) \\ &\dim W(L'_{\sigma+1}, \ldots, L'_i) \cap (\overline{V \setminus W}) \\ &\le n - i + \sigma - s - 1. \qquad (9) \end{aligned}$$

By Theorem 1 [9] if $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n) \in \mathcal{U}'_0$ then (9) holds (recall that (4), (5) are always fulfilled), $W_n(L')$ is the closure with respect to the Zariski topology of $W \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n) \setminus \mathcal{Z}(e_0, \ldots, e_r)$ and $D_n(L') = D_n = \deg p'_v \deg W'$. By Lemma 11 [9] if $L' \in \mathcal{U}'_0$ then $W_i(L')$ is exactly the union of all the irreducible over $\overline{k}$ components E of $W \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i)$ such that $E(\overline{k_v})$ contains at least one of $\deg p'_v \deg W'$ points from $W_v \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n) \setminus \mathcal{Z}(e_0, \ldots, e_r)$ (or, which is equivalent, E contains an irreducible component of $W_n(L')$).

Further, let $\sigma \le \alpha \le n$ be an integer and $L' \in \mathcal{U}'_0$. By Lemma 11 [9] condition (8) holds for all $\sigma \le i \le \alpha$ if and only if

$$W_i(L') = \overline{W \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i) \setminus \mathcal{Z}(e_0, \ldots, e_r)} \qquad (10)$$

for all $\sigma \le i \le \alpha$. In this case $W_i^{(3)}(L') = \emptyset$ for all $\sigma + 1 \le i \le \alpha$, and therefore, by (4) and (5) the algebraic variety $W_i^{(1)}(L')$, $\sigma+1 \le i \le \alpha$, coincides with the union of all the irreducible components of dimension $n - i + \sigma - s$ of $W \cap \mathcal{Z}(e_0, \ldots, e_r)$. Hence $W_i^{(1)}(L')$ and $J_i^{(1)}(L') = J_i^{(1)}$ do not depend on L' under the considered conditions.

One can decide whether $L' \in \mathcal{U}'_0$ by Lemma 25 Section 4 within the time polynomial in the size of input and $(dd')^n$, i.e., polynomial in the size of L' and in the values from the estimate for the working time of the algorithm of Theorem 2.

Let $\sigma \le \alpha \le n+1$ be an integer. Let us identify the set of all $(\alpha - \sigma)$–tuples $(L'_{\sigma+1}, \ldots, L'_\alpha)$ of linear forms $L'_j \in \overline{k}[X_0, \ldots, X_r]$ (respectively $L'_j \in \overline{k_v}[X_0, \ldots, X_r]$) with $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$ (respectively $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k_v})$). For convenience of notation for $\alpha = \sigma + 1$ we shall identify 1–tuples $(L'_{\sigma+1})$ with the linear form $L'_{\sigma+1}$. Recall that $L'_0 = L'_{n+1}$. *Denote by* $\mathcal{U}'_\alpha$, $\sigma \le \alpha \le n+1$, *the image of* $\mathcal{U}'_0$ *under the linear mapping* $\pi_\alpha : \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k}) \to \mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$,

$L' \mapsto (L'_{\sigma+1}, \dots, L'_\alpha)$. Then $\mathcal{U}'_\alpha$ is an open with respect to the Zariski topology set since the image under a linear projection of an open set is again open. Besides that, π_{n+1} is an isomorphism.

Let $\sigma \le \alpha \le n+1$ be an integer. Denote by $\mathcal{U}''_\alpha \subset \mathcal{U}'_\alpha$ the subset of all $(L'_{\sigma+1}, \dots, L'_\alpha)$ such that

$$D(L'_{\sigma+1}, \dots, L'_i) = D_i \tag{11}$$

for all $\sigma+1 \le i \le \alpha$. Set also $\mathcal{U}''_0 = \mathcal{U}'_0 \cap \pi_n^{-1}(\mathcal{U}''_n) = \pi_{n+1}^{-1}(\mathcal{U}''_{n+1})$. If $(L'_{\sigma+1}, \dots, L'_\alpha) \in \mathcal{U}''_\alpha$ then, see Lemma 12 Section 3, the indices of intersection $\iota_j(L'_{\sigma+1}, \dots, L'_i) = \iota_j$ for all $j \in J_i^{(1)}$, $\sigma+1 \le i \le \alpha$, and as we have noticed $\iota_j = 1$ for all $j \in J_i(L'_{\sigma+1}, \dots, L'_i)$, $\sigma+1 \le i \le \alpha$. Further, $\mathcal{U}''_\alpha$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$, see Lemma 13 Section 3. By Lemma 14 [9] applied to W_v, p'_v, α in place of W, p', $i+1$ if $D_i(L') = D_i$ for all $\sigma+1 \le i \le \alpha$ then $W_i^{(3)}(L') = \emptyset$ for all $\sigma+1 \le i \le \alpha$. Therefore, if $L' \in \mathcal{U}''_0$ then condition (8) and (10) hold for all $\sigma \le i \le n-1$.

If one extends the ground field then $U \in \mathcal{U}''$. Hence (6) implies

$$(d')^{n-\sigma+1} \deg W = \sum_{\sigma+1 \le i \le n+1} \Big(\sum_{j \in J_i^{(1)}} (d')^{n-i+1} \iota_j \deg W_j + \sum_{j \in J_i^{(2)}} (d')^{n-i+1} \iota_j \deg W_j \Big). \tag{12}$$

Let $\sigma \le \alpha \le n+1$ be an integer. Denote by $\mathcal{U}'''_\alpha \subset \mathcal{U}''_\alpha$ the subset of all $(L'_{\sigma+1}, \dots, L'_\alpha)$ such that

$$D^{(2)}(L'_{\sigma+1}, \dots, L'_i) = D_i^{(2)} \tag{13}$$

for all $\sigma+1 \le i \le \alpha$. Set also $\mathcal{U}'''_0 = \mathcal{U}''_0 \cap \pi_n^{-1}(\mathcal{U}'''_n)$, i.e., $\mathcal{U}'''_0$ is a set of all $L' \in \mathcal{U}''_0$ such that $(L'_{\sigma+1}, \dots, L'_n) \in \mathcal{U}'''_n$. Then $\mathcal{U}'''_\alpha$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$, see Lemma 16 Section 3. Further, $\pi_{n+1}(U) \in \mathcal{U}'''_{n+1}(\overline{K})$. Obviously $\pi_{n+1}(\mathcal{U}'''_0) \supset \mathcal{U}'''_{n+1}$, and if $s = \sigma$ then the last inclusion becomes the equality.

Denote for brevity $\mathcal{U}' = \mathcal{U}'_{\sigma+1}$ (respectively $\mathcal{U}'' = \mathcal{U}''_{\sigma+1}$, $\mathcal{U}''' = \mathcal{U}'''_{\sigma+1}$).

To formulate Theorem 2 without any restrictions on s and σ we need to consider also the case when W is a union of some defined over k and irreducible over k components of V (i.e., W is not necessarily irreducible over k) and $\dim W = n - s = 0$ (actually for $n = s$ Theorem 2 is trivial). In the considered case p, p', W', are defined as above. We have $W \cap \mathcal{Z}(e_0, \dots, e_r) = \emptyset$, $\dim W' = n - \sigma = 0$. Let $L' = (L'_0)$ where L'_0 is a linear form from $\overline{k}[X_0, \dots, X_r]$. The element $L' = (L'_0) \in \mathcal{U}'_0$, if and only if $W' \cap \mathcal{Z}(L'_0) = \emptyset$ in $\mathbb{P}^r(\overline{k})$. The morphism p' is dominant if and only if $L' \in \mathcal{U}'_0$. For arbitrary $L' \in \mathcal{U}'_0$ put $W_n(L') = W$; $W_{n+1}(L') = \emptyset$, $W_n^{(j)}(L') = \emptyset$, $W_{n+1}^{(j)}(L') = \emptyset$ for $j = 1, 2, 3$. Set $\mathcal{U}'''_0 = \mathcal{U}''_0 = \mathcal{U}'_0 = \mathcal{U}'''_{n+1} = \mathcal{U}''_{n+1} = \mathcal{U}'_{n+1}$. All the other objects defined above for irreducible W are defined in the considered case in the similar way.

REMARK 1 *The Bézout theorem implies that if $L' \in \mathcal{U}'_0$, $J_i^{(2)}(L') = \emptyset$ for all $s+2 \le i \le n$ and condition (10) holds for all $\sigma+1 \le i \le n-1$ then for every $\sigma+1 \le i \le n+1$ the degree $D_i(L') = D_i$ and $\iota_j(L') = \iota_j$ for every $j \in J_i^{(1)}$.*

REMARK 2 *Let K_1 be an algebraically closed field and $K_1 \supset k$ linearly disjoint with $k(W)$ over k. Let us choose an extension $K_2 \supset k$ such that W is irreducible over K_2 and the algebraic closure $\overline{K_2} = K_1$ (e.g., K_2 may be an arbitrary purely transcendental an extension of k such that $\overline{K} = K_1$). Let us define the sets $\mathcal{U}'_\alpha$, $\mathcal{U}''_\alpha$, $\mathcal{U}'''_\alpha$ for the ground field K_1 (and also K_2) replacing in the definitions of these sets the ground field k by K_2. Denote for brevity the obtained sets by $\mathcal{U}'_{K_1,\alpha}$, $\mathcal{U}''_{K_1,\alpha}$, $\mathcal{U}'''_{K_1,\alpha}$ (we use this more detailed notation only in this remark). Then obviously $\mathcal{U}'_{K_1,\alpha}$, $\mathcal{U}''_{K_1,\alpha}$, $\mathcal{U}'''_{K_1,\alpha}$ do not depend on the choice of the field K_2 such as above. In particular, the initial $\mathcal{U}'_\alpha$, $\mathcal{U}''_\alpha$, $\mathcal{U}'''_\alpha$ defined for the ground field k depend only on $\overline{k}$. Further, obviously $\mathcal{U}'_\alpha = \mathcal{U}'_{K_1,\alpha}(\overline{k})$, $\mathcal{U}''_\alpha = \mathcal{U}''_{K_1,\alpha}(\overline{k})$, $\mathcal{U}'''_\alpha = \mathcal{U}'''_{K_1,\alpha}(\overline{k})$ for all α. On the other hand, probably also $\mathcal{U}'_\alpha(K_1) = \mathcal{U}'_{K_1,\alpha}$, $\mathcal{U}''_\alpha(K_1) = \mathcal{U}''_{K_1,\alpha}$, $\mathcal{U}'''_\alpha(K_1) = \mathcal{U}'''_{K_1,\alpha}$. We leave to prove the last three equalities to the reader. We shall not need them in this paper. Similarly the reader may prove that if V and W are defined over a subfield k' of k (the extension $k \supset k'$ may be transcendental) then $\mathcal{U}'_\alpha$, $\mathcal{U}''_\alpha$, $\mathcal{U}'''_\alpha$ are also defined over k'.*

Recall some details from [8]. Let $\overline{\lambda} = (L_0, \dots, L_{n+1})$, and all $L_j \in k[X_0, \dots, X_n]$, $0 \le j \le n+1$, are linear forms. Suppose that $\overline{\lambda}$ (i.e., linear forms $L_0, \dots, L_{n+1}$) satisfy assertion of Theorem 1 [8] for the algebraic variety V. Then, see Section 2 [8], for every $1 \le j \le n-2$ such that V has an irreducible component of dimension $n-j$ the equality $v_j(\overline{\lambda}) = u_j(\overline{\lambda})$ holds, where $v_j(\overline{\lambda}), u_j(\overline{\lambda}) \le d^j(d^j - 1)$ are integers defined in [8]. But now the notations v_j and u_j are already used in the other sense. So we shall denote in this paper these integers by $b_j(\overline{\lambda}), a_j(\overline{\lambda})$ in place of $v_j(\overline{\lambda}), u_j(\overline{\lambda})$ from [8] respectively. More precisely, in Section 1 for arbitrary $\overline{\lambda}$ and V satisfying property (B), see the beginning of Section 1, we define the integers $b_j(\overline{\lambda}) = b_j(\overline{\lambda}, V)$, $a_j(\overline{\lambda}) = a_j(\overline{\lambda}, V)$ for all $0 \le j \le n$.

In [8] we did not construct $\overline{\lambda}$ in general position with maximal possible $b_j(\overline{\lambda})$, see Remark 3 [8]. We applied there Remark 2 [8] as a weak reduction to general position. In this paper it is also possible but it makes the presentation of results slightly awkward. So we manage to avoid the application of Remark 2 [8]. Instead of it we strengthen the result of [8] and construct $\overline{\lambda}$ in general position with maximal possible $b_j(\overline{\lambda})$ in see Section 1 using the elaborated method, see the beginning of the Introduction. So it is sufficient now to apply the simple algorithm of reduction to integer coefficients, see Proposition 2, [9].

Let $\lambda = (L_{s+1}, \dots, L_\sigma)$ where all $L_j \in k[X_0, \dots, X_n]$, $s+1 \le j \le \sigma$, are linear forms. Consider the following property

(A) For λ an element $\overline{\lambda}$ is given (so linear forms L_w, $s+1 \le w \le \sigma$, of λ and $\overline{\lambda}$ coincide); Theorem 1 [8] holds for the algebraic variety V and the linear forms $L_0, \dots, L_{n+1}$; and for every $0 \le j \le n$ such that V has an irreducible component of dimension $n-j$ the integer $b_j(\overline{\lambda}) = a_j(\overline{\lambda})$ is maximal possible (if one varies $\overline{\lambda}$, see Section 1 for details).

In this case we shall say for brevity that λ satisfies condition (A) for V, or that (A) holds for λ and V.

We shall suppose in what follows unless we state otherwise that the dimension $\dim W \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma) = n - \sigma$; the last intersection is transversal; if $n - \sigma > 0$ then the algebraic variety $W \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma)$ is irreducible over k. Denote $W_\lambda = W \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma)$, $V_\lambda = V \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma)$. We shall

assume in what follows that $\dim(W_\lambda \cap (\overline{V_\lambda \setminus W_\lambda}) < \dim W_\lambda$. Notice that if (A) holds for λ then all these properties are fulfilled. Denote by

$$p_\lambda \,:\, W_\lambda \setminus \mathcal{Z}(e_0,\dots,e_r) \to \mathbb{P}^r(\overline{k}), \tag{14}$$

$$p'_\lambda \,:\, W_\lambda \setminus \mathcal{Z}(e_0,\dots,e_r) \to W' \tag{15}$$

the morphisms induced by p and p'. Suppose that

$$\dim W_\lambda \cap \mathcal{Z}(e_0,\dots,e_r) < \dim W_\lambda \tag{16}$$

and the morphism p'_λ is dominant. Notice that if $n-\sigma > 0$ then W_λ is irreducible over k by our assumption. Hence if $n-\sigma > 0$ and the morphism p'_λ is dominant then condition (16) holds. Let us replace (V, W, p, p') by $(V_\lambda, W_\lambda, p_\lambda, p'_\lambda)$ in the definitions of all $W_i(L')$, $W(L'_{\sigma+1},\dots,L_n)$, $\mathcal{U}'_\alpha$, $\mathcal{U}''_\alpha$, $\mathcal{U}'''_\alpha$, $p_{L'}$ and so on. We shall denote the obtained objects by $W_{\lambda,i}(L')$, $W_\lambda(L'_{\sigma+1},\dots,L_n)$, $\mathcal{U}'_{\lambda,\alpha}$, $\mathcal{U}''_{\lambda,\alpha}$, $\mathcal{U}'''_{\lambda,\alpha}$, $p_{\lambda,L'}$ and so on. Notice that the field $k_{v_\lambda} = k$ and $p_{\lambda,v_\lambda,L'} = p_{\lambda,L'}$ since $s_\lambda = \sigma_\lambda = \sigma$. It will be convenient for us to consider the morphism p'_λ (set $q_\lambda = p'_\lambda$) together with p'_v, see Theorem 2 below. Let us define V_v, W_v and so on extending the ground field k till k_v. Notice that condition (A) holds for v and V if one extend k till $k_{\overline{v}}$.

Giving these definitions we proceed to the precise statements. Let homogeneous polynomials $f_1,\dots,f_m \in k[X_0,\dots,X_n]$ be given, $n \ge 0$, $m \ge 1$. Consider the closed algebraic set or algebraic variety (in this paper we do not distinguish these two concepts)

$$V = \{(x_0 : \dots : x_n) :\, f_i(x_0,\dots,x_n) = 0,\ x_i \in \overline{k},\ \forall 1 \le i \le m\} \subset \mathbb{P}^n(\overline{k})\,.$$

This is a set of all common zeros of polynomials $f_1,\dots,f_m$ in $\mathbb{P}^n(\overline{k})$, where $\overline{k}$ is an algebraic closure of k. Hence $V = \mathcal{Z}(f_1,\dots,f_m)$. Let $e_0,\dots,e_r \in k[X_0,\dots,X_n]$ be homogeneous polynomials such that the degree $\deg_{X_0,\dots,X_n} e_j = d'$ if $e_j \ne 0$ for every $0 \le j \le r$.

Now let the field $k = \mathbb{Q}(t_1,\dots,t_l,\theta)$ where $t_1,\dots,t_l$ are algebraically independent over the field $\mathbb{Q}$ and θ is algebraic over $\mathbb{Q}(t_1,\dots,t_l)$ with the minimal polynomial $F \in \mathbb{Q}[t_1,\dots,t_l,Z]$ and leading coefficient $\mathrm{lc}_Z F$ of F is equal to 1. We shall represent each polynomial $f = f_i$ (and similarly e_j) in the form

$$f = \frac{1}{a_0} \sum_{i_0,\dots,i_n} \sum_{0 \le j < \deg_Z F} a_{i_0,\dots,i_n,j} \theta^j X_0^{i_0} \cdots X_n^{i_n}\,,$$

where $a_0, a_{i_0,\dots,i_n,j} \in \mathbb{Z}[t_1,\dots,t_l]$, $\mathrm{G\,C\,D}\,_{i_0,\dots,i_n,j}(a_0, a_{i_0,\dots,i_n,j}) = 1$. Define the length $\mathrm{l}(a)$ of an integer a by the formula $\mathrm{l}(a) = \min\{s \in \mathbb{Z} :\, |a| < 2^{s-1}\}$. The length of coefficients $\mathrm{l}(f)$ of the polynomial f is defined to be the maximum of lengths of coefficients from $\mathbb{Z}$ of polynomials $a_0, a_{i_0,\dots,i_n,j}$ and the degree

$$\deg_{t_\gamma}(f) = \max_{i_0,\dots,i_n,j} \{\deg_{t_\gamma}(a_0), \deg_{t_\gamma}(a_{i_0,\dots,i_n,j})\}\,,$$

where $1 \le \gamma \le l$. In the similar way we shall define degrees and lengths of integer coefficients of other polynomials, in particular $\deg_{t_\gamma} F$ and $\mathrm{l}(F)$ are defined. We shall suppose that we have the following bounds

$$\begin{aligned}
&\deg_{X_0,\dots,X_n}(f_i) < d,\ \deg_{t_\gamma}(f_i) < d_2,\ \mathrm{l}(f_i) < M,\\
&\deg_{X_0,\dots,X_n}(e_j) = d' \text{ or } e_j = 0,\ \deg_{t_\gamma}(e_j) < d'_2,\ \mathrm{l}(e_j) < M',\\
&\deg_Z(F) < d_1,\ \deg_{t_\gamma}(F) < d_1,\ \mathrm{l}(F) < M_1\,.
\end{aligned}$$

for all $1 \leq i \leq m$, $0 \leq j \leq r$, $1 \leq \gamma \leq l$. The size $\mathrm{L}(f)$ of the polynomial f is defined to be the product of $\mathrm{l}(f)$ to the number of all the coefficients from $\mathbb{Z}$ of f in the dense representation. We have

$$\mathrm{L}(f_i) < (\binom{d+n}{n} d_1 + 1) d_2^l M$$

Similarly $\mathrm{L}(F) < d_1^{l+1} M_1$.

REMARK 3 *Unless we state otherwise, in what follows we suppose l to be fixed. The working time of the algorithm from Theorem 2, see below, is essentially the same as for solving systems of polynomial equations with a finite set of solutions in the projective space. So this theorem can be formulated also in the case when l is not fixed, see [2]. Notice that the constants $O(\ldots)$, see Theorem 2 and lemmas below, in the estimate of the lengths of integer coefficients of linear forms L'_j, L_j are absolute; they does not depend on l.*

We shall represent a point $z \in V$ with coordinates from a finite extension of k as follows. An index $0 \leq i_0 \leq n$ is known such that $X_{i_0}(z) \neq 0$ and an isomorphism of fields

$$k(z) = k((X_1/X_{i_0})(z), \ldots, (X_n/X_{i_0})(z)) = k[\eta] \simeq k[Z]/(\Phi) \qquad (17)$$

is given where $\eta = \sum_{0 \leq i \leq n} c_i (X_i/X_{i_0})(z)$, the coefficients $c_i \in \mathbb{Z}$ are given and $\Phi \in k[Z]$ is minimal polynomial of η over k with leading coefficient $\mathrm{lc}_Z \Phi = 1$ (so the point z is defined up to a conjugation over k).

Let $g \in k[\eta]$ be an arbitrary element. Then $g = G(\eta)$ for the uniquely defined polynomial $A \in k[Z]$ such that $\deg_Z G < \deg_Z \Phi$. The length of integer coefficients $\mathrm{l}(g)$, the size $\mathrm{L}(g)$ and the degrees $\deg_{t_\alpha} g$, $1 \leq \alpha \leq l$, of g are defined by the formulas

$$\mathrm{l}(g) = \mathrm{l}(G), \quad \mathrm{L}(g) = \mathrm{L}(G), \quad \deg_{t_\alpha} g = \deg_{t_\alpha} G.$$

Let us define the size of the point z to be $\mathrm{L}(\Phi) + \sum_{0 \leq i \leq n} \mathrm{L}(X_i/X_{i_0})$.

Let $W_1, W_2 \subset \mathbb{P}^n(\overline{k})$ be two quasiprojective algebraic varieties defined over k. Let W_3 be an arbitrary defined over k and irreducible over k component of $W_1 \cap W_2$. We shall say that the intersection $W_1 \cap W_2$ is transversal at W_3 if and only if there is a point $x \in W_3$ such that x is a smooth point of every W_i, $i = 1, 2$, and the intersection of the tangent spaces $T_{x,W_1} \cap T_{x,W_2}$ of W_1 and W_2 at the point x is transversal, i.e., $\dim(T_{x,W_1} \cap T_{x,W_2}) = n - \dim(T_{x,W_1}) - \dim(T_{x,W_2})$. We shall say that the intersection $W_1 \cap W_2$ is transversal if and only if it is transversal at every its defined over k and irreducible over k component. In the similar way one defines transversality for intersection of a finite number of algebraic varieties.

By Lemma 21 if the intersection $W_1 \cap W_2$ is transversal and y is an arbitrary smooth point of $W_1 \cap W_2$ then y is a smooth point of every W_i, $i = 1, 2$ and the intersection of the tangent spaces $T_{y,W_1} \cap T_{y,W_2}$ of W_1 and W_2 at the point y is transversal.

THEOREM 1 *Let the field k, the algebraic varieties V, W, W', and polynomials F, $f_1, \ldots, f_m$, $e_0, \ldots, e_r$, and the point $x^{(0)} \in W$ be given as above. One can compute the dimensions $\dim W = n - s$, $\dim W' = n - \sigma$ (hence $n \geq \sigma \geq s \geq 0$).*

Let $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n) \in \mathcal{U}'_0$ be given. Suppose that all the linear forms L'_j, $j \in \{0, \sigma+1, \ldots, n\}$ have coefficients from a finitely generated extension k' of k which is given by the minimal polynomial of a primitive element of k' over a purely trancendental extension of k of fixed transcendency degree. In the statement of the theorem we assume that all the points y', y'', x, see below, are given similarly to (17) but with the field k' in place of k. Set $L'_{n+1} = L'_0$, $e'_j = L'_j(e_0, \ldots, e_r)$ for all j. Put $\widetilde{V}_i = (V \cup \mathcal{Z}(e'_{i+1})) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i)$ and $\iota = i - \sigma + s$ for all $\sigma \le i \le n$. Then one can construct an element $\overline{\lambda} = (L_0, L_1, \ldots, L_{n+1})$, where $L_0, L_1, \ldots, L_{n+1}$ are linear forms in $X_0, \ldots, X_n$ with integer coefficients of length $O(n \log d + (n-\sigma) \log d')$ such that the following assertions (a) and (b) hold.

- *(a) For every $\sigma \le i \le n$ the intersection of the algebraic varieties $\widetilde{V}_i$ and $\mathcal{Z}(L_{\iota+1}, \ldots, L_n)$ is transversal, $W_i(L') \cap \mathcal{Z}(L_0, L_{\iota+1}, \ldots, L_n) = \emptyset$, and the number of elements $\#(L_{n+1}/L_0)(W_i(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)) = D_i(L')$ in $\mathbb{P}^n(\overline{k})$. Further, for every $\sigma \le i \le n$ the assertion of Theorem 3, see Section 1, holds for $\widetilde{V}_i$ in place of V. In particular, one can construct all the points from $W_i(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$ for every $\sigma \le i \le n$.*

- *(b) For every $\sigma + 1 \le i \le n+1$ the intersection of the algebraic varieties $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ and $\mathcal{Z}(L_{\iota+1}, \ldots, L_n)$ is transversal, $W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_0, L_{\iota+1}, \ldots, L_n) = \emptyset$, and $\#(L_{n+1}/L_0)(W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)) = D_i(L') + D_i^{(1)}(L') + D_i^{(2)}(L') + D_i^{(3)}(L')$ in $\mathbb{P}^n(\overline{k})$. For every $\sigma+1 \le i \le n+1$ the assertion of Theorem 3, see Section 1, holds for $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ in place of V (notice that here $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ is not given explicitly by a system of polynomial equations). In particular, one can construct the finite set $W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$, and for every irreducible over $\overline{k}$ component E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ the intersection $E \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_{n-1})$ is irreducible over $\overline{k}$. Further, for any two points $y', y'' \in W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$ one can decide whether y' and y'' belong to the same defined over k' and irreducible over k' (respectively defined over $\overline{k}$ and irreducible over $\overline{k}$) component of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ (in the case of irreducible over $\overline{k}$ components we assume that the composite $k'(y', y'')$ of the fields $k'(y')$ and $k'(y'')$ over k' is given explicitly similarly to Theorem 2 [8], and the bound for the size of the composite holds, which polynomial in the size of input, cf. [8]). Finally, for every $y' \in W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$ one can compute the number of points $y'' \in W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$ such that y' and y'' belong to the same defined over k' and irreducible over k' (respectively defined over $\overline{k}$ and irreducible over $\overline{k}$) component of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$.*

Let $\overline{\lambda}$ be an arbitrary element satisfying (a) and (b) (not necessarily constructed by the algorithm) and $\sigma + 1 \le i \le n+1$ be an integer. Then the following assertions holds.

- *(c) Let a point $y' \in W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$ be given. Denote by E the unique defined over k' and irreducible over k' (respectively defined over $\overline{k}$ and irreducible over $\overline{k}$) component of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that $y' \in W_{i-1}(L') \cap \mathcal{Z}(e'_i)$. Then for an arbitrary point $x \in \mathbb{P}^n(\overline{k})$ one can decide whether $x \in E$, and, more generally one can compute the multiplicity $\mu(x, E)$ of the point x at E (again if E is irreducible over $\overline{k}$ then we*

assume that the composite $k'(x,y')$ of the fields $k'(x)$ and $k'(y')$ over k is given explicitly similarly to (b)).

(d) For a given point $y' \in W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1},\ldots,L_n)$ one can decide whether y' is an element of $W_i(L')$ (respectively $W_i^{(1)}(L')$, $W_i^{(2)}(L')$, $W_i^{(3)}(L')$). Hence by (b) and (c) one can compute the degrees $\deg W_i(L')$, $\deg W_i^{(1)}(L')$, $\deg W_i^{(2)}(L')$, $\deg W_i^{(3)}(L')$. Besides that, for every irreducible over $\overline{k}$ component E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ we have $E \subset \mathcal{Z}(e_0,\ldots,e_r)$ if and only if $E \cap \mathcal{Z}(L_{\iota+1},\ldots,L_n) \subset \mathcal{Z}(e_0,\ldots,e_r)$.

(e) By (b), (c), (d) for an arbitrary point $x \in \mathbb{P}^n(\overline{k})$ one can compute the multiplicities $\mu(x, W_i(L'))$ (respectively $\mu(x, W_i^{(1)}(L'))$, $\mu(x, W_i^{(2)}(L'))$, $\mu(x, W_i^{(3)}(L'))$) of the point x at $W_i(L')$ (respectively $W_i^{(1)}(L')$, $W_i^{(2)}(L')$, $W_i^{(3)}(L')$).

(f) Under conditions of (c) suppose that E is an defined over $\overline{k'}$ and irreducible over $\overline{k'}$ component. Then one can compute the index of intersection $i(W_{i-1}(L'), \mathcal{Z}(e'_i); E)$ of the algebraic varieties $W_{i-1}(L')$ and $\mathcal{Z}(e'_i)$ at the component E in $\mathbb{P}^n(\overline{k})$.

(g) Let us identify the set of all $(n+2)$–tuples of linear forms from $\overline{k}[X_0,\ldots,X_n]$ with the affine space $\mathbb{A}^{(n+1)(n+2)}(\overline{k})$. Let $\mathfrak{l} \in \mathbb{A}^{(n+1)(n+2)}(\overline{k})$ be a line defined over k' such that $\overline{\lambda} \in \mathfrak{l}$. Then for all $\lambda^* \in \mathfrak{l}(k')$ (here $\mathfrak{l}(k')$ is the set of k'–points of $\mathfrak{l}$; we assume that $\lambda^* = (L_0^*,\ldots,L_{n+1}^*)$, where all $L_j^* \in \overline{k}[X_0,\ldots,X_n]$ are linear forms), except at most a polynomial in $d^n(d')^{n-\sigma}$ number, assertions (a) and (b) hold with λ^* in place of $\overline{\lambda}$. For every element $\lambda^* \in \mathfrak{l}(k')$ one can decide whether assertions (a) and (b) hold with λ^* in place of $\overline{\lambda}$.

The working time of the algorithm for computing s, σ and constructing $\overline{\lambda}$ (respectively of the algorithm from (a), (b), (d), (f)) is polynomial in d^n, $(d')^n$, d_1, d_2, d'_2, M, M', M_1, m, r, the size $\mathrm{L}(x^{(0)})$ of the point $x^{(0)}$, the size $\mathrm{L}(L')$ of the element L' (including the size of the minimal polynomial of the primitive element of k', see above). The working times of the algorithms from (c) and (e) (respectively (g)) are polynomial in the same values and in the size $\mathrm{L}(x)$ of the point x (respectively in the same values and in the size $\mathrm{L}(\lambda^*)$ of the element λ^*).

Notice here that the difficulty in the proof of assertion (b) of Theorem 1 in comparison with its assertion (a) (and Theorem 3) is that some irreducible components of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ may be contained in $\overline{V \setminus W}$.

THEOREM 2 *Let the field k of zero–characteristic, the algebraic variety V, the polynomials F, $f_1,\ldots,f_m$ and $e_0,\ldots,e_r$, $r \geq 2$, be given as above. Let a point $x^{(0)} \in V$ be given and $x^{(0)}$ belongs to only one irreducible over k component of V. Denote this component by W. Let W' be as above, see (1). Then one can compute the dimensions $\dim W$, $\dim W'$. Suppose that $\dim W = n - s$ and $\dim W' = n - \sigma$. Hence the open in the Zariski topology subset $\mathcal{U}_0''' \subset \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$ is defined, see above.*

(a) Then one can construct an element $L' = (L'_0, L'_{\sigma+1},\ldots,L'_n) \in \mathcal{U}_0'''$ such that linear forms L'_j have integer coefficients with lengths $O(n\log d + (n -$

$\sigma)\log d')$ *for all* $j \in \{0, \sigma+1, \ldots, n\}$. *Hence conditions (4), (5),(8)–(13) hold for* $\sigma+1 \le i \le n$. *One can compute the degrees* D_i, $D_i^{(1)}$, $D_i^{(2)}$, $\sigma \le i \le n$.

(b) *For a given* $L' \in \mathcal{U}_0'''$ *satisfying the assertion from (a) (but not necessarily computed in (a); one can verify that* $L' \in \mathcal{U}_0'''$ *by remarks given at the end of the Introduction) one can construct* $\lambda = (L_{s+1}, \ldots, L_\sigma)$, *where all* $L_j \in k[X_0, \ldots, X_n]$ *are linear forms with integer coefficients of length* $O(n\log d + (n-\sigma)\log d')$ *such that the following statements (c) and (d) hold.*

(c) λ *satisfies conditions (A) and (16), see above; the morphism* p'_λ, *see (15), is dominant; the degree* $\deg p'_\lambda = \deg p'_v$; $D_{\lambda,i} = D_i$, $D_{\lambda,i}^{(1)} = D_i^{(1)}$, $D_{\lambda,i}^{(2)} = D_i^{(2)}$ *for all* $\sigma \le i \le n$.

(d) $L' \in \mathcal{U}'''_{\lambda,0} \subset \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$.

(e) *Let* λ *and* L' *satisfy assertions (c) and (d). Then* $L' \in \mathcal{U}_0'''$. *Further,* $W_{\lambda,i}(L') = W_i(L') \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$, $W_{\lambda,i}^{(1)} = W_i^{(1)} \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$, $W_{\lambda,i}^{(2)}(L') = W_i^{(2)}(L') \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ *for all* $\sigma \le i \le n$ *and all these intersections are transversal. Therefore,* $D_{\lambda,i} = D_{\lambda,i}(L') = D_i(L') = D_i$, $D_{\lambda,i}^{(1)} = D_i^{(1)}$, $D_{\lambda,i}^{(2)} = D_{\lambda,i}^{(2)}(L') = D_i^{(2)}(L') = D_i^{(2)}$ *for all* $\sigma \le i \le n$.

(f) *Let* $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n) \in \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$ *be an arbitrary element (i.e., not necessarily the one constructed by the algorithm from this theorem), where all linear forms* $L'_j \in \overline{k}[X_0, \ldots, X_r]$. *Let* $\lambda = (L_{s+1}, \ldots, L_\sigma) \in \mathbb{A}^{(n+1)(\sigma-s)}(k)$ *be an arbitrary element (not necessarily constructed in (b)), where all* $L_j \in k[X_0, \ldots, X_n]$. *Set* $\nu_0 = (r+1)(n-\sigma+1) + (n+1)(\sigma-s)$. *Consider the pair* (λ, L') *as an element of* $\mathbb{A}^{\nu_0}(\overline{k})$. *Suppose that assertions (c) and (d) hold for* (λ, L'). *Let us identify* $\mathbb{A}^1(\overline{k}) = \overline{k}$ *using a coordinate function on* $\mathbb{A}^1(\overline{k})$ *defined over* k. *Let* $\iota_0 : \mathbb{A}^1(\overline{k}) \to \mathbb{A}^{\nu_0}(\overline{k})$ *be an affine linear embedding. For every* $y \in \overline{k}$ *let us represent* $\iota_0(y) = (\lambda_0, L^{(0)})$ *where* $\lambda_0 = (L_{0,s+1}, \ldots, L_{0,\sigma})$, $L^{(0)} = (L_0^{(0)}, L_{\sigma+1}^{(0)}, \ldots, L_n^{(0)})$, *all* $L_{0,j} \in \overline{k}[X_0, \ldots, X_n]$, $L_w^{(0)} \in \overline{k}[X_0, \ldots, X_r]$ *are linear forms. Assume that for every* $y \in k$ *all* $L_{0,j}$ *have coefficients from* k. *Suppose that* $\iota_0(0) = (\lambda, L')$. *Then for all* $y \in k$, *except at most a polynomial in* $d^n(d')^{n-\sigma}$ *number,* $\iota_0(y) = (\lambda_0, L^{(0)})$ *satisfies assertions (c) and (d) in place of* (λ, L'). *Besides that, see condition (A), for* $\alpha \in \{0, \ldots, s, \sigma+1, \ldots, n+1\}$ *the linear forms* L_α *of* $\overline{\lambda}$ *and* $L_{0,\alpha}$ *of* $\overline{\lambda_0}$ *coincide.*

The working time of the algorithm from (a) is polynomial in d^n, $(d')^n$, d_1, d_2, d'_2, M, M', M_1, m, r, *and the size* $\mathrm{L}(x^{(0)})$ *of the point* $x^{(0)}$. *The working time of the algorithm from (b) is polynomial in the same values and the size* $\mathrm{L}(L')$ *of* L'.

This theorem will be proved in Section 7. Notice that if one replaces k by k_v then v (in place of λ) satisfies assertions (c) and (d). For more detailed information see the algorithms from Section 4 and Section 5. Namely, for a given element $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n) \in \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$, where all $L'_j \in k[X_0, \ldots, X_n]$ one can decide whether $L' \in \mathcal{U}'_0$ using Lemma 25. Using the algorithms from

Theorem 1 one can check whether $D_i(L') = D_i$, $D_i^{(2)}(L') = D_i^{(2)}$, $\sigma \le i \le n$, and hence to decide whether $L' \in \mathcal{U}_0''$ and $L' \in \mathcal{U}_0'''$. For more details related to assertion (f) see also Lemma 19, Lemma 18, Lemma 20.

REMARK 4 *Actually in Section 6 we construct an element $L' \in \mathcal{U}_0'''$ from assertion (a) of Theorem 2 such that $\pi_{n+1}(L') \in \mathcal{U}_{n+1}'''$.*

REMARK 5 *In what follows in many cases, when one says that $a_1, \dots, a_\mu$ (here may be any other family of variables or coefficients of some polynomials, may be only one variable) are infinitesimals, par abus de langage it is convenient to understand that every $a_1, \dots, a_\mu$ is an infinitesimals or zero. Usually we state explicitly that $a_1, \dots, a_\mu$ can be equal to zero. But even if we forgot to state this explicitly somewhere, the reader will see it easily from the context. This will not lead to an ambiguity. For example, par abus de langage it is more convenient to say that all the coefficients of a linear form $\widetilde{L}_0$ are infinitesimals, but still we usually say that all the nonzero coefficients of a linear form $\widetilde{L}_0$ are infinitesimals in this case.*

1 Constructing linear sections in general position of an algebraic variety

Recall some notations from [8]. Let $V \subset \mathbb{P}^n(\overline{k})$ be an arbitrary projective algebraic variety (not necessarily the one from the Introduction). For every $0 \le j \le n$ denote by V_j the union of all the irreducible components of V of dimension j. Let $L_0, L_1, \dots, L_{n+1} \in k[X_0, \dots, X_n]$ be linear forms, $L_w = \sum_{0 \le i \le n} l_{w,i} X_i$, where all $l_{w,i} \in k$, $0 \le w \le n+1$, $0 \le i \le n$. Put $\overline{\lambda} = (L_0, L_1, \dots, L_{n+1})$. Denote by $\widetilde{\lambda}$ the family of coefficients $l_{w,i}$, $0 \le w \le n+1$, $0 \le i \le n$. Consider the following condition.

(B) For every $0 \le j \le n$ the set $A_j = V_j \cap \mathcal{Z}(L_{j+1}, \dots, L_n)$ is finite (or empty if $V_j = \emptyset$) in $\mathbb{P}^n(\overline{k})$, consists of smooth points of V, and the intersections of tangent spaces of V_j and $\mathcal{Z}(L_{j+1}, \dots, L_n)$ at all these points are transversal. The linear form L_0 does not vanish at any point of A_j (i.e., $A_j \cap \mathcal{Z}(L_0) = \emptyset$), and the number of elements $\#(L_{n+1}/L_0)(A_j) = \#A_j = \deg V_j$.

If V is from the Introduction then using the algorithms from [4] and [2] for given linear forms $L_0, \dots, L_n$ (or which is the same for $\overline{\lambda}$) one can decide whether condition (B) holds within the time polynomial in d^n, d_1, d_2, M, M_1, m, and the sum of sizes $\sum_{0 \le j \le n+1} \mathrm{L}(L_j)$ of linear forms L_j.

In what follows we shall consider also property (B) of linear forms with coefficients from an extension of k replacing the ground field k by its extension. In particular, the integers $a_j(\overline{\lambda})$, $b_j(\overline{\lambda})$, see below, are also defined if the coefficients of the linear forms L_j are from an extension of k and property (B) holds over the algebraic closure of this extension.

Suppose that $\overline{\lambda}$ satisfies condition (B). Let $Z_0, \dots, Z_{n+1}$ be independent variables. For every $0 \le j \le n$ denote by $\Phi_j \in k[Z_0, Z_{j+1}, Z_{j+2}, \dots, Z_{n+1}] = B_j$ the nonzero separable polynomial of minimal degree such that $\Phi_j(L_0, L_{j+1}, L_{j+2}, \dots, L_{n+1})$ is vanishing on V_j. Then $\deg_{Z_{n+1}} \Phi_j = \deg \Phi_j = \deg V_j$. Unless we state

otherwise, we choose and fix Φ_j such that the leading coefficient of Φ_j with respect to Z_{n+1} is equal to 1. Denote by $R_j \in k[Z_0, Z_{j+1}, Z_{j+2}, \ldots, Z_n]$ the discriminant of Φ_j with respect to of Z_{n+1} and by $R_{j,0}$ square free part of R_j (i.e., $R_{j,0}$ is a separable polynomial of the most degree dividing R_j). Put $\Phi_{j,*} = \Phi_j(Z_0, 0, \ldots, 0, Z_n, Z_{n+1}) \in k[Z_0, Z_n, Z_{n+1}]$, $R_{j,*} = R_j(Z_0, 0, \ldots, 0, Z_n) \in k[Z_0, Z_n]$, $R_{j,0,*} = R_{j,0}(Z_0, 0, \ldots, 0, Z_n) \in k[Z_0, Z_n]$. If $V_j = \emptyset$ then $\Phi_j = R_j = 1$. If $j = n$ then Φ_j, R_j, $\Phi_{j,*}$, $R_{j,*}$ do not depend on Z_n. Denote by $R_{j,*,0}$ square free part of $R_{j,*}$. If the dependence on $\overline{\lambda}$ is essential then we shall denote the introduced objects by $\Phi_{\overline{\lambda},j}$, $R_{\overline{\lambda},j}$, $R_{\overline{\lambda},j,0}$, $\Phi_{\overline{\lambda},j,*}$, $R_{\overline{\lambda},j,*}$, $R_{\overline{\lambda},j,0,*}$ and $R_{\overline{\lambda},j,*,0}$. In particular extending the field k till $k_{\overline{v}}$, see the Introduction, we get $\Phi_{\overline{v},j}$, $R_{\overline{v},j}$, $R_{\overline{v},j,0}$, $\Phi_{\overline{v},j,*}$, $R_{\overline{v},j,*}$, $R_{\overline{v},j,0,*}$ and $R_{\overline{v},j,*,0}$. Besides that, we choose the last elements to be the separable polynomials from the ring $k[\widetilde{v}, Z_0, \ldots, Z_{n+1}]$.

LEMMA 1 *Under previous conditions for every $0 \le j \le n$ the polynomial $\Phi_{\overline{v},j}(\widetilde{\lambda}, Z_0, Z_{j+1}, Z_{j+2}, \ldots, Z_{n+1})$ is nonzero and coincides with $\Phi_{\lambda,j}$ up to a nonzero from k.*

PROOF Let $\nu \in k$. Then $v_0, v_{s+1}, \ldots, v_n, v_{n+1} + \nu v_0$ are generic linear forms in $X_0, \ldots, X_n$ over k. More generally, each linear automorphism γ of $k[v_0, v_{j+1}, \ldots, v_{n+1}]$ over k induces the uniquely defined linear automorphism $\gamma' : k[\widetilde{v}] \to k[\widetilde{v}]$ such that (denote by $\gamma'(\widetilde{v})$ the family of elements obtained under the action of γ' to the elements of $\widetilde{v}$) the polynomial $\Phi_{\overline{v},j}(\gamma'(\widetilde{v}), \gamma(v_0), \gamma(v_{j+1}), \ldots, \gamma(v_{n+1})) = \Phi_{\overline{v},j}$.

Hence it is sufficient to prove the lemma for $L_0, L_{j+1}, \ldots, L_n, L_{n+1} + \nu L_0$ in place of $L_0, L_{j+1}, \ldots, L_n, L_{n+1}$. Let us choose $\nu \in k$ such that $V_j \cap \mathcal{Z}(L_{j+1}, \ldots, L_n, L_{n+1} + \nu L_0) = \emptyset$. Then the conditions of Lemma 13 [6] are satisfied for j, V_j and linear forms $L_0, L_{j+1}, \ldots, L_n, L_{n+1} + \nu L_0$ in place of s, $\mathcal{Z}(\mathfrak{P})$ and $L_0, L_s, \ldots, L_n$. The required assertion follows immediately from Lemma 13 [6] since $\deg \Phi_{\overline{\lambda},j} = \deg \Phi_{\overline{v},j}$. The lemma is proved.

REMARK 6 *In the similar way one can strengthen Lemma 13 from [6] replacing there conditions (63), (64) by the one $\mathcal{Z}(\mathfrak{P}) \cap \mathcal{Z}(L_0, L_s, \ldots, L_n) = \emptyset$.*

Now set $b_j(\overline{\lambda}) = b_j(\overline{\lambda}, V) = \deg_{Z_0, Z_{j+1}, \ldots, Z_n} R_{j,0}$; $a_j(\overline{\lambda}) = a_j(\overline{\lambda}, V) = \deg_{Z_0, Z_n} R_{j,*,0}$. If condition (B) holds for $\overline{\lambda}$, and V is from the Introduction, then, see [8], using [2] one can construct all $\Phi_{j,*}$, $R_{j,*}$, $0 \le j \le n$ within the time polynomial in d^n and the size of the input. After that one can compute $R_{j,*,0}$ and $a_j(\overline{\lambda}, V)$ within the same time. Let $\lambda^* = (L_0^*, \ldots, L_{n+1}^*)$ where all L_j^* are linear forms with coefficients from a finitely generated extension k^* of k. If condition (B) holds for λ^* then extending the ground field in one can compute $a_j(\lambda^*, V)$ in the similar way.

Notice that $b_j(\overline{\lambda}) = \deg_{Z_0, Z_n} R_{j,0,*}$. We have $a_j(\overline{\lambda}) \le b_j(\overline{\lambda}) \le \deg V_j(-1 + \deg V_j)$ for all j, see [8]. Obviously $a_j(L') = b_j$ for $j = n-1, n$. Further, see [8] Theorems 1 and 3 if $a_j(\overline{\lambda}) = b_j(\overline{\lambda})$ (or which is the same $\deg_{Z_0, Z_n} R_{j,*,0} = \deg_{Z_0, Z_n} R_{j,0,*}$) for all $1 \le j \le n-2$ then assertions (a), (b), (c) of Theorem 1 [8] holds. For a given $\overline{\lambda}$ satisfying assertions (a), (b), (c) Theorem 1 [8] one can construct by [8] all the objects from assertions (d) and (e) of this theorem and prove (d) and (e) (provided that all L_w have integer coefficients of length $O(n \log d)$).

Put $b_j = b_j(\overline{v}) = b_j(V) = b_j(\overline{v}, V) = \deg_{Z_0, Z_{j+1}, \ldots, Z_n} R_{\overline{v},j,0}$; $a_j = a_j(\overline{v}) = a_j(V) = a_j(\overline{v}, V) = \deg_{Z_0, Z_n} R_{\overline{v},j,*,0}$. Then $b_j = a_j$, see [8] for all $0 \le j \le n-1$.

We shall identify the set of all $(n+2)$–tuples of linear forms from $\overline{k}[X_0,\ldots,X_n]$ with $\mathbb{A}^{(n+2)(n+1)}(\overline{k})$. The following lemma is straightforward, cf. [4].

LEMMA 2 *There is a defined over k open in the Zariski topology subset $\mathcal{V}' \subset \mathbb{A}^{(n+2)(n+1)}(\overline{k})$ such that for every $\overline{\lambda} \in \mathcal{V}'$ the property (B) holds and $b_j(\overline{\lambda}) = a_j(\overline{\lambda}) = b_j$ for all $0 \le j \le n$.*

LEMMA 3 *Let K be an arbitrary extension of k. Assume that all the linear forms $L_w \in \overline{K}[X_0,\ldots,X_n]$, $0 \le w \le n+1$. Let $\overline{\lambda}$ satisfies property (B) (over $\overline{K}$), see above. Let $\mathfrak{l} \subset \mathbb{A}^{(n+2)(n+1)}(\overline{K})$ be a line containing $\overline{\lambda}$. Then for all $\lambda^* \in \mathfrak{l}$, apart of at most a polynomial in $2^n \sum_{0\le j\le n} \deg V_j$ number of exceptions, the element λ^* satisfies (B), and the inequalities $b_j(\overline{\lambda}) \le b_j(\lambda^*)$, $a_j(\overline{\lambda}) \le a_j(\lambda^*)$ hold for all $0 \le j \le n$.*

PROOF The proof is straightforward by the Bézout theorem, cf. [4]. The lemma is proved.

THEOREM 3 *Let V be from the Introduction. Under conditions of Theorem 1 [8] one can construct all the objects from the statement of this theorem such that additionally $b_j = b_j(\overline{\lambda}) = a_j(\overline{\lambda})$ for all $0 \le j \le n$ (and hence all L_w, $0 \le w \le n+1$, are linear forms with integer coefficients of length $O(n \log d)$, and the integers $b_j = b_j(\overline{\lambda}) = a_j(\overline{\lambda}) = a_j$ are computed).*

PROOF We shall suppose without loss of generality that $\dim V < n$, and hence it is sufficient to prove the assertion of the theorem for all $1 \le j \le n-1$. By the previous remarks it is sufficient to construct $\overline{\lambda}$ such that all L_w, $0 \le w \le n+1$, are linear forms with integer coefficients of length $O(n \log d)$, and $b_j = b_j(\overline{\lambda})$. The construction of the required $\overline{\lambda}$ is recursive. We take as the base of the recursion an arbitrary element $\overline{\lambda}$ satisfying (B), i.e., all the required conditions but not necessarily to the condition $b_j = b_j(\overline{\lambda})$. (for example one can take as $\overline{\lambda}$ the element constructed in Theorem 1 [8]).

Let us describe the general step of the recursion. Let $1 \le j \le n-2$ be maximal such that for all $1 \le w < j$ the integers b_w are computed, $b_w = a_w(\overline{\lambda})$, and $V_j \ne \emptyset$ (the last condition can be verified by the algorithm from [4]). Let us compute using the algorithms from [4] and [2] the polynomials $\Phi_{\overline{\lambda},j,*}$, $R_{\overline{\lambda},j,*}$ and $R_{\overline{\lambda},j,*,0}$. We shall suppose without loss of generality in what follows that Z_0 does not divide $R_{\overline{\lambda},j,*,0}$ replacing if it is necessary Z_0 by $Z_0 - \mu Z_n$ for an appropriate integer μ bounded from above by d^{2n}.

Let $1 \le \tau \le 4$ be an integer. Let $0 \le j_1 < \ldots < j_\tau \le n$ be integers. Assume that the field k is supplied with a real structure, see [3]. Let the field $k_j = k(\varepsilon_1,\ldots,\varepsilon_j)$, $0 \le j \in \mathbb{Z}$, where $\varepsilon_j > 0$ for $j \ge 1$ is an infinitesimal with respect to the field k_{j-1}. Let $X_{w,i}$, $1 \le w \le 4$, $0 \le i \le n$; $Z_{w,i}$, $i \in \{0, j+1,\ldots,n+1\} = I_j$, $1 \le w \le \tau$, be variables. Denote for brevity $L'_i = L_i + \sum_{1\le w\le\tau} Z_{w,i} X_{j_w}$, $i \in I_j$ (so all L'_i depend on the choice of $j_1,\ldots,j_\tau$). Denote $X^{(i)} = (X_{i,0},\ldots,X_{i,n})$, and for an arbitrary polynomial c in $n+1$ variables $c(X^{(i)}) = c(X_{i,0},\ldots,X_{i,n})$, $1 \le i \le 4$. Assume that $j \le n-2$. Consider the following system of polynomial equations and inequalities with squares of absolute values with coefficients from k_3 with respect to the variables $X_{w,i}$, $1 \le w \le 4$, $0 \le i \le n$; and $Z_{w,i}$,

$i \in \{0, j+1, \ldots, n+1\} = I_j$, $1 \le w \le \tau$.

$$\left\{\begin{array}{lll}
\sum_{i\in I_j}\sum_{1\le w\le\tau}|Z_{i,w}|^2 \le \varepsilon_1, & & \\
\varepsilon_2 \le |L'_n(X^{(1)}) - L'_n(X^{(3)})|^2 \le \varepsilon_1, & & \\
|L'_{n+1}(X^{(i)}) - L'_{n+1}(X^{(i+1)})|^2 = \varepsilon_3, & i = 1,3, & \\
L'_\nu(X^{(i_1)}) = L'_\nu(X^{(i_2)}) = 0, & 1 \le i_1 < i_2 \le 4, & j+1 \le \nu \le n-1, \\
L'_n(X^{(i)}) - L'_n(X^{(i+1)}) = 0, & i = 1,3, & \\
L'_0(X^{(i)}) - 1 = 0, & 1 \le i \le 4, &
\end{array}\right. \tag{18}$$

LEMMA 4 *Let $1 \le j \le n-2$ be an integer. The inequality $a_j(\overline{\lambda}) < b_j$ holds if and only if there exist $1 \le \tau \le 4$ and $0 \le j_1 < \ldots < j_\tau \le n$ satisfying the following property. System (18) has a solution $X_{w,i} = x^*_{w,i} \in \overline{k_3}$, $1 \le w \le 4$, $0 \le i \le n$; $Z_{w,i} = z^*_{w,i} \in \overline{k_3}$, $i \in \{0, j+1, \ldots, n+1\}$, $1 \le w \le \tau$ such that*

$$(x^*_{w,0} : \ldots : x^*_{w,n}) \in V_j(\overline{k_3}), \quad 1 \le w \le 4. \tag{19}$$

PROOF There exists $\varepsilon' > 0$ such that ε' is an infinitesimal with respect to the field k_1 and ε_2 is an infinitesimal with respect to the field $k_1(\varepsilon')$.

Suppose that $a_j(\overline{\lambda}) < b_j$. By Lemma 2 there are linear forms $L''_w \in \overline{k(\varepsilon')}[X_0, \ldots, X_n]$, $0 \le w \le n+1$, such that all the nonzero coefficients of $L''_w - L_w$, $0 \le w \le n+1$, are infinitesimals with respect to the field k and $a_j(\lambda') = b_j$, where $\lambda' = (L''_0, \ldots, L''_{n+1})$. Let us replace in system (18) all L'_i by L''_i and delete the first inequality. Denote the obtained system by (**).

By Lemma 1 all the nonzero coefficients of the polynomial $R_{\lambda',j,0,*}$ are infinitely close to the corresponding coefficients of $R_{\overline{\lambda},j,0,*}$, and $R_{\lambda',j,0,*}(1, Z_n)$ has more different roots than $R_{\overline{\lambda},j,0,*}(1, Z_n)$. Hence by the transfer principle, see [1], there are roots $Z_n = z_{\alpha,n} \in k(\varepsilon')$, $\alpha = 1, 3$, of $R_{\lambda',j,0,*}(1, Z_n)$ such that $z_{1,n} - z_{3,n}$ is an infinitesimal from $\overline{k(\varepsilon')}$. Recall that $\mathrm{lc}_{Z_{n+1}}\Phi_{\lambda',j} = 1$. Hence for $\alpha = 1, 3$ by the transfer principle, applied to the polynomial $\Phi_{\lambda',j,*}(1, z_{\alpha,n}, Z)$, there are $\widetilde{z}_{\alpha,n} \in \overline{k(\varepsilon')}$, $\widetilde{z}_{\beta,n+1} \in \overline{k(\varepsilon', \varepsilon_3)}$, $\beta = \alpha, \alpha+1$ such that $\widetilde{z}_{\alpha,n} - z_{\alpha,n} \in \overline{k(\varepsilon', \varepsilon_3)}$, $\alpha = 1, 3$, is infinitesimal with respect to the field $k_2(\varepsilon')$, $|\widetilde{z}_{\alpha,n+1} - \widetilde{z}_{\alpha+1,n+1}|^2 = \varepsilon_3$, and $\Phi_{\lambda',j,*}(1, \widetilde{z}_{\alpha,n}, \widetilde{z}_{\beta,n+1}) = 0$ for $\beta = \alpha, \alpha+1$. Set $\widetilde{z}_{\beta,n} = \widetilde{z}_{\alpha,n}$ for $\alpha = 1, 3$, $\beta = \alpha, \alpha+1$. The morphism $V_j \to \mathcal{Z}(\Phi_{\lambda',j})$, $(X_0 : \ldots : X_n) \mapsto (L''_0 : L''_{j+1} : \ldots : L''_n)$, is finite dominant. Therefore, there is a solution of system (**) giving a point from $V_j(\overline{k(\varepsilon', \varepsilon_3)})^4$ and satisfying additional equations $L''_\nu(X^{(\beta)}) = 0$, $j+1 \le \nu \le n-1$, $L''_j(X^{(\beta)}) = \widetilde{z}_{\beta,j}$, $j = n, n+1$, $\beta = 1, 2, 3, 4$. Let this solution be $X_{w,i} = \widetilde{x}_{w,i}$, $1 \le w \le 4$, $0 \le i \le n$.

Now we would like to replace all L''_i by $L_i + \sum_{1\le w\le\tau}\widetilde{z}_{w,i}X_{j_w}$, $i \in \{0, j+1, \ldots, n\}$ for appropriate $1 \le \tau \le 4$, $0 \le j_1 < \ldots < j_\tau \le n$ and $\widetilde{z}_{w,i} \in \overline{k(\varepsilon', \varepsilon_3)}$ that are infinitesimals with respect to the field k (if they are nonzero). Let $\tau \ge 1$ be the rank of the matrix $(\widetilde{x}_{w,i})$. Let us choose a $(\tau \times \tau)$–minor $\Delta_0 \ne 0$ of the considered matrix such that for any other $(\tau\times\tau)$–minor Δ_1 of this matrix Δ_1/Δ_0 is not infinitely large with respect to the field k. Let $0 \le j_1 < \ldots < j_\tau \le n$ be the indices of the columns of Δ_0. Then by Cramer's rule one can represent

$$\widetilde{x}_{w,q} = \sum_{1\le\nu\le\tau}\mu_{w,q,\nu}\widetilde{x}_{w,j_\nu}, \tag{20}$$

$1 \le w \le 4$, where every $\mu_{w,q,\nu} \in \overline{k(\varepsilon', \varepsilon_3)}$ is not infinitely large with respect to the field k. By (20) one can find $\widetilde{z}_{w,i}$ for all $1 \le w \le \tau$, $i \in I_j$, such that

$X_{w,i} = \widetilde{x}_{w,i}$, $1 \leq w \leq 4$, $0 \leq i \leq n$, and $Z_{w,i} = \widetilde{z}_{w,i}$, $1 \leq w \leq \tau$, $i \in I_j$, is a solution of system (18) giving a point from $V_j^4(\overline{k(\varepsilon', \varepsilon_3)})$. Thus, we have constructed the required solution for some $1 \leq \tau \leq 4$ and $0 \leq j_1 < \ldots < j_\tau \leq n$ but over the field $\overline{k(\varepsilon', \varepsilon_3)}$. Now by transfer principle, see [1], or directly one sees that there is also the required solution over the field $\overline{k_3}$ for the considered $1 \leq \tau \leq 4$ and $0 \leq j_1 < \ldots < j_\tau \leq n$.

Conversely, suppose that system (18) has a solution for some $1 \leq \tau \leq 4$ and $0 \leq j_1 < \ldots < j_\tau \leq n$. Then, as it follows from the description of the algorithm below, $a_j(\overline{\lambda}) < b_j$. The lemma is proved.

Let us return to the description of the algorithm for Theorem 3. Let us enumerate integers τ and $j_1, \ldots, j_\tau$ such that $1 \leq \tau \leq 4$ and $0 \leq j_1 < \ldots < j_\tau \leq n$. Applying Theorem 2 [9] to the algebraic variety $\mathrm{con}(V) \subset \mathbb{A}^{n+1}(\overline{k})$ (which is the set of all zeroes of the homogeneous ideal of V in $\mathbb{A}^{n+1}(\overline{k})$) and using [4] we decide whether system (18) has a solution satisfying (19). If it is true, we construct such a solution. If system (18) does not have a solution satisfying (19) then we proceed to the next τ and $j_1, \ldots, j_\tau$.

Suppose that system (18) does not have a solution satisfying (19) for all $1 \leq \tau \leq 4$ and $0 \leq j_1 < \ldots < j_\tau \leq n$. Then $b_j = a_j(\overline{\lambda})$ by Lemma 4. If $j + 1 = n - 1$ the element $\overline{\lambda}$ satisfying assertion of Theorem 3 is constructed. Suppose that $j + 1 \leq n - 2$. Then we replace j by $j + 1$ and proceed to the next step of the described recursion.

Suppose that (18) has a solution satisfying (19) for some $1 \leq \tau \leq 4$ and $0 \leq j_1 < \ldots < j_\tau \leq n$. Let us compute, see [3], $z_{w,i} = \mathrm{st}_{\varepsilon_3}(z^*_{w,i})$, $1 \leq w \leq \tau$, $i \in I_j$, where all $z^*_{w,i}$ are from the statement of Lemma 4. Set $L^*_i = L_i$ for $i \in \{1, \ldots, j\}$, and $L^*_i = L_i + \sum_{1 \leq w \leq \tau} z_{w,i} X_{j_w}$ for $i \in I_j$. So L^*_i are linear forms with coefficients from $\overline{k_3}$. Put $\lambda^* = (L^*_0, \ldots, L^*_n)$. By Lemma 2 the element λ^* satisfies property (B). From system (18) we get that the polynomial $R_{\lambda^*, j, *}(1, Z_n)$ has more different roots than $R_{\overline{\lambda}, j, *}(1, Z_n)$. So $a_j(\overline{\lambda}) < a_j(\lambda^*)$ and $a_w(\overline{\lambda}) = a_w(\lambda^*)$ for $1 \leq w < j$. Applying Lemma 3 and the algorithm of reduction to integer coefficients, see [9] Proposition 2, we replace subsequently all the coefficients of linear forms L^*_i by integers of length $O(n \log d)$ and obtain new linear forms L^{**}_i, $0 \leq i \leq n + 1$, with integer coefficients of length $O(n \log d)$ satisfying condition (B) and such that (denote $\lambda^{**} = (L^{**}_0, \ldots, L^{**}_{n+1})$) $a_w(\lambda^{**}) \geq a_w(\lambda^*)$ for all $1 \leq w \leq j$. In particular $a_j(\lambda^{**}) > a_j(\overline{\lambda})$. Now we replace $\overline{\lambda}$ by λ^{**} and proceed to the next step of the described recursion.

Thus, finally we shall construct the element $\overline{\lambda}$ satisfying the statement of Theorem 3. Notice that by the Bézout theorem the total number of steps of the described recursion is bounded from above by a polynomial in d^n. The estimation for the working time of the algorithm for Theorem 3 follows immediately from the ones of the applied algorithms. The theorem is proved.

2 Standard parts of algebraic varieties and their intersections

Let K' be an arbitrary extension of the field $\overline{k}$ by infinitesimals, and the algebraic closure $\overline{K'}$ is supplied with the real structure, see [3]. Denote by $\mathrm{st} : \mathbb{P}^n(\overline{K'}) \to \mathbb{P}^n(\overline{k})$ the mapping of the standard part, see, e.g., [3] (the standard part of an element $x' \in \mathbb{P}^n(\overline{K'})$ is the element $x \in \mathbb{P}^n(\overline{k})$ which is infinitely close to

x'). Let us identify $\mathbb{A}^n(\overline{K'}) \subset \mathbb{P}^n(\overline{K'}) \setminus \mathcal{Z}(X_0)$. Then for every $x' \in \mathbb{A}^n(\overline{K'})$ the standard part $\mathrm{st}(x') \in \mathbb{P}^n(\overline{K'})$. Let us define for an arbitrary subset $S \subset \mathbb{A}^n(\overline{K'})$ the affine standard part $\mathrm{sta}(E) = \mathrm{st}(E) \cap \mathbb{A}^n(\overline{K'}) = \mathrm{st}(E) \setminus \mathcal{Z}(X_0)$. In the next three general lemmas the variety W is not necessarily the one from the Introduction.

LEMMA 5 *Let $\widetilde{W} \subset \mathbb{P}^n(\overline{K'})$ (respectively $\widetilde{W} \subset \mathbb{A}^n(\overline{K'})$) be an arbitrary irreducible over $\overline{K'}$ projective algebraic variety (respectively affine algebraic variety such that $\mathrm{sta}(\widetilde{W}) \neq \emptyset$). Let $W = \mathrm{st}(\widetilde{W})$ (respectively $W = \mathrm{sta}(\widetilde{W})$). Then W is a nonempty projective (respectively affine) algebraic variety in $\mathbb{P}^n(\overline{k})$ (respectively $\mathbb{A}^n(\overline{k})$), for every irreducible component W_1 of W the dimension $\dim W_1 = \dim \widetilde{W}$, and the degree $\deg \widetilde{W} \geq \deg W$.*

PROOF Let $\widetilde{W} \subset \mathbb{A}^n(\overline{K'})$ be an affine algebraic variety. Denote by $\widetilde{W}_1$ the closure with respect to the Zariski topology of $\widetilde{W}$ in $\mathbb{P}^n(\overline{K'})$. Then obviously $\mathrm{st}(\widetilde{W}) = \mathrm{st}(\widetilde{W}_1)$, and, therefore, the assertions related to the affine varieties follow from the corresponding ones about projective varieties. Hence we shall assume in what follows in the proof that $\widetilde{W} \subset \mathbb{P}^n(\overline{K})$ is a projective algebraic variety.

For $n = 0$ the assertion of the lemma is trivial. Using the induction on n we shall suppose that this assertion is proved for projective algebraic varieties in $\mathbb{P}^{n-1}(\overline{K'})$.

We can suppose without loss of generality that $\widetilde{W} \neq \mathbb{P}^n(\overline{K'})$. Then there is an irreducible polynomial $\Phi \in \overline{K'}[X_0, \ldots, X_n]$ such that $\widetilde{W} \subset \mathcal{Z}(\Phi)$. Multiplying Φ by a nonzero factor from K' we shall suppose without loss of generality that each coefficient of Φ is not infinitely large with respect to k and there is a coefficient of Φ that is not infinitesimal with respect to the field k. Let $\mathrm{st}(\Phi) \in \overline{k}[X_0, \ldots, X_n]$ be the polynomial such that all the nonzero coefficients of $\Phi - \mathrm{st}(\Phi)$ are infinitesimals with respect to k. Then $\mathrm{st}(\Phi) \neq 0$ and $W \subset \mathcal{Z}(\mathrm{st}(\Phi))$.

Suppose that $\dim \widetilde{W} = n - 1$. Then $\deg \Phi = \deg \mathrm{st}(\Phi)$, $\widetilde{W} = \mathcal{Z}(\Phi)$ and $W \subset \mathcal{Z}(\mathrm{st}(\Phi))$. Let $x \in \mathcal{Z}(\mathrm{st}(\Phi))$. Let us show that $x \in W$ and hence $W = \mathrm{st}(\mathcal{Z}(\Phi))$. If $n \geq 2$ then, intersecting $\widetilde{W}$ and W with a hyperplane in general position trough x, and applying the inductive assumption we get $x \in W$. If $n = 1$ then one can prove that $x \in W$ directly or using the Veronese embedding, see [10], $\mathbb{P}^1 \to \mathbb{P}^{\deg \Phi}$ of degree $\deg \Phi$ and applying Proposition 6 [3]. Hence $\deg \widetilde{W} = \deg \Phi = \deg \mathrm{st}(\Phi) \geq \deg W$. The lemma is proved for $\dim \widetilde{W} = n - 1$.

Suppose that $\dim \widetilde{W} \leq n - 2$ (hence $n \geq 2$). The set W is nonempty since for every point $z \in \widetilde{W}$ the point $\mathrm{st}(z) \in W$. Denote by $\overline{W}$ the closure in the Zariski topology of W.

Let $x \in \mathbb{P}^n(\overline{k})$ be a point. Denote by $\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$ the linear projection with the center x. Let us choose a point x such that $\mathrm{st}(\Phi)(x) \neq 0$ and $\deg \widetilde{W} = \deg \pi(\widetilde{W})$ (these are conditions of general position for x). Then $x \notin \overline{W}$ and hence $\pi(\widetilde{W})$ and $\pi(\overline{W})$ are closed with respect to the Zariski topology in $\mathbb{P}^{n-1}(\overline{K'})$ and $\mathbb{P}^{n-1}(\overline{k})$ respectively.

Further, for every $z \in \widetilde{W}$ the equality $\mathrm{st}(\pi(z)) = \pi(\mathrm{st}(z))$ holds since the morphism $\widetilde{W} \to \pi(\widetilde{W})$ is finite dominant due to the choice of the point x. Therefore, $\mathrm{st}(\pi(\widetilde{W})) = \pi(W)$. Hence by induction $\pi(W)$ is a projective algebraic variety and $\dim \pi(W) = \dim \pi(\widetilde{W})$. Considering the finite morphism

$\overline{W} \to \pi(\overline{W})$ induced by π we get $\pi(\overline{W}) = \pi(W)$. But $\dim \pi(\overline{W}) = \dim \overline{W}$ and $\dim \pi(\widetilde{W}) = \dim \widetilde{W}$ by the choice of the point x. Hence $\dim \overline{W} \leq n-2$.

Let $y \in \overline{W}$. We fixe this point for some time. Since $\dim \overline{W} \leq n-2$ one can choose in advance the point x such that additionally for every irreducible component W_1 of $\overline{W}$ the morphism $W_1 \to \pi(W_1)$ induced by π is finite dominant; $\deg \pi(W_1) = \deg W_1$; the irreducible components of $\overline{W}$ and $\pi(\overline{W})$ are in the one–to–one correspondence by the rule $W_1 \mapsto \pi(W_1)$; the number of elements $\#\pi^{-1}\pi(y) \cap \overline{W} = 1$ (these are conditions of general position for x). As it is proved there is a point $\widetilde{y} \in \widetilde{W}$ such that $\mathrm{st}(\pi(\widetilde{y})) = \pi(y)$. Now $\mathrm{st}(\widetilde{y}) \in W$ and $\pi(\mathrm{st}(\widetilde{y})) = \pi(y)$. Hence $\mathrm{st}(\widetilde{y}) = y \in W$. Since $y \in \overline{W}$ is arbitrary the algebraic variety $\overline{W} = W$.

Finally, by the inductive assumption $\dim \pi(\widetilde{W}) = \dim \pi(W_1)$ and $\deg \pi(\widetilde{W}) \geq \deg \pi(W)$. On the other hand, $\dim W_1 = \dim \pi(W_1)$. The lemma is proved.

LEMMA 6 *Under conditions of Lemma 5 let $g \in \overline{k}[X_0, \ldots, X_n]$ be a polynomial and E be an irreducible over $\overline{k}$ component of W such that $E \cap \mathcal{Z}(g)$ is a union of some irreducible components of $W \cap \mathcal{Z}(g)$ and $\dim(E \cap \mathcal{Z}(g)) = \dim(W) - 1$. Then $\dim(\widetilde{W} \cap \mathcal{Z}(g)) = \dim(\widetilde{W}) - 1$, and $\mathrm{st}(\widetilde{W} \cap \mathcal{Z}(g)) \supset E$ in $\mathbb{P}^n(\overline{k})$ (respectively $\mathrm{sta}(\widetilde{W} \cap \mathcal{Z}(g)) \supset E$ in $\mathbb{A}^n(\overline{k})$). In particular, if $\dim(W \cap \mathcal{Z}(g)) = \dim(W) - 1$ then $\mathrm{st}(\widetilde{W} \cap \mathcal{Z}(g)) = W \cap \mathcal{Z}(g)$ in $\mathbb{P}^n(\overline{k})$ (respectively $\mathrm{sta}(\widetilde{W} \cap \mathcal{Z}(g)) = W \cap \mathcal{Z}(g)$ in $\mathbb{A}^n(\overline{k})$).*

PROOF Similarly to the proof of Lemma 5 the case of affine algebraic varieties is reduced to the projective one. So we shall assume in what follows in the proof that $\widetilde{W}$ and W are projective varieties.

The polynomial g does not vanish on $\widetilde{W}$ since, otherwise, by Lemma 5 it vanishes on every irreducible component of W. Therefore, $\dim(\widetilde{W} \cap \mathcal{Z}(g)) = \dim(\widetilde{W}) - 1$. We shall assume without loss of generality that $\widetilde{W} \neq \mathbb{P}^n(\overline{K'})$. Using the Veronese isomorphism, see [10], we can suppose without loss of generality that g is a linear form.

Suppose at first that $\dim W \cap \mathcal{Z}(g) = \dim W - 1$. In this case we shall prove that $\mathrm{st}(\widetilde{W} \cap \mathcal{Z}(g)) = W \cap \mathcal{Z}(g)$ in $\mathbb{P}^n(\overline{k})$ using the induction on $n - \dim \widetilde{W}$. The base is $\dim \widetilde{W} = n-1$. For the base $\widetilde{W} = \mathcal{Z}(\Phi)$ and $W = \mathcal{Z}(\mathrm{st}(\Phi))$, see the proof of Lemma 5. Further, $\dim \mathcal{Z}(\mathrm{st}(\Phi), g) = n-2$, since $\mathcal{Z}(g)$ is not an irreducible component of $\mathrm{st}(\Phi)$. One sees that $\mathrm{st}(\mathcal{Z}(\Phi, g)) = \mathcal{Z}(\mathrm{st}(\Phi), g)$, and the base of the required assertion is proved.

Let $\dim W \cap \mathcal{Z}(g) = \dim W - 1$ and $\dim W < n-1$. Let $x \in \mathbb{P}^n(\overline{k})$ be an arbitrary point, and $\pi : \mathbb{P}^n(\overline{k}) \to \mathbb{P}^{n-1}(\overline{k})$ be the linear projection with the center x. Let $y \in W \cap \mathcal{Z}(g)$. By Lemma 5 it is sufficient to prove that $y \in \mathrm{st}(\widetilde{W} \cap \mathcal{Z}(g))$. Let $\mathfrak{l} \subset \mathbb{P}^n(\overline{k})$ be a line such that $y \in \mathfrak{l}$. If we choose $\mathfrak{l}$ in general position then $\mathfrak{l} \cap W = \{y\}$. Hence one can find x such that $x \in \mathcal{Z}(g) \setminus (W \cup \widetilde{W})$ and the number of elements $\#\pi^{-1}\pi(y) \cap W = 1$. In what follows we assume that x satisfies these properties. Then the morphisms $W \to \pi(W)$ and $\widetilde{W} \to \pi(\widetilde{W})$ induced by π are finite dominant. The standard part $\mathrm{st}(\pi(\widetilde{y})) = \pi(\mathrm{st}(\widetilde{y}))$ for arbitrary $\widetilde{y} \in \widetilde{W}$ since the morphism $\widetilde{W} \to \pi(\widetilde{W})$ induced by π is finite. By the inductive assumption applied to $\pi(\widetilde{W}) \subset \mathbb{P}^{n-1}(\overline{K'})$ we have $\mathrm{st}(\pi(\widetilde{W}) \cap \mathcal{Z}(g)) = \pi(W) \cap \mathcal{Z}(g)$. But $\pi(W) \cap \mathcal{Z}(g) = \pi(W \cap \mathcal{Z}(g))$, $\pi(\widetilde{W}) \cap \mathcal{Z}(g) = \pi(\widetilde{W} \cap \mathcal{Z}(g))$ since $x \in \mathcal{Z}(g)$. Hence there is $\widetilde{y} \in \widetilde{W} \cap \mathcal{Z}(g)$ such that $\mathrm{st}(\pi(\widetilde{y})) = \pi(y)$. Therefore, $\pi(\mathrm{st}(\widetilde{y})) = \pi(y)$. Obviously $\mathrm{st}(\widetilde{y}) \in W$.

Finally, $\widetilde{y} = y$ since $\#(\pi^{-1}\pi(y) \cap W) = 1$. The required assertions are proved for the case $\dim W \cap \mathcal{Z}(g) = \dim W - 1$.

Consider the general case. We need to prove that $\mathrm{st}(\widetilde{W} \cap \mathcal{Z}(g)) \supset E$ in $\mathbb{P}^n(\overline{k})$. Let $y \in W \cap \mathcal{Z}(g) \setminus \overline{W \setminus E}$. By Lemma 5 it is sufficient to prove that $y \in \mathrm{st}(\widetilde{W} \cap \mathcal{Z}(g))$. There is a linear subspace $\mathcal{L} \subset \mathbb{P}^n(\overline{k})$ such that $y \in \mathcal{L}$, $\dim \mathcal{L} + \dim W - 1 = n$, y is an isolated point of $\mathcal{W} \cap \mathcal{L} \cap \mathcal{Z}(g)$, $\dim \widetilde{W} \cap \mathcal{L} = 1$, $\dim W \cap \mathcal{L} = 1$. Denote $\widetilde{C} = \widetilde{W} \cap \mathcal{L}$, $C = W \cap \mathcal{L}$. By the proved $\mathrm{st}(\widetilde{C}) = C$. So it is sufficient to show that $y \in \mathrm{st}(\widetilde{C} \cap \mathcal{Z}(g))$.

Let us represent $C = C_1 \cup C_2$ where C_1 (respectively C_2) is a union of all the irreducible components C' of C such that $C' \not\subset \mathcal{Z}(g)$ (respectively $C' \subset \mathcal{Z}(g)$). Performing a linear transformation of the coordinates we shall assume without loss of generality that $(\widetilde{C} \cup C_1) \cap \mathcal{Z}(X_0, g) = \emptyset$ in $\mathbb{P}^n(\overline{K'})$. Consider the finite dominant morphisms $\widetilde{\tau} : \widetilde{C} \to \mathbb{P}^1(\overline{K'})$, and $\tau_1 : C_1(\overline{K'}) \to \mathbb{P}^1(\overline{K'})$ given by the same formula $(X_0 : \ldots : X_n) \mapsto (X_0 : g)$. Hence for all $a \in \overline{K'}$, except at most a finite number, $\#\widetilde{\tau}^{-1}((1 : a)) = \deg \widetilde{\tau}$ and $\#\tau_1^{-1}((1 : a)) = \deg \tau_1$. Let $\overline{K'} = R + R\sqrt{-1}$, where R is the real closed and real ordered field. For arbitrary two points $z', z'' \in \mathbb{P}^n(\overline{K'}) \setminus \mathcal{Z}(X_0)$ put the distance $0 \leq d(z', z'') = (\sum_{1 \leq i \leq n} |(X_i/X_0)(z') - (X_i/X_0)(z'')|^2)^{1/2} \in R$. Hence if $z', z'' \in \mathbb{P}^n(\overline{k}) \setminus \mathcal{Z}(X_0)$ then $d(z', z'') \in R \cap \overline{k}$.

Since y is an isolated point of $C \cap \mathcal{Z}(g)$ there is $0 < m_2 \in R \cap \overline{k}$ such that for every $y \neq z' \in C \cap \mathcal{Z}(g)$ the distance $d(y, z') \geq m_2$. There is $0 < m_1 \in R \cap \overline{k}$ so small that for all $a \in R$ satisfying the inequalities $-m_1 \leq a \leq m_1$, $a \neq 0$,

- $\#\widetilde{\tau}^{-1}((1 : a)) = \deg \widetilde{\tau}$ and $\#\tau_1^{-1}((1 : a)) = \deg \tau_1$,
- for every $z' \in \tau_1^{-1}((1 : a))$ there is $z'' \in \tau_1^{-1}((1 : 0))$ such that the distance $d(z', z'') \leq m_2/8$, and, conversely, for every $z'' \in \tau_1^{-1}((1 : 0))$ there is $z' \in \tau_1^{-1}((1 : a))$ such that the distance $d(z', z'') \leq m_2/8$

Denote by c_1 the semialgebraically connected component, see [1], of $C \cap \{x \in \mathbb{P}^n(\overline{K'}) \setminus \mathcal{Z}(X_0) : |(g/X_0)(x)|^2 \leq m_1\} = S_{m_1}$ such that $y \in c_1$. Denote by c_2 the union of all the semialgebraically connected components c' of S_{m_1} such that $c' \neq c_1$. Then obviously $c_1 \subset C_1$, for all $z' \in c_1$, $z'' \in c_2$ the distance $d(z', z'') \geq (3/4)m_2$, and $c_1 \cap \mathcal{Z}(g) = \{y\}$.

For every semialgebraic set $A \subset \mathbb{P}^n(\overline{K'}) \setminus \mathcal{Z}(X_0)$ and $z' \in \mathbb{P}^n(\overline{K'}) \setminus \mathcal{Z}(X_0)$ the distance $d(z', A) = \inf\{d(z', z'') : z'' \in A\}$ is a continuous function of z' with respect to the topology of the real closed field R, see [1]. Set for every $0 \neq a \in R$, $|a| \leq m_1$,

$$\psi(a) = \sum_{z \in \widetilde{\tau}^{-1}(a)} d(z, c_2) \in R. \tag{21}$$

Then by the implicit function theorem ψ is a continuous function of a with respect to the topology of the real closed field R (we leave the details to the reader here).

Let $\varepsilon > 0$ be an infinitesimal with respect to the field $\overline{K'}$, and $\widetilde{R(\varepsilon)}$ be the real closure of $R(\varepsilon)$. Hence $\widetilde{R(\varepsilon)}$ is a real ordered field, and the mapping of the standard part $\mathrm{st}_\varepsilon : \mathbb{P}^n(\overline{K'(\varepsilon)}) \to \mathbb{P}^n(\overline{K'})$ is defined. Obviously $\mathrm{st}_\varepsilon(\widetilde{C}(\overline{K'(\varepsilon)}) = \widetilde{C}$. Hence $\mathrm{st}(\mathrm{st}_\varepsilon(\widetilde{C}(\overline{K'(\varepsilon)})) = C$. We extend the ground field K' till $K'(\varepsilon)$. Hence now $\psi : \{a \in \widetilde{R(\varepsilon)} : |a| \leq m_1 \ \& \ a \neq 0\} \to \widetilde{R(\varepsilon)}$ is a continuous function.

Let us show that there is a point $\widetilde{y}_\varepsilon \in \widetilde{C}(\overline{K'(\varepsilon)}) \cap \mathcal{Z}(g - \varepsilon X_0)$ such that $\widetilde{y}_\varepsilon$ is infinitely close to y with respect to the field k, i.e., such that $\mathrm{st}(\mathrm{st}_\varepsilon(\widetilde{y}_\varepsilon)) = y$. Indeed, suppose contrary. Then $\mathrm{st}(\mathrm{st}_\varepsilon(z)) \in c_2$ for every $z \in \widetilde{\tau}^{-1}(\varepsilon)$, and, hence $|\psi(\varepsilon)| \leq m_2/3$. On the other hand, $C \cap \mathcal{Z}(g - m_1 X_0)$ is a finite set. Hence as we have proved $\mathrm{st}(\widetilde{C} \cap \mathcal{Z}(g - m_1 X_0)) = C \cap \mathcal{Z}(g - m_1 X_0)$. We have $c_1 \cap \mathcal{Z}(g - m_1 X_0) \setminus \mathcal{Z}(X_0) \neq \emptyset$ by the choice of m_1 and m_2. Therefore, $\psi(m_1) \geq 2m_2/3$. Now the image of the interval $\psi([\varepsilon, m_1])$ is a semialgebraically connected set in $\widetilde{R(\varepsilon)}$, see [1]. Hence $\psi([\varepsilon, m_1])$ contains the interval $[\psi(\varepsilon), \psi(m_1)] \supset [m_2/3, 2m_2/3]$, see [1]. Therefore, there is $a \in [\varepsilon, m_1]$ such that $m_2/3 < \psi(a) < 2m_2/3$. But $\psi(a) \geq 2m_2/3$ if $\mathrm{st}(\mathrm{st}_\varepsilon(z)) \in c_1$ for at least one $z \in \widetilde{\tau}^{-1}(a)$ and, otherwise, $\psi(a) \leq m_2/3$. This contradiction proves the existence of $\widetilde{y}_\varepsilon$. Now put $\widetilde{y} = \mathrm{st}_\varepsilon(\widetilde{y}_\varepsilon)$. Then obviously $\widetilde{y} \in \widetilde{C} \cap \mathcal{Z}(g)$ and $\mathrm{st}(\widetilde{y}) = y$. The lemma is proved.

LEMMA 7 *Let the field K' be as above. Let $\widetilde{W}_i \subset \mathbb{P}^n(\overline{K'})$ (respectively $\widetilde{W}_i \subset \mathbb{A}^n(\overline{K'})$), $i = 1, 2$, be projective (respectively affine) algebraic varieties. Suppose that all the irreducible component of $\widetilde{W}_i$ have the same dimension $n - s_i$, $i = 1, 2$. Put $W_i = \mathrm{st}(\widetilde{W}_i) \subset \mathbb{P}^n(\overline{k})$ (respectively $W_i = \mathrm{sta}(\widetilde{W}_i) \subset \mathbb{A}^n(\overline{k})$), $i = 1, 2$ (hence the dimension of every irreducible component of W_i is $n - s_i$ by Lemma 5). Then the following assertions hold.*

(a) Let E be an irreducible over $\overline{k}$ component of $W_1 \cap W_2$. Assume that $\dim E = n - s_1 - s_2 \geq 0$. Then there is an irreducible over $\overline{K'}$ component $\widetilde{E}$ of $\widetilde{W}_1 \cap \widetilde{W}_2$ such that $\dim \widetilde{E} = n - s_1 - s_2$ and $\mathrm{st}(\widetilde{E}) \supset E$ (respectively $\mathrm{sta}(\widetilde{E}) \supset E$).

(b) Suppose that $\dim(W_1 \cap W_2) = n - s_1 - s_2$ (if $n - s_1 - s_2 < 0$ then the intersection is empty). Then $\mathrm{st}(\widetilde{W}_1 \cap \widetilde{W}_2) = W_1 \cap W_2$ (respectively $\mathrm{sta}(\widetilde{W}_1 \cap \widetilde{W}_2) = W_1 \cap W_2$) and therefore, $\dim(\widetilde{W}_1 \cap \widetilde{W}_2) = n - s_1 - s_2$ (respectively the dimension of every irreducible component $\widetilde{E}$ of $\widetilde{W}_1 \cap \widetilde{W}_2$ such that $\mathrm{sta}(\widetilde{E}) \neq \emptyset$ is $n - s_1 - s_2$) by Lemma 5.

PROOF Let us prove (a). By Lemma 5 it is sufficient to consider the case of affine algebraic varieties. Set $\Delta = \{(x, x) \in \mathbb{A}^{2n}(\overline{k}) : x \in \mathbb{A}^n(\overline{k})\}$. Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ be the coordinate functions on $\mathbb{A}^{2n}(\overline{k})$ such that $\Delta = \mathcal{Z}(X_1 - Y_1, \ldots, X_n - Y_n)$. Put $\delta_i = X_i - Y_i$, $1 \leq i \leq n$. One identifies $W_1 \cap W_2 = (W_1 \times W_2) \cap \mathcal{Z}(\delta_1, \ldots, \delta_n)$ in $\mathbb{A}^{2n}(\overline{k})$ and $\widetilde{W}_1 \cap \widetilde{W}_2 = (\widetilde{W}_1 \times \widetilde{W}_2) \cap \mathcal{Z}(\delta_1, \ldots, \delta_n)$ in $\mathbb{A}^{2n}(\overline{K'})$. There are irreducible over $\overline{k}$ algebraic varieties $E_0 \supset E_1 \supset \ldots \supset E_n = E$ such that E_i is an irreducible component of $(W_1 \times W_2) \cap \mathcal{Z}(\delta_1, \ldots, \delta_i)$ and $\dim E_i = (n - s_1) + (n - s_2) - i$ for every $0 \leq i \leq n$.

Let us show that that there are irreducible over $\overline{K'}$ algebraic varieties $\widetilde{E}_0 \supset \widetilde{E}_1 \supset \ldots \supset \widetilde{E}_n = \widetilde{E}$ such that $\widetilde{E}_i$ is an irreducible component of $(\widetilde{W}_1 \times \widetilde{W}_2) \cap \mathcal{Z}(\delta_1, \ldots, \delta_i)$, $\dim \widetilde{E}_i = (n - s_1) + (n - s_2) - i$ and $\mathrm{sta}(\widetilde{E}_i) \supset E_i$ for every $1 \leq i \leq n$. We shall use the induction on i. The base of the induction $i = 0$ follows from the equality $\mathrm{sta}(\widetilde{W}_1 \times \widetilde{W}_2) = W_1 \times W_2$ and Lemma 5. Suppose that we have proved that there are $\widetilde{E}_0, \ldots, \widetilde{E}_i$ for some $0 \leq i \leq n - 1$. Then obviously

$$E_{i+1} \subset \mathrm{sta}(\widetilde{E}_i) \cap \mathcal{Z}(\delta_{i+1}) \subset (W_1 \times W_2) \cap \mathcal{Z}(\delta_1, \ldots, \delta_{i+1}).$$

Therefore, E_{i+1} is an irreducible component of $\mathrm{sta}(\widetilde{E}_i) \cap \mathcal{Z}(\delta_{i+1})$. Hence E_{i+1} is an irreducible component of $\mathrm{sta}(\widetilde{E}_i \cap \mathcal{Z}(\delta_{i+1}))$ by Lemma 6. By Lemma 5 there

is an irreducible over $\overline{K'}$ component E' of $\widetilde{E}_i \cap \mathcal{Z}(\delta_{i+1})$ such that $\mathrm{sta}(E') \supset E_{i+1}$. Then $\dim E' = \dim \widetilde{E}_i \cap \mathcal{Z}(\delta_{i+1}) = (n-s_1)+(n-s_2)-i-1$ by Lemma 5 and since $\widetilde{E}_i$ is irreducible over $\overline{K'}$. Now

$$E_{i+1} \subset \mathrm{sta}(E') \subset \mathrm{sta}((\widetilde{W}_1 \times \widetilde{W}_2) \cap \mathcal{Z}(\delta_1, \ldots, \delta_{i+1})) \subset$$
$$(W_1 \times W_2) \cap \mathcal{Z}(\delta_1, \ldots, \delta_{i+1}).$$

Therefore, E_{i+1} is an irreducible component of $\mathrm{sta}((\widetilde{W}_1 \times \widetilde{W}_2) \cap \mathcal{Z}(\delta_1, \ldots, \delta_{i+1}))$. We have $E' \subset (\widetilde{W}_1 \times \widetilde{W}_2) \cap \mathcal{Z}(\delta_1, \ldots, \delta_{i+1})$ and $\mathrm{sta}(E') \supset E_{i+1}$. Hence E' is an irreducible component of $(\widetilde{W}_1 \times \widetilde{W}_2) \cap \mathcal{Z}(\delta_1, \ldots, \delta_{i+1})$ by Lemma 5. Now one can put $\widetilde{E}_{i+1} = E'$. The existence of $\widetilde{E}_0, \ldots, \widetilde{E}_n$ is proved by the induction. Assertion (a) is proved.

Let us prove (b). Obviously $\mathrm{st}(\widetilde{W}_1 \cap \widetilde{W}_2) \subset W_1 \cap W_2$ (respectively $\mathrm{sta}(\widetilde{W}_1 \cap \widetilde{W}_2) \subset W_1 \cap W_2$). So one can assume without loss of generality that $W_1 \cap W_2 \neq \emptyset$. Now (b) follows from (a) immediately. The lemma is proved.

3 Several lemmas

In assertion (d) of Theorem 1 [9] we claim that $\mathcal{U}'_0$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(n-s+1)}(\overline{k})$ (recall that there $s = \sigma$). In the next lemma we shall consider a similar but more general situation. Below we shall not use Lemma 8 and Lemma 10. But still we decided to present them here for the completeness.

LEMMA 8 *Let $E \subset W$ be an arbitrary closed in the Zariski topology subvariety and E' be the closure in W' of the image $p(E \setminus \mathcal{Z}(e_0, \ldots, e_r))$. Suppose that $\dim E' \leq n-i$, where $\sigma \leq i \leq n$. Let $L'_0, L'_{i+1}, \ldots, L'_n \in \overline{k}[X_0, \ldots, X_r]$ be linear forms and $e'_w = L'_w(e_0, \ldots, e_r)$ for all w. Denote by $p_{E,L'} : E \setminus \mathcal{Z}(e'_0, e'_{i+1}, \ldots, e'_n) \to \mathbb{P}^{n-i}(\overline{k})$, $(X_0 : \ldots : X_n) \mapsto (e'_0 : e'_{i+1}, \ldots, e'_n)$, the regular morphism. Let E_1 be an arbitrary irreducible over $\overline{k}$ component of E. Denote by E'_1 (respectively E''_1) the closure with respect to Zariski topology of the image $p(E_1 \setminus \mathcal{Z}(e_0, \ldots, e_r))$ (respectively $p_{E,L'}(E_1 \setminus \mathcal{Z}(e'_0, e'_{i+1}, \ldots, e'_n))$). Then the following assertions are equivalent.*

(a) $E \cap \mathcal{Z}(e'_0, e'_{i+1}, \ldots, e'_n) = E \cap \mathcal{Z}(e_0, \ldots, e_r)$ in $\mathbb{P}^n(\overline{k})$, and for every irreducible component E_1 of E the dimension $\dim E''_1 = \dim E'_1$.

(b) $(E \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_0, e'_{i+1}, \ldots, e'_n) = \emptyset$ in $\mathbb{P}^n(\overline{k})$, and for every irreducible component E_1 of E the dimension $\dim E''_1 = \dim E'_1$.

(c) $E' \cap \mathcal{Z}(L'_0, L'_{i+1}, \ldots, L'_n) = \emptyset$ in $\mathbb{P}^r(\overline{k})$.

Further, denote by $\mathcal{U}_{E,i}$ the subset of all $(n-i+1)$–tuples $(L'_0, L'_{i+1}, \ldots, L'_n) \in \mathbb{A}^{(r+1)(n-i+1)}(\overline{k})$ satisfying (a), (b) and (c). Then $\mathcal{U}_{E,i}$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(n-i+1)}(\overline{k})$.

PROOF Obviously (a) and (b) are equivalent, and (c) implies (a). Let us show that (a) implies (c). We shall use the induction on $j = n-i$. If $j = 0$ the assertions of the lemma are trivial. The case $j = 1$ will be considered separately. If $j > 1$ then at the inductive step we reduce the case of j to the one of $j-1$. For an arbitrary i we shall suppose without loss of generality that

E is irreducible over $\overline{k}$. Intersecting E with an appropriate linear subspace of codimension $\dim E - \dim E'$ in $\mathbb{P}^n(\overline{k})$ we shall assume without loss of generality that $\dim E = \dim E' = n - i_1$ (here we leave the details to the reader). Replacing $L'_0, L'_{i+1}, \ldots, L'_n$ by $n - i_1 + 1$ appropriate linear combinations of these linear forms, we shall assume without loss of generality that (a) holds with i_1 in place of i. So we shall suppose without loss of generality at the beginning of the inductive step (and also if $j = 1$) that $i_1 = i$ and E is irreducible over $\overline{k}$. Hence the morphism $p_{E,L'}$ is dominant. At the end of the inductive step one can obtain again the general case when E is not necessarily irreducible over $\overline{k}$.

Suppose that $j = 1$. Let $\xi' \in E' \cap \mathcal{Z}(L'_0, L'_n)$. We shall assume without loss of generality that $X_0(\xi) \neq 0$. Let $\widetilde{E}$ be the normalization of E. Then, see, e.g., [10], we have the dominant regular morphism $\widetilde{p} : \widetilde{E} \to E'$ which is a composite of the natural projection $\pi : \widetilde{E} \to E$ and the morphism $E \setminus \mathcal{Z}(e_0, \ldots, e_r) \to E'$ induced by p. Since $\widetilde{E}$ and E are complete varieties $\widetilde{p}(\widetilde{E}) = E'$. Therefore, there is $\widetilde{\xi} \in \widetilde{E}$ such that $\widetilde{p}(\widetilde{\xi}) = \xi'$. Set $\xi = \pi(\widetilde{\xi})$. Then obviously $\xi \in E \cap \mathcal{Z}(e'_0, e'_n) \setminus \mathcal{Z}(e_0)$. This contradiction proves the lemma for $j = n - i = 1$.

Suppose that $j > 1$, and $E' \cap \mathcal{Z}(L'_0, L'_{i+1}, \ldots, L'_n) = F \neq \emptyset$. Suppose that $\dim F < n - i - 1$ (respectively $\dim F = n - i - 1$). Let L''_{i+1} be a linear combination of $L'_0, L'_{i+1}, \ldots, L'_n$ (respectively $X_0, \ldots, X_r$) with coefficients from $\overline{k}$. Denote $e''_{i+1} = L''_{i+1}(e_0, \ldots, e_r)$. Let us choose L''_{i+1} such that $\dim E' \cap \mathcal{Z}(L''_{i+1}) = \dim E \cap \mathcal{Z}(e''_{i+1}) = n - i - 1$; the morphism $E \cap \mathcal{Z}(e''_{i+1}) \to E' \cap \mathcal{Z}(L''_{i+1})$ induced by p is dominant; the morphism $E \cap \mathcal{Z}(e''_{i+1}) \to \mathcal{Z}(L''_{i+1}) = \mathbb{P}^{n-i-1}(\overline{k})$ induced by $p_{E,L'}$ is dominant; and if $\dim F = n - i - 1$ then $\dim F \cap \mathcal{Z}(L''_{i+1}) = n - i - 2$ (these are conditions of general position for L''_{s+1} in the both considered cases). Notice that $E \cap \mathcal{Z}(e''_{i+1}) \cap \mathcal{Z}(e_0, \ldots, e_r) = E \cap \mathcal{Z}(e''_{i+1}) \cap \mathcal{Z}(e'_0, e'_{i+1}, \ldots, e'_n)$ and $E' \cap \mathcal{Z}(L''_{i+1}) \cap \mathcal{Z}(L'_0, L'_{i+1}, \ldots, L'_n) \neq \emptyset$. There are appropriate linear combinations $L''_0, L''_{i+2}, \ldots, L''_n$ of $L'_0, L'_{i+1}, \ldots, L'_n$ such that (a) holds with $i + 1$, $E \cap \mathcal{Z}(e''_{i+1})$ and $L''_0, L''_{i+2}, \ldots, L''_n$ in place of i, E and $L'_0, L'_{i+1}, \ldots, L'_n$. Further, $E' \cap \mathcal{Z}(L''_{i+1}) \cap \mathcal{Z}(L''_0, L''_{i+2}, \ldots, L''_n) \neq \emptyset$. Thus, by induction we get a contradiction. Therefore, (a) implies (c). The equivalence of (a), (b) and (c) is proved.

Now the last assertion of the lemma follows from [11] (and also [6] Section 7). The lemma is proved.

LEMMA 9 (A CRITERION FOR DOMINANTNESS OF A MORPHISM) *Let $\pi : V_1 \to V_2$ be a defined over k morphism of defined over k algebraic varieties. Suppose that all the irreducible components of V_1 have the same dimension n_1, and V_2 is an irreducible over k algebraic variety of dimension n_2. Let $n_1 \geq n_2$. Let $z \in V_2$. Suppose that for every irreducible over k component E of V_1 there is an irreducible over k (or which is the equivalent irreducible over $\overline{k}$) component E_1 of the fiber $\pi^{-1}(z)$ such that $E_1 \subset E$ and $\dim E_1 = n_1 - n_2$. Then the morphism π is dominant. In particular, if V_1 is irreducible over k and $\dim \pi^{-1}(z) = n_1 - n_2$ then the morphism π is dominant.*

PROOF This follows from the theorem about dimensions of fibers, see [12] and also the Introduction.

LEMMA 10 *Let $\sigma + 1 \leq \alpha \leq n + 1$. Then the set of all $(L'_{\sigma+1}, \ldots, L'_\alpha) \in \mathcal{U}'_\alpha$ such that $J^{(1)}(L'_{\sigma+1}, \ldots, L'_i) = J_i^{(1)}$ for all $\sigma + 1 \leq i \leq \alpha$ is a nonempty open in the Zariski topology subset of $\mathcal{U}'_\alpha$.*

PROOF Let $(L'_{\sigma+1},\dots,L'_{\alpha})\in\mathcal{U}'_{\alpha}$. Let us extend the ground field k till $k_{\overline{v}}$, see the Introduction. Let $\mathcal{L}=\mathcal{Z}(v_{s+1},\dots,v_{n-\alpha+\sigma+1})\subset\mathbb{P}^n(\overline{k_{\overline{v}}})$ be a generic linear subspace. Hence the intersection of $\mathcal{L}$ and

$$W(L'_{\sigma+1},\dots,L'_{\alpha-1})\cup(W\cap\mathcal{Z}(e_0,\dots,e_r))$$

is transversal. In particular, by Theorem 1 [9] the number of elements $\#(W(L'_{\sigma+1},\dots,L'_{\alpha-1})\cap\mathcal{L})<+\infty$. Further, e'_{α} does not vanish at any point from $W(L'_{\sigma+1},\dots,L'_{\alpha-1})\cap\mathcal{L}$.

Set $E=W\cap\mathcal{L}$. The dimension of every irreducible component of E is $\alpha-\sigma-1$. As an easy consequence of Theorem 1 [9] the algebraic variety $W(L'_{\sigma+1},\dots,L'_{\alpha-1})$ has a nonempty intersection with every irreducible over $\overline{k}$ component of W. Hence the same is true for the algebraic variety $W(L'_{\sigma+1},\dots,L'_{\alpha-1})\cap\mathcal{L}$. Therefore, by Lemma 9 the morphism $E\setminus\mathcal{Z}(e'_{\sigma+1},\dots,e'_{\alpha})\to\mathbb{P}^{\alpha-\sigma-1}(\overline{k_{\overline{v}}})$, $(X_0:\ldots:X_n)\mapsto(e'_{\sigma+1}:\ldots:e'_{\alpha})$, is dominant. Let us replace in the statement of Lemma 8 element $(L'_0,L'_{i+1},\dots,L'_n)$ by $(L'_{\sigma+1},\dots,L'_{\alpha})$ (hence $n-i+1$ is replaced by $\alpha-\sigma-1$) Then condition (a) of Lemma 8 holds with such a replacement if and only if for every $\sigma+1\le i\le\alpha$ for every $j\in J_i^{(1)}$ the algebraic variety W_j is an irreducible component of $W\cap\mathcal{Z}(e'_{\sigma+1},\dots,e'_{\alpha})$. Now the required assertion follows from Lemma 8. The lemma is proved.

LEMMA 11 *In the notation from the Introduction let $L'=(L'_0,L'_{\sigma+1}\dots,L'_n)$, where all $L'_j\in\overline{k}[X_0,\dots,X_n]$ are linear forms. Let $j\in J_i^{(1)}(L')\cap J_i^{(1)}$ (i.e., W_j is an irreducible component of $W_i^{(1)}(L')$ and $W\cap\mathcal{Z}(e_0,\dots,e_r)$ simultaneously) for some $\sigma+1\le i\le n+1$. Then the index of intersection $\iota_j(L')\ge\iota_j$, see (7). Further, let $\sigma+1\le\alpha\le n+1$ be an integer. Denote by $\mathcal{U}_{\alpha}^{(\mathrm{IV})}\subset\mathcal{U}'_{\alpha}$ the subset of all $(L'_{\sigma+1},\dots,L'_{\alpha})$ such that $J_i^{(1)}(L'_{\sigma+1},\dots,L'_{\alpha})=J_i^{(1)}$ for all $\sigma+1\le i\le\alpha$ and $\iota_j(L'_{\sigma+1},\dots,L'_i)=\iota_j$ for all $j\in J_i^{(1)}$, $\sigma+1\le i\le\alpha$. Then $\mathcal{U}_{\alpha}^{(\mathrm{IV})}$ is a defined over k nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$. Finally, for any line $l\subset\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$ such that $l\cap\mathcal{U}_{\alpha}^{(\mathrm{IV})}\ne\emptyset$ the number of elements $\#(l\setminus\mathcal{U}_{\alpha}^{(\mathrm{IV})})$ is bounded from above by a polynomial in $d^n(d')^{n-\sigma}$.*

PROOF Set $L'_{n+1}=L'_0$ and the coefficients $l'_{n+1,\nu}=l'_{0,\nu}$ for all $0\le\nu\le r$, see the Introduction.

Let $\mathbb{A}^{(r+1)(i-\sigma)}(\overline{k})$ has coordinate functions $Z_{\beta,\nu}$, $\sigma+1\le\beta\le i$, $0\le\nu\le r$. Let $z'_i\in\mathbb{A}^{(r+1)(i-\sigma)}(\overline{k})$ (respectively $z_i\in\mathbb{A}^{(r+1)(i-\sigma)}(\overline{K})$) be the point with the coordinates $Z_{\beta,\nu}=l'_{\beta,\nu}$ (respectively $Z_{\beta,\nu}=u_{\beta,\nu}$) for all β, ν, where $l'_{\beta,\nu}$ (respectively $u_{\beta,\nu}$) are the coefficients of the linear forms L'_{ν} (respectively U_{ν}), see the Introduction. Consider the following closed algebraic subvarieties of $\mathbb{P}^n(\overline{K})\times\mathbb{A}^{(r+1)(i-\sigma)}(\overline{K})$:

$$\begin{aligned}
E^{(1)}&=W\times\mathbb{A}^{(r+1)(i-\sigma)}(\overline{k}),\\
E^{(2)}&=\mathcal{Z}\Big(\sum_{0\le\nu\le r}Z_{\beta,\nu}e_{\nu},\quad\sigma+1\le\beta\le i\Big),\\
E^{(3)}&=\mathcal{Z}(Z_{\beta,\nu}-l'_{\beta,\nu},\quad\sigma+1\le\beta\le i,\ 0\le\nu\le r\,),\\
E^{(4)}&=\mathcal{Z}(Z_{\beta,\nu}-u_{\beta,\nu},\quad\sigma+1\le\beta\le i,\ 0\le\nu\le r\,).
\end{aligned}$$

Notice that $E^{(\gamma)}$ for $\gamma=1,2,3$ are defined over $\overline{k}$. The variety $W_j\times\{z'_i\}$, $j\in J_i^{(1)}(L')$, is an irreducible component of the intersection $E^{(1)}\cap E^{(2)}\cap E^{(3)}$

and the codimension of $W_j \times \{z'_i\}$ is the sum of the codimensions of $E^{(1)}$, $E^{(2)}$, $E^{(3)}$ (all the codimensions are considered in $\mathbb{P}^n(\overline{K}) \times \mathbb{A}^{(r+1)(i-\sigma)}(\overline{K})$), i.e., the last intersection is proper at $W_j \times \{z'_i\}$.

For arbitrary algebraic varieties V', V'' let $i(V', V''; V''')$ be the index of intersection of algebraic varieties V' and V'' at an irreducible component V''' of $V' \cap V''$ when this index is defined, see [10]. Denote by W_γ, $\gamma \in \Gamma_i$, the family of all irreducible over $\overline{k}$ components of the intersection $E^{(1)} \cap E^{(2)}$. Let $\Gamma_j \subset \Gamma_i$ be the subset of all γ such that $W_\gamma \supset W_j \times \{z'_i\}$. There is $\gamma_j \in \Gamma_j$ such that $W_{\gamma_j} = W_j \times \mathbb{A}^{(r+1)(i-\sigma)}(\overline{k})$. Denote by W_γ, $\gamma \in \Gamma'_j$ the family of all irreducible component of $\mathcal{Z}(e'_{\sigma+1}, \dots, e'_i)$ such that $W_\gamma \supset W_j$. Further, since z_i is a generic point of $\mathbb{A}^{(r+1)(i-\sigma)}(\overline{k})$, the algebraic variety $W_j \times \mathbb{A}^{(r+1)(i-\sigma)}(\overline{K})$ is the unique irreducible component E of $E^{(1)} \cap E^{(2)}$ such that E contains $W_j \times \{z_i\}$.

Now, see [10], by the properties of indices of intersection (mainly by the one corresponding to the associativity of intersection) we have

$$
\begin{aligned}
&i(W, \mathcal{Z}(e'_{\sigma+1}, \dots, e'_i); W_j) = i(E^{(1)}, \mathcal{Z}(e'_{\sigma+1}, \dots, e'_i) \times \{z'_i\}; W_j \times \{z'_i\}) = \\
&\sum_{\gamma \in \Gamma'_j} i(E^{(1)}, W_\gamma \times \{z'_i\}; W_j \times \{z'_i\}) i(E^{(2)}, E^{(3)}; W_\gamma \times \{z'_i\}) = \\
&\sum_{\gamma \in \Gamma_j} i(E^{(1)}, E^{(2)}; W_\gamma) i(W_\gamma, E^{(3)}; W_j \times \{z'_i\}) \geq \\
&i(E^{(1)}, E^{(2)}; W_j \times \mathbb{A}^{(r+1)(i-\sigma)}(\overline{k})) \times \\
&i(W_j \times \mathbb{A}^{(r+1)(i-\sigma)}(\overline{k}), E^{(3)}; W_j \times \{z'_i\}) = \\
&i(E^{(1)}, E^{(2)}; W_j \times \mathbb{A}^{(r+1)(i-\sigma)}(\overline{k})) = i(E^{(1)}, E^{(2)}; W_j \times \mathbb{A}^{(r+1)(i-\sigma)}(\overline{K})) \times \\
&i(W_j \times \mathbb{A}^{(r+1)(i-\sigma)}(\overline{K}), E^{(4)}; W_j \times \{z_i\}) = \\
&i(E^{(1)}, \mathcal{Z}(u_{\sigma+1}, \dots, u_i) \times \{z_i\}; W_j \times \{z_i\}) = i(W, \mathcal{Z}(u_{\sigma+1}, \dots, u_i); W_j).
\end{aligned}
$$

Let $(L'_{\sigma+1}, \dots, L'_\alpha) \in \mathcal{U}^{(\mathrm{IV})}_\alpha$ (respectively $L' \in \mathcal{U}^{(\mathrm{IV})}_{n+1}$) and for $\sigma+1 \leq i \leq \alpha$. Then $\Gamma_j = \{\gamma_j\}$ for all $j \in J^{(1)}_i$ and $\sigma+1 \leq i \leq \alpha$. Therefore, for every $j \in J^{(1)}_i$ for every $\sigma+1 \leq i \leq \alpha$ there is $w_j \in W_j$ such that $(w_j, z'_i) \notin W_\gamma$ for every $\gamma \in \Gamma_i \setminus \{\gamma_j\}$. Denote by $p_{\alpha,i} : \mathcal{U}'_\alpha \to \mathcal{U}'_i$ the natural projection. The set of $z \in \mathcal{U}'_\alpha$ such that $(w_j, p_{\alpha,i}(z)) \notin W_\gamma$ for all $\gamma \in \Gamma_i \setminus \{\gamma_j\}$, $j \in J^{(1)}_i$, $\sigma+1 \leq i \leq \alpha$ is a neighborhood with respect to the Zariski topology of z'_α. This neighborhood is contained in $\mathcal{U}^{(\mathrm{IV})}_\alpha$. If one extends the ground field k till K then $(U_{\sigma+1}, \dots, U_\alpha) \in \mathcal{U}^{(\mathrm{IV})}_\alpha$. Hence $\mathcal{U}^{(\mathrm{IV})}_\alpha \neq \emptyset$. Obviously $\mathcal{U}^{(\mathrm{IV})}_\alpha$ is invariant with respect to the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$ and, therefore, is defined over k.

To prove the last assertion we shall suppose that $(L'_{\sigma+1}, \dots, L'_\alpha) \in \mathfrak{l} \cap \mathcal{U}^{(\mathrm{IV})}_\alpha$. By the Bézout theorem $\sum_{\gamma \in \Gamma_i} \deg W_\gamma \leq d^s (d')^{i-\sigma}$ for $\sigma+1 \leq i \leq \alpha$. Therefore, by Theorem 1 (d) [9] there is at most a polynomial in $d^n (d')^{n-\sigma}$ number of points $z \in \mathfrak{l}$ such that $z \notin \mathcal{U}'_\alpha$ or $(w_j, p_{\alpha,i}(z)) \in \bigcup_{\gamma \in \Gamma_i \setminus \{\gamma_j\}} W_\gamma$ for some $j \in J^{(1)}_i$, $\sigma+1 \leq i \leq \alpha$. The lemma is proved.

LEMMA 12 *In the notation from the Introduction let $L' = (L'_0, L'_{\sigma+1} \dots, L'_n)$, where all $L'_j \in \overline{k}[X_0, \dots, X_n]$ are linear forms. Then $D_i(L'_{\sigma+1}, \dots, L'_i) \leq D_i$ for $i \leq n$, and $\iota_j(L') = \iota_j$ for every $j \in J^{(1)}_{n+1}$ (recall that $D_{n+1}(L') = D_{n+1} = 0$ and $J^{(1)}_{n+1}(L') \subset J^{(1)}_{n+1}$, see the Introduction). Let $j \in J^{(1)}_i(L') \cap J^{(1)}_i$ (i.e., W_j*

is an irreducible component of $W_i^{(1)}(L')$ *and* $W \cap \mathcal{Z}(e_0, \dots, e_r)$ *simultaneously) for some* $\sigma+1 \le i \le n$ *and* $\iota_j(L'_{\sigma+1}, \dots, L'_i) > \iota_j$. *Then* $D(L'_{\sigma+1}, \dots, L'_i) < D_i$.

PROOF We shall suppose without loss of generality that the coefficients of the linear forms $U_{\sigma+1}, \dots, U_\alpha$ belong to a field with a real structure (i.e., the field $\overline{K}$ is supplied with a real structure) and are infinitely close to the corresponding coefficients of linear forms $L'_{\sigma+1}, \dots, L'_\alpha$. Obviously condition (8) holds for U and K (in place of L' and k) and hence (10) for U and K is satisfied. Now by the definitions and Lemma 7 the inclusion $\mathrm{st}(W_i(U_{\sigma+1}, \dots, U_i)) \supset W_i(L'_{\sigma+1}, \dots, L'_i)$ holds. Hence by Lemma 5 the degree $D_i \ge D_i(L'_{\sigma+1}, \dots, L'_i)$.

Further, let $j \in J_i^{(1)}$ and $\sigma+1 \le i \le n+1$. Assume that $\iota_j(L'_{\sigma+1}, \dots, L'_i) > \iota_j$ if $i \le n$ and $\iota_j(L') > \iota_j$ if $i = n+1$. We shall use the notation from the proof of Lemma 11. There is $\gamma \in \Gamma_j \setminus \{\gamma_j\}$, see the proof of Lemma 11. Set $\widetilde{W} \times \{z_i\} = W_\gamma \cap \mathcal{Z}(Z_{\beta,\nu} - u_{\beta,\nu},\ 0 \le \nu \le r,\ \sigma+1 \le \beta \le i)$, where $\widetilde{W} \subset \mathbb{P}^n(\overline{K})$ and $\dim \widetilde{W} = n - i$. Then $\widetilde{W} \ne W_j$ since, otherwise, $\gamma = \gamma_j$. Obviously $\widetilde{W}$ is irreducible over $\overline{k}(u)$ and $\dim \widetilde{W} = n - i + \sigma - s$.

Put $\iota = i - \sigma + s$. Let us choose linear forms $L_0, L_{\iota+1}, \dots, L_n \in k[X_0, \dots, X_n]$ (this family of linear forms depend on i; it will not lead to an ambiguity) such that the intersection $W_\gamma \cap \mathcal{Z}(Z_{\beta,\nu} - l'_{\beta,\nu},\ 0 \le \nu \le r,\ \sigma+1 \le \beta \le i)$ and $\mathcal{Z}(L_{\iota+1}, \dots, L_n)$ is transversal,

$$\mathrm{st}(\widetilde{W}) \cap \mathcal{Z}(L_0, L_{\iota+1}, \dots, L_n) = \emptyset, \tag{22}$$

$$W(L'_{\sigma+1}, \dots, L'_{i-1}) \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_0, L_{\iota+1}, \dots, L_n) = \emptyset, \tag{23}$$

$$W(L'_{\sigma+1}, \dots, L'_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n) \cap (\mathcal{Z}(e_0, \dots, e_r) \cup (\overline{V \setminus W})) = \emptyset, \tag{24}$$

$$\#(W(L'_{\sigma+1}, \dots, L'_{i-1}) \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)) = \deg(W(L'_{\sigma+1}, \dots, L'_{i-1}) \cap \mathcal{Z}(e'_i)) \tag{25}$$

(these are conditions of general position for linear forms $L_0, L_{\iota+1}, \dots, L_n$; some of these conditions will be used later in the proof of next lemmas).

Let us apply Lemma 7 and Lemma 5 to the affine algebraic varieties $W_\gamma \setminus \mathcal{Z}(L_0)$ and $\mathcal{Z}(Z_{\beta,\nu} - u_{\beta,\nu},\ 0 \le \nu \le r, \sigma+1 \le \beta \le i) \setminus \mathcal{Z}(L_0)$. We get $\mathrm{st}(\widetilde{W}) \times \{z'_i\} = W_\gamma \cap \mathcal{Z}(Z_{\beta,\nu} - l'_{\beta,\nu},\ 0 \le \nu \le r, \sigma+1 \le \beta \le i) \supset W_j \times \{z'_i\}$. Hence $\mathrm{st}(\widetilde{W}) \supset W_j$. Further, $\mathrm{st}(\widetilde{W} \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)) = \mathrm{st}(\widetilde{W}) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n) \supset W_j \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)$ by Lemma 7 and (22). Let us choose points $\xi_j \in W_j \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)$ and $\widetilde{\xi}_j \in \widetilde{W} \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)$ such that $\mathrm{st}(\widetilde{\xi}_j) = \xi_j$.

Now $\mathrm{st}(\widetilde{W}) \supset W_j$, $\widetilde{W} \ne W_j$, and $\dim \widetilde{W} = \dim W_j$, see above. Therefore, $\widetilde{W} \not\subset \mathcal{Z}(e_0, \dots, e_r)$ in $\mathbb{P}^n(\overline{K})$ since W_j is an irreducible component of $W \cap \mathcal{Z}(e_0, \dots, e_r)$. Hence if $i = n+1$ we get a contradiction, which implies that $\iota_j(L') = \iota_j$. In what follows $i \le n$ in the proof. Since condition (10) is satisfied for U and K, the variety $\widetilde{W}$ is a union of some components of $W(U_{\sigma+1}, \dots, U_i)$ irreducible over $\overline{K}$.

We have proved that $\mathrm{st}(W(U_{\sigma+1}, \dots, U_i)) \supset W(L'_{\sigma+1}, \dots, L'_i)$. Hence $\mathrm{st}(W(U_{\sigma+1}, \dots, U_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)) \supset W(L'_{\sigma+1}, \dots, L'_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)$ by Lemma 5 and Lemma 7. Hence for every point $\xi \in W(L'_{\sigma+1}, \dots, L'_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)$ there is $\widetilde{\xi} \in W(U_{\sigma+1}, \dots, U_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)$ such that $\widetilde{\xi}$ is infinitely close to ξ. Notice that $\mathrm{st}(\widetilde{\xi}_j) = \xi_j \notin W(L'_{\sigma+1}, \dots, L'_i)$ by (24).

Hence by the Bézout theorem $D_i > D(L'_{\sigma+1}, \ldots, L'_i)$ if $i \le n$. The lemma is proved.

LEMMA 13 *Let $\sigma+1 \le \alpha \le n+1$ be an integer, $\mathcal{U}_\alpha^{(\mathrm{IV})}$ be the set from the statement of Lemma 11 and $\mathcal{U}''_\alpha$ the set defined in the Introduction. Then $\mathcal{U}''_\alpha$ is a defined over k nonempty open in the Zariski topology subset of $\mathcal{U}_\alpha^{(\mathrm{IV})}$. Further, let $\mathfrak{l} \subset \mathbb{A}^{(r+1)(\alpha-s)}(\overline{k})$ be a line such that $\mathfrak{l} \cap \mathcal{U}''_\alpha \ne \emptyset$. Then the number of elements $\#(\mathfrak{l} \setminus \mathcal{U}''_\alpha)$ is bounded from above by a polynomial in $d^n(d')^{n-\sigma}$.*

PROOF Recall that $D_n = D_n(L')$ for every $L' \in \mathcal{U}'_0$ and $D_{n+1}(L') = D_{n+1} = 0$, see the Introduction. Hence it is sufficient to prove the lemma for $\alpha \le n-1$. Recall that V_v and W_v are defined, see the Introduction. Replacing if necessary (V, W, k) by (V_v, W_v, k_v) we shall suppose without loss of generality in what follows in the proof that $s = \sigma$.

Now there are linear forms $L_0, L_{s+2}, \ldots, L_n \in k[X_0, \ldots, X_r]$ such that for every $s+1 \le i \le \alpha$ conditions (23)–(25) hold and additionally

- the intersection $W \cap \mathcal{Z}(L_{i+1}, \ldots, L_n)$ is transversal and is an defined over k and irreducible over k algebraic variety.

The set U''_α is nonempty since if one extends the ground field k till K then $(U_{s+1}, \ldots, U_\alpha) \in \mathcal{U}''_\alpha(\overline{K})$. Now the last two assertions of the lemma follows from Theorem 1 (d) [9] applied to the morphisms

$$W \cap \mathcal{Z}(L_{i+1}, \ldots, L_n) \to \mathbb{P}^r(\overline{k}),$$
$$(X_0 : \ldots : X_n) \mapsto (e_0 : \ldots : e_r), \quad s+1 \le i \le \alpha,$$

where $W \cap \mathcal{Z}(L_{i+1}, \ldots, L_n)$ is considered as an irreducible component the algebraic variety $V \cap \mathcal{Z}(L_{i+1}, \ldots, L_n)$ (in place of the component W of the variety V). The lemma is proved.

LEMMA 14 *Let $\sigma+1 \le \alpha \le n+1$ be an integer. Let $(L'_0, L'_{\sigma+1}, \ldots, L'_n) \in \mathcal{U}'_0$ and $(L'_{\sigma+1}, \ldots, L'_{\alpha-1}) \in \mathcal{U}''_{\alpha-1}$. Let $\mathfrak{l} \subset \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$ be a line such that $(L'_0, L'_{\sigma+1}, \ldots, L'_n) \in \mathfrak{l}$. Then there is at most a polynomial in $d^n(d')^{n-\sigma}$ number of elements $L'' = (L''_0, L''_{\sigma+1}, \ldots, L''_n) \in \mathfrak{l}$ such that at least one of the following three conditions fails*

- $(L''_0, L''_{\sigma+1}, \ldots, L''_n) \in \mathcal{U}'_0$,
- $(L''_{\sigma+1}, \ldots, L''_{\alpha-1}) \in \mathcal{U}''_{\alpha-1}$,
- $D_\alpha(L'') \ge D_\alpha(L')$.

PROOF We shall assume without loss of generality that $\alpha \le n$. Similarly to the proof of Lemma 13 we shall suppose without loss of generality that $s = \sigma$. Now the required assertion follows from Lemma 13, assertion (d) Theorem 1 [9], and Lemma 13 [9] The lemma is proved.

LEMMA 15 *Let $\sigma+1 \le \alpha \le n+1$ be an integer and $(L'_{\sigma+1}, \ldots, L'_\alpha) \in \mathcal{U}''_\alpha$. Let the field K' and the mapping of the standard part* st *be from Section 2. Let us extend the ground field k till K'. Let $L''_{\sigma+1}, \ldots, L''_\alpha \in \overline{K'}[X_0, \ldots, X_r]$ be linear forms such that all the nonzero coefficients of linear forms $L''_j - L'_j$,*

$\sigma+1\le j\le\alpha$ are infinitesimal with respect to the field k. Put $L''_0=L''_{n+1}$ and $L''=(L''_0,L''_{\sigma+1},\dots,L''_n)$ if $\alpha=n+1$. Then for all $\sigma\le i\le\alpha$

$$\mathrm{st}(W(L''_{\sigma+1},\dots,L''_i))=W(L'_{\sigma+1},\dots,L'_i), \tag{26}$$

$$\mathrm{st}(W^{(2)}(L''_{\sigma+1},\dots,L''_i))=W^{(2)}(L'_{\sigma+1},\dots,L'_i). \tag{27}$$

PROOF By Lemma 13 and transfer principle $(L''_{\sigma+1},\dots,L''_\alpha)\in\mathcal{U}''_\alpha$. Hence $\deg W(L''_{\sigma+1},\dots,L''_i)=\deg W(L'_{\sigma+1},\dots,L'_i)=D_i$ for all $\sigma\le i\le\alpha$.

By Lemma 5 and Lemma 7 the inclusion $\mathrm{st}(W(L''_{\sigma+1},\dots,L''_i))\supset W(L'_{\sigma+1},\dots,L'_i)$ holds. Hence (26) is fulfilled by Lemma 5 since $\deg W(L''_{\sigma+1},\dots,L''_i)=\deg W(L'_{\sigma+1},\dots,L'_i)$.

In particular we have $\mathrm{st}(W(L''_{\sigma+1},\dots,L''_{i-1},L'_i))=W(L'_{\sigma+1},\dots,L'_i)$ and $\mathrm{st}(W(L''_{\sigma+1},\dots,L''_{i-1})=W(L'_{\sigma+1},\dots,L'_{i-1})$. Further, $W^{(1)}(L''_{\sigma+1},\dots,L''_{i-1},L'_i)=W^{(1)}(L'_{\sigma+1},\dots,L'_i)$, see the Introduction. Every irreducible component E of $W^{(2)}(L'_{\sigma+1},\dots,L'_i)$ (respectively of $W^{(2)}(L''_{\sigma+1},\dots,L''_i)$) is an irreducible component of $W(L'_{\sigma+1},\dots,L'_{i-1})\cap\mathcal{Z}(e'_i)$ (respectively of $W(L''_{\sigma+1},\dots,L''_{i-1})\cap\mathcal{Z}(e'_i)$) such that $E\subset W\cap\mathcal{Z}(e_0,\dots,e_r)$ and E is not an irreducible component of $W\cap\mathcal{Z}(e_0,\dots,e_r)$. Now (27) follows from Lemma 6 and the previous remarks. The lemma is proved.

LEMMA 16 *Let $\sigma+1\le\alpha\le n+1$ be an integer. Let $(L'_0,L'_{\sigma+1},\dots,L'_n)\in\mathcal{U}''_0$ and $(L'_{\sigma+1},\dots,L'_{\alpha-1})\in\mathcal{U}'''_{\alpha-1}$. Let $\mathfrak{l}\subset\mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$ be a line such that $(L'_0,L'_{\sigma+1},\dots,L'_n)\in\mathfrak{l}$. Then there are at most a polynomial in $d^n(d')^{n-\sigma}$ number of elements $L''=(L''_0,L''_{\sigma+1},\dots,L''_n)\in\mathfrak{l}$ such that at least one of the following three conditions fails:*

- $(L''_0,L''_{\sigma+1},\dots,L''_n)\in\mathcal{U}''_0$,
- $(L''_{\sigma+1},\dots,L''_{\alpha-1})\in\mathcal{U}'''_{\alpha-1}$,
- $D^{(2)}_\alpha(L'')\ge D^{(2)}_\alpha(L')$.

In particular, $D^{(2)}_\alpha(L')\le D^{(2)}_\alpha$. The set $\mathcal{U}'''_\alpha$ is a defined over k nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$.

PROOF Let the field K' and the mapping st be as above. Let us extend the ground field k till K'. Suppose at first that $(L''_{\sigma+1},\dots,L''_\alpha)\in\mathcal{U}''_\alpha$, and all the nonzero coefficients of the linear forms $L''_j-L'_j$, $\sigma+1\le j\le\alpha$ are infinitesimal with respect to the field k. Then by Lemma 15, Lemma 5 and (27) the degree $D^{(2)}_\alpha(L')\le D^{(2)}_\alpha(L'')$.

Now suppose that $K'=K$ and the family of all the coefficients of linear forms L''_j, $\sigma+1\le j\le\alpha$, is algebraically independent over k. We get $D^{(2)}_\alpha(L')\le D^{(2)}_\alpha$. Further, the set $\mathcal{U}'''_\alpha$ is nonempty since $(U_{\sigma+1},\dots,U_\alpha)\in\mathcal{U}'''_\alpha(\overline{K})$ if $\alpha\le n$ (respectively $U\in\mathcal{U}'''_\alpha$ if $\alpha=n+1$).

It is not difficult to prove that one can define $\mathcal{U}'''_\alpha$ by a formula with quantifiers of the first order theory of algebraically closed fields (we leave the details to the reader). Hence $\mathcal{U}'''_\alpha$ is a constructive set, see the definition in [10]. As we have proved each point $(L'_{\sigma+1},\dots,L'_\alpha)\in\mathcal{U}'''_\alpha$ has an infinitely small with respect to k neighborhood $\mathcal{W}_0$ in $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{K})$ such that $\mathcal{W}_0\subset\mathcal{U}'''_\alpha$. Obviously that for the ground field $\mathbb{C}$ any constructive set $\mathcal{W}$ of an affine space is an open

in the Zariski topology subset of this affine space, provided that $\mathcal{W}$ contains a neighborhood in the classic topology of any its point. Now it follows from the transfer principle, see [1], that in our situation $\mathcal{U}'''_\alpha$ is an open in the Zariski topology subset of $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$.

It remains to prove the first statement of the lemma. Let $\varepsilon > 0$ be an infinitesimal with respect to the field k. Now suppose that $K' = \overline{k(\varepsilon)}$. By the induction on α and Lemma 14 it is sufficient to prove that $D^{(2)}_\alpha(L'') < D^{(2)}_\alpha(L')$ for at most a polynomial in $d^n(d')^{n-\sigma}$ number of points $L'' \in \mathfrak{l}$. Let $L' = (L'_{\sigma+1}, \ldots, L'_n) \in \mathfrak{l}$ and $(L'_{\sigma+1}, \ldots, L'_{\alpha-1}) \in \mathcal{U}'''_{\alpha-1}$. Suppose that $\widetilde{L} = (\widetilde{L}_0, \widetilde{L}_{\sigma+1}, \ldots, \widetilde{L}_n) \in \mathfrak{l}(\overline{k(\varepsilon)})$, and $\widetilde{L}_j = L'_j + \varepsilon L'''_j$, where $L'''_j \in \overline{k}[X_0, \ldots, X_r]$ are linear forms for all $j \in \{0, \sigma+1, \ldots, n\}$. Hence $\widetilde{L}$ is a generic point of $\mathfrak{l}$.

Let linear forms $L_{\iota+1}, \ldots, L_n$ be as above, and $i = \alpha$, $\iota = \alpha - \sigma + s$. Therefore, (25) holds with $i = \alpha$. Hence by (25) and Lemma 6 applied $n - \alpha$ times the number of elements

$$+\infty > \#(W^{(2)}_\alpha(\widetilde{L}) \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)) \geq D^{(2)}_\alpha(L').$$

Let ξ be an arbitrary point from $W^{(2)}_\alpha(\widetilde{L}) \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$, and, say $X_0(\xi) \neq 0$. Using, e.g., the algorithm from [2] or directly one obtains that the nonhomogeneous coordinates $(X_w/X_0)(\xi) = P_w(\varepsilon, \theta)/P(\varepsilon, \theta)$, $1 \leq w \leq n$, where θ is an algebraic over $\overline{k}(\varepsilon)$ element with a minimal polynomial $0 \neq g \in \overline{k}[\varepsilon, Z]$, $g(\varepsilon, \theta) = 0$ and $P_w, P \in k[\varepsilon, Z]$. Further, the degrees with respect to ε, Z of all the polynomials P_w, P, g are bounded from above by a polynomial in $d^s(d')^{i-\sigma}$. Hence considering the specializations of the generic point we obtain by Theorem 1 from [9] (or directly) that the number of elements $L'' \in \mathfrak{l}$ such that $D^{(2)}_\alpha(L'') < D^{(2)}_\alpha(L')$ is bounded from above by a polynomial in $d^n(d')^{n-\sigma}$. The lemma is proved.

4 Construction of an element from $\mathcal{U}'_0$

In this section under conditions of Theorem 2 we shall construct an elements $L' \in \mathcal{U}'_0$ and λ satisfying assertion (a)–(d) of this theorem but with $\mathcal{U}'_0$ in place of $\mathcal{U}'''_0$ (obviously if the constructed element $L' \in \mathcal{U}'''_0$ then the original assertions (a)–(d) holds). Let $e'_j = L'_j(e_0, \ldots, e_r)$, $j \in \{0, \sigma+1, \ldots, n\}$, for linear forms $L'_j \in \overline{k}[X_0, \ldots, X_r]$ (in Lemma 17 and Lemma 19 below we shall consider also the case when L'_j have coefficients from $\overline{K}$; all the corresponding objects are defined in this case in the similar way with the replacement k by K). Let $\lambda = (L_{s+1}, \ldots, L_\sigma)$ where all $L_j \in \overline{K}[X_0, \ldots, X_n]$ are linear forms. Set $V_\lambda = V \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$, $W_\lambda = W \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ in $\mathbb{P}^n(\overline{K})$. Denote by $A_{\lambda,L'}$ the set of isolated points of $W_\lambda \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n) \setminus \mathcal{Z}(e_0, \ldots, e_r)$ in $\mathbb{P}^n(\overline{K})$. Let $N_{\lambda,L'} = \#A_{\lambda,L'}$ be the number of elements. Let us define $p_{\lambda,L'}$, (respectively p_λ, p'_λ) by (3), (respectively (14), (15)) replacing there the field k_v by K (respectively k by K). Recall that $q_v = p'_v$. Set as previously $q_\lambda = p'_\lambda$.

Let us replace in the definition of the morphism $p_{L'}$ the field $\overline{k}$ by $\overline{K}$, see (2). Let $\sigma \leq i \leq n$ be an integer. Similarly to the Introduction $W_i(L')$ is a union of all the irreducible over $\overline{K}$ components E of $W \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i)$ such that $\dim E = n - s + \sigma - i$ and the morphism $E \setminus \mathcal{Z}(e'_{i+1}, \ldots, e'_n) \to \mathcal{Z}(L'_{\sigma+1}, \ldots, L'_i) = \mathbb{P}^{n-i}(\overline{K})$ induced by $p_{L'}$ is dominant. For arbitrary λ the

variety $W_{\lambda,i}(L')$ is a union of all the irreducible over the algebraic closure of the ground field components E of $W_\lambda \cap \mathcal{Z}(e'_{\sigma+1},\dots,e'_i)$ such that $\dim E = n-i$ and the morphism $E \setminus \mathcal{Z}(e'_{i+1},\dots,e'_n) \to \mathcal{Z}(L'_{\sigma+1},\dots,L'_i)$ induced by $p_{\lambda,L'}$ is dominant (here $\mathcal{Z}(L'_{\sigma+1},\dots,L'_i)$ is a linear subspace of dimension $n-i$ in the projective space).

LEMMA 17 *Let $L'' \in \mathbb{A}^{(n-\sigma+1)(r+1)}(\overline{K})$ and $\lambda' = (L'_{s+1},\dots,L_\sigma) \in \mathbb{A}^{(\sigma-s)(n+1)}(\overline{K})$ where $L''_j \in \overline{K}[X_0,\dots,X_r]$, $j \in \{0,\sigma+1,\dots,n\}$, and $L'_i \in \overline{K}[X_0,\dots,X_n]$, $s+1 \le i \le \sigma$, are arbitrary linear forms. Then the following assertions hold.*

(i) $N_{\lambda',L''} \le \deg q_v \deg W' = D_n$.

(ii) Let $N_{\lambda',L''} = \deg q_v \deg W' = D_n$, and x be an arbitrary point from $A_{\lambda',L''}$. Put $e''_j = L''_j(e_0,\dots,e_r)$, $j \in \{0,\sigma+1,\dots,n\}$. Then x is a smooth point of each algebraic variety W, $\mathcal{Z}(L'_{s+1}),\dots,\mathcal{Z}(L'_\sigma)$, $\mathcal{Z}(e''_{\sigma+1}),\dots,\mathcal{Z}(e''_n)$, and the intersection of tangent spaces of these $n-s+1$ algebraic varieties at the point x is transversal, i.e., is $\{0\}$. In particular the intersection $W \cap \mathcal{Z}(L'_{s+1},\dots,L'_\sigma)$ is transversal.

PROOF We assume without loss of generality replacing if it is necessary the linear form L''_0 for a new one that $A_{\lambda',L''} \cap \mathcal{Z}(L''_0(e_0,\dots,e_r)) = \emptyset$.

Let us prove (i). We know then $D_n = \deg q_v \deg W'$, see the Introduction. Obviously the variety $W_n(L'')$ is the union of all the irreducible over $\overline{K}$ components E of $W \cap \mathcal{Z}(e''_{\sigma+1},\dots,e''_n)$ such that $\dim E = \sigma - s$ and $E \not\subset \mathcal{Z}(e''_0,e''_{\sigma+1}\dots,e''_n)$. Then $A_{\lambda',L''}$ is a subset of isolated points of the intersection $W_n(L'') \cap \mathcal{Z}(L'_{s+1},\dots,L'_\sigma)$. Hence $N_{\lambda',L''} \le \deg W_n(L'')$. Consider the morphism $p_{v,L'',\sigma} : W_{v,\sigma}(L'') \setminus \mathcal{Z}(e''_0,e''_{\sigma+1},\dots,e''_n) \to \mathbb{P}^{n-\sigma}(\overline{K_v})$, $(X_0 : \dots : X_n) \mapsto (e''_0,e''_{\sigma+1} : \dots : e''_n)$ defined over $\overline{K_v}$, where $W_{v,\sigma}(L'')$ is defined in the Introduction (now one should only extend the field $\overline{k}$ till $\overline{K_v}$). Hence $W_{v,\sigma}(L'')$ is a union of some irreducible components of W_v. Then the morphism $p_{v,L'',\sigma}$ is dominant, see the Introduction, and obviously $W_n(L'') \cap \mathcal{Z}(v_{s+1},\dots,v_\sigma)$ is contained in the set of isolated points of $p^{-1}_{v,L'',\sigma}((1:0:\dots:0))$. Hence $\deg W_n(L'') \le \deg p_{v,L'',\sigma}$ by Lemma 2 [9]. Denote by $q_{v,L'',\sigma}$ the restriction of the morphism q_v to $W_{v,\sigma}(L'') \setminus \mathcal{Z}(e_0,\dots,e_r)$.

Consider the morphism of the linear projection

$$\begin{aligned} q_{L''} : W' \setminus \mathcal{Z}(L''_0,L''_{\sigma+1},\dots,L''_n) &\to \mathbb{P}^{n-\sigma}(\overline{K}), \\ (X_0 : \dots : X_r) &\mapsto (L''_0 : L''_{\sigma+1} : \dots : L''_n) \end{aligned} \tag{28}$$

defined over the field $\overline{K}$. Denote by $\widetilde{W'}$ the union of all the irreducible component E_2 of W' such that the restriction $q_{L''}$ to $E_2 \setminus \mathcal{Z}(L''_0,L''_{\sigma+1},\dots,L''_n)$ is dominant. Denote by $\widetilde{q}$ the restriction of $q_{L''}$ to $\widetilde{W'}$. Then the degree of the linear projection $\deg \widetilde{q} \le \deg \widetilde{W'} \le \deg W'$. The definitions imply $p_{v,L'',\sigma} = \widetilde{q} \circ q_{v,L'',\sigma}$. Hence $\deg p_{v,L'',\sigma} = \deg q_{v,L'',\sigma} \deg \widetilde{q} \le \deg q_v \deg W'$. Assertion (i) is proved.

To prove (ii) let us choose a linear form $L'_0 \in k[X_0,\dots,X_n]$ such that $A_{\lambda',L''} \cap \mathcal{Z}(L'_0) = \emptyset$. Consider the morphism

$$\begin{aligned} &\overline{p}_{\lambda',L''} : W \setminus (\mathcal{Z}(e''_0,e''_{\sigma+1},\dots,e''_n) \cup \mathcal{Z}(L'_0,L'_{s+1},\dots,L'_\sigma)) \\ &\longrightarrow (\mathbb{P}^{\sigma-s} \times \mathbb{P}^{n-\sigma})(\overline{k}), \\ &\overline{p}_{\lambda',L''}(z) = \big((L'_0(z) : L'_{s+1}(z) : \dots : L'_\sigma(z)), (e''_0(z) : e''_{\sigma+1}(z) : \dots : e''_n(z)) \big). \end{aligned}$$

Hence $\overline{p}_{\lambda',L''}$ depend on the choice of L'_0. According to the proof of (i) if $N_{\lambda',L''} = \deg q_v \deg W'$ then $N_{\lambda',L''} = \deg W_n(L'') = \deg p_{v,L'',\sigma} = \deg q_v \deg W'$; the intersection $W_n(L'') \cap \mathcal{Z}(L'_{s+1}, \ldots, L'_\sigma)$ is transversal and coincides with $A_{\lambda',L''}$; $W_v = W_{v,\sigma}(L'')$; every irreducible component of $W_{v,\sigma}(L'')$ contains a point from $W_{v,n}(L'') = W_n(L'') \cap \mathcal{Z}(v_{s+1}, \ldots, v_\sigma)$ (recall that the last is by Lemma 2 [9]). Hence every irreducible component of $W_\sigma(L'')$ contains an irreducible component of $W_n(L'')$. But $W = W_\sigma(L'')$ since $W_v = W_{v,\sigma}(L'')$. Therefore, every irreducible component of W contains a point from $A_{\lambda',L''}$. Thus, by Lemma 9 the morphism $\overline{p}_{\lambda',L''}$ is dominant.

Let $\nu_j, \mu_i \in k$, $s+1 \le j \le \sigma$, $\sigma+1 \le i \le n$. Set $\lambda'' = (L'_{s+1} - \nu_{s+1}L'_0, \ldots, L'_\sigma - \nu_\sigma L'_0)$, $L''' = (L''_0, L''_{\sigma+1} - \mu_{\sigma+1}L''_0, \ldots, L''_n - \mu_n L''_0)$. By Lemma 2 [9] there are $\nu_j, \mu_i \in \overline{k}$ such that $\deg q_v \deg W' = N_{\lambda',L''} \le N_{\lambda'',L'''} = \deg \overline{p}_{\lambda',L''}$. On the other hand, $N_{\lambda'',L'''} \le \deg q_v \deg W'$ by (i). Hence $\deg \overline{p}_{\lambda',L''} = \deg q_v \deg W'$. Now (ii) follows from Lemma 2 [9] applied to $\overline{p}_{\lambda',L''}$. The lemma is proved.

LEMMA 18 *Let $\lambda = (L_{s+1}, \ldots, L_\sigma)$ where all $L_j \in k[X_0, \ldots, X_n]$ are linear forms. Suppose that $\dim W_\lambda = n - \sigma > 0$, and W_λ is irreducible over k. Let $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n)$ and all linear forms $L'_w \in \overline{k}[X_0, \ldots, X_r]$, $w \in \{0, \sigma+1, \ldots, n\}$, see above. Assume that $N_{\lambda,L'} \ge 1$. Then the following assertions hold.*

(i) If e'_0 does not vanish on $A_{\lambda,L'}$ then the morphism $p_{\lambda,L'}$, see the Introduction, is dominant.

(ii) The morphism q_λ, see the Introduction, is dominant.

(iii) $N_{\lambda,L'} \le \deg p_{\lambda,L'} \le \deg q_\lambda \deg W'$. Denote by $\mathcal{V}_\lambda$ the set of all $L' \in \mathbb{A}^{(n-\sigma+1)(r+1)}(\overline{k})$ such that $N_{\lambda,L'} = \deg q_\lambda \deg W'$ and $A_{\lambda,L'} \cap \mathcal{Z}(e'_0) = \emptyset$. Then $\mathcal{V}_\lambda$ is a nonempty defined over k open in the Zariski topology subset of $\mathbb{A}^{(n-\sigma+1)(r+1)}(\overline{k})$.

(iv) Let $N_{\lambda,L'} = \deg q_\lambda \deg W'$, and x be an arbitrary point from $A_{\lambda,L'}$. Then x is a smooth point of each algebraic variety W_λ, $\mathcal{Z}(e'_{\sigma+1}), \ldots, \mathcal{Z}(e'_n)$, and the intersection of the tangent spaces of these $n-\sigma+1$ algebraic varieties at the point x is transversal, i.e., is $\{0\}$.

(v) $\deg q_\lambda \le \deg q_v$.

PROOF Let us prove (i). Denote by $B_{\lambda,L'}$ the subset of all points $x \in A_{\lambda,L'}$ such that $e'_0(x) \ne 0$. Suppose that e'_0 does not vanish on $A_{\lambda,L'}$. Then the set $B_{\lambda,L'}$ is nonempty. The set $B_{\lambda,L'}$ is invariant with respect to the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$. Therefore, every irreducible over k component of W_λ has a nonempty intersection with $B_{\lambda,L'}$. Hence by Lemma 9 the morphism $p_{\lambda,L'}$, see (3), is dominant. Let us define the morphism $q_{L'} : W' \setminus \mathcal{Z}(L'_0, L'_{\sigma+1}, \ldots, L'_n) \to \mathbb{P}^{n-\sigma}(\overline{k})$ replacing in (28) the element L'' by L' and the field K by k. Hence $q_{L'}$ is defined over k. Now $p_{\lambda,L'}$ can be represented as a composition of the morphisms $W_\lambda \setminus \mathcal{Z}(e'_0, e'_{\sigma+1}, \ldots, e'_n) \to W' \xrightarrow{q_{L'}} \mathbb{P}^{n-\sigma}(\overline{k})$, where $W_\lambda \setminus \mathcal{Z}(e'_0, e'_{\sigma+1}, \ldots, e'_n) \to W'$ is the restriction of q_λ. Hence q_λ and $q_{L'}$ are also dominant, and $\deg p_{\lambda,L'} = \deg q_\lambda \deg q_{L'} \le \deg q_\lambda \deg W'$. Assertion (i) is proved.

In what follows in the proof of (ii)–(v) replacing, if it is necessary, the linear form L'_0 for a new one we shall assume without loss of generality that $A_{\lambda,L'} \cap \mathcal{Z}(e'_0) = \emptyset$. Hence $A_{\lambda,L'} = B_{\lambda,L'}$. Now (ii) follows from (i).

Notice that $\deg q_{L'} = \deg W'$ if and only if $W' \cap \mathcal{Z}(L'_0, L'_{\sigma+1}, \dots, L'_n) = \emptyset$ in $\mathbb{P}^n(\overline{k})$. Hence the set of all L' such that $\deg q_{L'} = \deg W'$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(n-\sigma+1)(r+1)}(\overline{k})$. Therefore, the set $\mathcal{V}_\lambda$ is also a nonempty (by Lemma 2 [9]) open in the Zariski topology subset of this affine space. Assertion (iii) is proved. The set $\mathcal{V}_\lambda$ is invariant with respect to the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$, and hence defined over k. Now by Lemma 2 [9] assertion (iv) holds.

Let us define the set $\mathcal{V}_v$ replacing in the definition of $\mathcal{V}_\lambda$ the field k by k_v and λ by v. So $\mathcal{V}_v$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(n-\sigma+1)(r+1)}(\overline{k_v})$. To prove (v) we shall suppose without loss of generality that $L' \in \mathcal{V}_\lambda \cap \mathcal{V}_v$. Then (v) follows from assertion (i) of Lemma 17(vii). The lemma is proved.

LEMMA 19 *Let the field $\overline{K}$ be supplied with a real structure, see [3], and the mapping of standard part* $\mathrm{st} : \mathbb{P}^n(\overline{K}) \to \mathbb{P}^n(\overline{k})$ *is defined. Let* $\lambda = (L_{s+1}, \dots, L_\sigma)$, $L' = (L'_0, L'_{\sigma+1}, \dots, L'_n)$ *where all* $L_j \in \overline{K}[X_0, \dots, X_n]$, $L'_w \in \overline{K}[X_0, \dots, X_r]$ *are linear forms. Let* $\mathfrak{l} \subset \mathbb{A}^{(\sigma-s)(n+1)}(\overline{K})$ *be a line such that* $\lambda \in \mathfrak{l}$. *Then the following assertions hold.*

(a) For all $\lambda' \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, the inequality $N_{\lambda',L'} \geq N_{\lambda,L'}$ is satisfied. More than that, if $A_{\lambda,L'} \cap (\mathcal{Z}(e'_0) \cup \overline{V \setminus W}) = \emptyset$ then for all $\lambda' \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, $\#(A_{\lambda',L'} \setminus (\mathcal{Z}(e'_0) \cup \overline{V \setminus W})) \geq N_{\lambda,L'}$.

(b) If λ' is infinitely close to λ with respect to the field k then $\mathrm{st}(A_{\lambda',L'}) \supset A_{\lambda,L'}$ and hence $N_{\lambda',L'} \geq N_{\lambda,L'}$.

(c) If $N_{v,L'} = \deg q_v \deg W'$ (i.e., if L' belongs to the nonempty open in the Zariski topology subset $\mathcal{V}_v$ of the affine space $\mathbb{A}^{(n-\sigma+1)(r+1)}(\overline{k_v})$, see the proof of Lemma 18) then the set $\mathcal{V}_{L'}$ of all $\lambda \in \mathbb{A}^{(\sigma-s)(n+1)}(\overline{k})$ such that $N_{\lambda,L'} = \deg q_v \deg W'$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(\sigma-s)(n+1)}(\overline{k})$ defined over k.

PROOF Let $\lambda'' = (L''_{s+1}, \dots, L''_\sigma) \in \mathfrak{l}$ and $\lambda'' \neq \lambda$. Let $\mathbb{A}^1(\overline{K})$ has the coordinate function Z. Consider the subvariety

$$(W \times \mathbb{A}^1(\overline{K})) \cap \mathcal{Z}(L_{s+1} + ZL''_{s+1}, \dots, L_\sigma + ZL''_\sigma) \cap \mathcal{Z}(e'_{\sigma+1}, \dots, e'_n) \qquad (29)$$

of $\mathbb{P}^n(\overline{K}) \times \mathbb{A}^1(\overline{K})$. Let C be the union of all the irreducible components C_1 of (29) such that $C_1 \cap (A_{\lambda,L'} \times \{0\}) \neq \emptyset$. Then, since $A_{\lambda,L'}$ is finite every irreducible component C_1 of C is a curve, and $\deg C \leq d^s 2^{\sigma-s}(d')^{n-\sigma}$ by the Bézout theorem (the degree $\deg C$ is by definition the degree of $C \setminus \mathcal{Z}(L_0)$ in the affine space $\mathbb{A}^n(\overline{K}) \times \mathbb{A}^1(\overline{K})$ where $L_0 \in k[X_0, \dots, X_n]$ is an arbitrary linear form which does not vanish on any irreducible component of C). The natural linear projection $C \to \mathbb{A}^1(\overline{K})$, $((X_0 : \dots : X_n), Z) \mapsto Z$, is dominant. Hence the number of elements $\#(C \cap \mathcal{Z}(Z - a)) \geq \#(C \cap \mathcal{Z}(Z)) = N_{\lambda,L'}$ for all $a \in \overline{K}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number. Denote by C' the union of all the irreducible components C_2 of variety (29) such that $C_2 \not\subset C$. Then by the

Bézout theorem $\#(C \cap C')$ (respectively under the condition $A_{\lambda,L'} \cap (\mathcal{Z}(e'_0) \cup \overline{V \setminus W}) = \emptyset$ the number of elements $\#(C \cap (C' \cup \mathcal{Z}(e'_0) \cup (\overline{V \setminus W} \times \mathbb{A}^1(\overline{K}))))$ is bounded from above by a polynomial in $d^n(d')^{n-\sigma}$. Hence for all $a \in \overline{K}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, $C \cap \mathcal{Z}(Z-a) \subset A_{\lambda^{(a)},L'} \times \{a\}$ (respectively $C \cap \mathcal{Z}(Z-a) \subset (A_{\lambda^{(a)},L'} \setminus (\mathcal{Z}(e'_0) \cup (\overline{V \setminus W} \times \mathbb{A}^1(\overline{K}))) \times \{a\}$), where $\lambda^{(a)} = (L_{s+1} + aL''_{s+1}, \ldots, L_\sigma + aL''_\sigma)$. This implies (a). Assertion (b) follows from (a) since λ' from (b) is a generic point of $\mathfrak{l}$ (here we leave the details to the reader).

Using assertion (a) one can replace subsequently all the coefficients of the linear forms v_j, $s+1 \le j \le \sigma$, by the coefficients from k and obtain linear forms L_j, $s+1 \le j \le \sigma$, such that $N_{\lambda,L'} = \deg q_v \deg W'$, cf. the algorithm of reduction to integer coefficients, Proposition 2 [9]. Hence $\mathcal{V}_{L'}$ is nonempty. It is not difficult to prove (we leave the details to the reader) that $\mathcal{V}_{L'}$ is a constructive set. Hence $\mathcal{V}_{L'}$ is an open in the Zariski topology subset of $\mathbb{A}^{(\sigma-s)(n+1)}(\overline{k})$ by the implicit function theorem and assertion (ii) Lemma 17 (cf. the proof of Lemma 16). Finally, $\mathcal{V}_{L'}$ is invariant with respect to the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$ and hence is defined over k. The lemma is proved.

LEMMA 20 *Under conditions of Theorem 2 suppose that $\lambda = (L_{s+1}, \ldots, L_\sigma)$, where all $L_j \in k[X_0, \ldots, X_n]$ are linear forms. Suppose that $\dim W_\lambda = n - \sigma$, the intersection $W \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ is transversal, and if $n - \sigma > 0$ then W_λ is irreducible over k. Suppose that $\dim(W_\lambda \cap (\overline{V_\lambda \setminus W_\lambda}) < \dim W_\lambda$, condition (16) holds, the morphism q_λ, see (15), is dominant, and the degree $\deg q_\lambda = \deg q_v$. Further, let $L' \in \mathcal{U}'_{\lambda,0}$. Then $L' \in \mathcal{U}'_0$.*

PROOF By definition, see assertion (d) of Theorem 1 [9], $L' \in \mathcal{U}'_{\lambda,0}$ (respectively $L' \in \mathcal{U}'_0$) if and only if

$$\#\big((W_\lambda \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n)\big) = \deg(q_\lambda) \deg W', \quad (30)$$

$$(W_\lambda \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n) \cap \mathcal{Z}(e'_0) = \emptyset, \quad (31)$$

$$(W_\lambda \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n) \cap (\overline{V_\lambda \setminus W_\lambda}) = \emptyset, \quad (32)$$

$$\#\big(W' \cap \mathcal{Z}(L'_{\sigma+1}, \ldots, L'_n)\big) = \deg W'. \quad (33)$$

(respectively the similar equalities hold with W_v, V_v, q_v in place of W_λ, V_λ, q_λ; and all the varieties are considered in this case over the field $\overline{k_v}$). We can assume without loss of generality that the coefficients of all the linear forms v_j are infinitely close to the corresponding coefficients of the linear forms L_j, $s+1 \le j \le \sigma$. Now (30)–(33) with V_v, W_v in place of V_λ, W_λ follow from (30)–(33) and Lemma 17 and the implicit function theorem. Thus, $L' \in \mathcal{U}'_0$. The lemma is proved.

LEMMA 21 *Let $W_1, \ldots, W_\nu$, $\nu \ge 2$, be arbitrary quasiprojective algebraic varieties in $\mathbb{P}^n(\overline{k})$ (or in $\mathbb{A}^n(\overline{k})$). Let E be an irreducible over $\overline{k}$ component of $W_1 \cap \ldots \cap W_\nu$. Suppose that the intersection $W_1 \cap \ldots \cap W_\nu$ is transversal at E, the definition see in the Introduction, and $x \in E$ is a smooth point of $W_1 \cap \ldots \cap W_\nu$. Then x is a smooth point of every W_i, $i = 1, \ldots, \nu$, and the intersection $W_1 \cap \ldots \cap W_\nu$ is transversal at x.*

PROOF Obviously for every $2 \le i \le \nu$ there is only one irreducible component E_i of $W_1 \cap \ldots \cap W_i$ such that $x \in E_i$, and the intersection $W_1 \cap \ldots \cap W_i$ transversal at E_i. Further, x is a smooth point of the intersection $W_1 \cap \ldots \cap W_i$.

Hence using the induction on i it is sufficient to consider only the case $i = 2$. In what follows $\nu = 2$ in the proof.

We shall suppose without loss of generality that W_1, W_2 are subvarieties of $\mathbb{A}^n(\overline{k})$. The variety $W_1 \cap W_2$ is isomorphic to $(W_1 \times W_2) \cap \Delta$, where $(W_1 \times W_2) \cap \Delta \subset \mathbb{A}^n(\overline{k}) \times \mathbb{A}^n(\overline{k})$, and $\Delta = \{(z, z) : z \in \mathbb{A}^n(\overline{k})\}$. The intersection $(W_1 \times W_2) \cap \Delta$ is transversal at $(E \times E) \cap \Delta$ if and only if $W_1 \cap W_2$ is transversal at E. Hence we shall assume without loss of generality that W_2 is an affine linear subspace. Obviously x is a smooth point of E. Let us choose an affine linear subspace $\mathcal{L}$ such that $x \in \mathcal{L}$, the intersection $E \cap \mathcal{L}$ is transversal, the intersection $W_2 \cap \mathcal{L}$ is transversal, and x is an isolated point of $E \cap \mathcal{L}$. Now the index of intersection $i(W_1, W_2 \cap \mathcal{L}; x) = i(W_1, W_2; E) i(E, \mathcal{L}; x) = 1$, see [10]. Hence x is a smooth point of W_1, and the intersection of W_1 and the affine linear space $W_2 \cap \mathcal{L}$ at x is transversal. This implies the assertion of the lemma. The lemma is proved.

We shall suppose that the field $\overline{k}$ is supplied with a real structure. Let $\varepsilon_j > 0$ be an infinitesimal with respect to the field $k_{j-1} = k(\varepsilon_1, \ldots, \varepsilon_{j-1})$ for every positive integer j. Then the algebraic closure $\overline{k}_j$ (of the field k_j) is supplied with a real structure in the natural way, and one can consider and solve systems of polynomial equations with squares of absolute values with coefficients from $\overline{k}_j$, see [3]. The mapping of the standard part $\mathrm{st}_j : \mathbb{P}^n(\overline{k_j}) \to \mathbb{P}^n(\overline{k})$ is defined in the natural way, see [3].

Let λ be from the statement of Lemma 18. Suppose that the linear forms $L'_j \in k[X_0, \ldots, X_r]$, $j \in \{0, \sigma + 1, \ldots, n\}$. Let $0 \le \beta \le n$ be an integer. Consider the system of equations and inequalities with respect to the variables $X_0, \ldots, X_n$, $Y_{\sigma+1}, \ldots, Y_n$, $Z_{s+1}, \ldots, Z_\sigma$ with coefficients from k_1

$$\left\{ \begin{array}{ll} e'_j - Y_j e'_0 = 0, & \sigma + 1 \le j \le n, \\ L_j - Z_j X_\beta = 0, & s + 1 \le j \le \sigma, \\ \sum_{\sigma+1 \le j \le n} |Y_j|^2 + \sum_{s+1 \le j \le \sigma} |Z_j|^2 \le \varepsilon_1, & \\ \sum_{0 \le i \le r} |e_j|^2 \ge \varepsilon_1^{-1}, & \\ e'_0 = 1. & \end{array} \right. \tag{34}$$

LEMMA 22 *Suppose that* $\dim W_\lambda = n - \sigma > 0$, *the algebraic variety* W_λ *is irreducible over* k, *the intersection* $W \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ *is transversal, and all the linear forms* $L'_j \in k[X_0, \ldots, X_r]$, $j \in \{0, \sigma+1, \ldots, n\}$. *Let* $\deg q_{L'} = \deg W'$, *see the proof of Lemma 18, and* $1 \le N_{\lambda, L'} = \deg p_{\lambda, L'} < \deg q_v \deg W'$ *(hence* $A_{\lambda, L'} \cap \mathcal{Z}(e'_0) = \emptyset$ *and the morphisms* q_λ *and* $p_{\lambda, L'}$ *are dominant by Lemma 18 (ii) and (i)). Then there is* $0 \le \beta \le n$ *such that system (34) has a solution*

$$(x_0^*, \ldots, x_n^*, y_{\sigma+1}^*, \ldots, y_n^*, z_{s+1}^*, \ldots, z_\sigma^*) \in \overline{k_1}^{2n+1-s} \tag{35}$$

with $x^* = (x_0^* : \ldots : x_n^*) \in W(\overline{k_1})$.

PROOF Obviously it is sufficient to prove the lemma with k_2 in place of k_1. Let $\widetilde{L}_j \in \overline{k(\varepsilon_2)}[X_0, \ldots, X_n]$, $s + 1 \le j \le \sigma$, be linear forms. Denote $\widetilde{\lambda} = (\widetilde{L}_{s+1}, \ldots, \widetilde{L}_\sigma)$ (we shall specify $\widetilde{\lambda}$ below), $\widetilde{L} = (\widetilde{L}_0, \widetilde{L}_{\sigma+1}, \ldots, \widetilde{L}_n)$, where $\widetilde{L}_0 = L'_0$ and $\widetilde{L}_j = L'_j - y_j^* L'_0$ for all $\sigma + 1 \le j \le n$. Let us replace in system (34) every equation $L_j - Z_j X_\beta = 0$ by $\widetilde{L}_j = 0$, $s + 1 \le j \le \sigma$. Denote the obtained system by (*). We have $\deg p_{v, L'} = \deg q_v \deg W'$ since $\deg q_{L'} = \deg W'$.

Therefore by Lemma 2 [9] (applied over the field $\overline{k_v(\varepsilon_2)}$ there are $y_j^* \in \overline{k(\varepsilon_2)}$, $s+1 \le j \le \sigma$, such that all nonzero y_j^* are infinitesimal with respect to the field k; and the number of elements $N_{v,\widetilde{L}} = \deg q_v \deg W'$. By Lemma 19 (c) there are linear forms $\widetilde{L}_j \in \overline{k(\varepsilon_2)}[X_0, \dots, X_n]$, $s+1 \le j \le \sigma$, such that all the nonzero coefficients of the linear forms $L_j - \widetilde{L}_j$, $s+1 \le j \le \sigma$, are infinitesimal with respect to the field k, and the number of elements $N_{\widetilde{\lambda},\widetilde{L}} = \deg q_v \deg W'$. We have $W' \cap \mathcal{Z}(L'_0, L'_{\sigma+1}, \dots, L'_n) = \emptyset$ since $\deg q_{L'} = \deg W'$. Now obviously $W' \cap \mathcal{Z}(\widetilde{L}_0, \widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_n) = \emptyset$. Hence $A_{\widetilde{\lambda},\widetilde{L}} \cap \mathcal{Z}(\widetilde{e}_0) = \emptyset$, where $\widetilde{e}_0 = e'_0$.

By Lemma 18 (iv) and Lemma 21 every point $z \in A_{\lambda,L'}$ is a smooth point of each algebraic variety W, $\mathcal{Z}(L_{s+1}), \dots, \mathcal{Z}(L_\sigma)$, $\mathcal{Z}(e'_{\sigma+1}), \dots, \mathcal{Z}(e'_n)$, and the intersection of tangent spaces of these $n-s+1$ algebraic varieties at the point z is transversal, i.e., is $\{0\}$. We have $N_{\lambda,L'} = \deg p_{\lambda,L'}$. Therefore, $A_{\lambda,L'} = W_\lambda \cap \mathcal{Z}(e'_0, e'_{\sigma+1}, \dots, e'_n)$ by Lemma 2 [9]. Hence for every $z \in A_{\widetilde{\lambda},\widetilde{L}}$ the standard part $\mathrm{st}_2(z) \in A_{\lambda,L'} \cup \mathcal{Z}(e_0, \dots, e_r)$. Since $N_{\widetilde{\lambda},\widetilde{L}} > N_{\lambda,L'}$ the implicit function theorem implies that system (*) has a solution $(x_0^*, \dots, x_n^*, y_{\sigma+1}^*, \dots, y_n^*) \in \overline{k_2}^{2n+1-\sigma}$ with $x^* \in W(\overline{k_2})$. There is $0 \le \beta \le n$ such that $X_\beta(x^*) \ne 0$ and $(X_i/X_\beta)(x^*)$ are not infinitely large with respect to k for all $0 \le i \le n$. Set $z_j^* = ((\widetilde{L}_j - L_j)/X_\beta)(x^*)$. Then (35) with k_2 in place of k_1 is a solution of system (34) with $x^* \in W(\overline{k_2})$. The lemma is proved.

Let us proceed to the describe of the algorithm. Recall that the point $x^{(0)} \in W$ is given. We can suppose without loss of generality that $x^{(0)} \notin \mathcal{Z}(e_0, \dots, e_r)$, cf. the algorithm from [9]. Performing permutations of $X_0, \dots, X_n$ and $e_0, \dots, e_r$ we shall assume without loss of generality that $(e_0 X_0)(x^{(0)}) \ne 0$. Using Theorem 5 [4] we construct indices $1 \le i_\sigma < \ldots < i_n \le r$ and $1 \le j_{s+1} < \ldots < j_\sigma \le n$ such that the family $X_{j_{s+1}}/X_0, \dots, X_{j_\sigma}/X_0$, $e_{i_\sigma}/e_0, \dots, e_{i_n}/e_0 \in k(W)$ is algebraically independent, and σ is minimal possible. For convenience of notation we shall assume without loss of generality that $i_w = w$ for $\sigma \le w \le n$ and $j_w = w$ for $s+1 \le w \le \sigma$. Now $\dim W = n - s$ and $\dim W' = n - \sigma$. Let us apply Theorem 3 to $V \cup \mathcal{Z}(e_0)$ and construct linear forms $L_0, \dots, L_{n+1}$. At the beginning set $\lambda = (L_{s+1}, \dots, L_\sigma)$, $\overline{\lambda} = (L_0, \dots, L_{n+1})$. Hence condition (A) holds for V and λ and also condition (16) is satisfied, see the Introduction.

Assume that $n - \sigma = 0$. Then put $L'_0 = X_0$. Obviously $W' \cap \mathcal{Z}(L'_0) = \emptyset$. Hence $L' = (L'_0) \in \mathcal{U}'_{\lambda,0} = \mathcal{U}''_{\lambda,0} = \mathcal{U}'''_{\lambda,0} = \mathcal{U}'_{\lambda,n+1} = \mathcal{U}''_{\lambda,n+1} = \mathcal{U}'''_{\lambda,n+1}$ in this case. By Lemma 20 the required λ and L' are constructed if $n - \sigma = 0$.

Suppose that $n - \sigma > 0$. The constructed λ is the base of the described below recursion. Our aim is to construct at the end of this recursion on λ

(†) an element (λ, L') such that $\lambda = (L_{s+1}, \dots, L_\sigma)$ satisfies conditions of Lemma 20, condition (A) for V, see the Introduction, all the linear forms $L_j \in k[X_0, \dots, X_n]$, $s+1 \le j \le \sigma$; the element L' satisfies conditions (30)–(33) (with $\deg q_\lambda = \deg q_v$) and all the linear forms $L'_w \in k[X_0, \dots, X_n]$, $w \in \{0, \sigma+1, \dots, n\}$. Hence $L' \in \mathcal{U}'_{\lambda,0}$ and $L' \in \mathcal{U}'_0$.

Besides that all the linear forms $L_j \in k[X_0, \dots, X_n]$, $s+1 \le j \le \sigma$, and $L'_w \in k[X_0, \dots, X_n]$, $w \in \{0, \sigma+1, \dots, n\}$, will have integer coefficients of length $O(n \log d + (n-\sigma)\log(d'))$.

Consider the morphism

$$\pi : W \setminus \mathcal{Z}(X_0 e_0) \to \mathbb{P}^{n-s}(\overline{k}),$$

$$(X_0 : \ldots : X_n) \mapsto (1 : X_{s+1}/X_0 : \ldots : X_\sigma/X_0,\, e_{\sigma+1}/e_0 : \ldots : e_n/e_0).$$

Let us construct using the algorithm from Theorem 1 [9] (b) a point $z^{(0)} \in \mathbb{P}^{n-s}(\overline{k})$ such that

$$\#\pi^{-1}(z^{(0)}) = \deg \pi \quad \& \quad \pi^{-1}(z^{(0)}) \cap (\overline{V \setminus W}) = \emptyset. \tag{36}$$

For convenience of notation, performing a linear transformation of the coordinates of $\mathbb{P}^{n-s}(\overline{k})$, we shall suppose without loss of generality that $z^{(0)} = (1 : 0 : \ldots : 0)$. Put $X^{(0)} = (X_0, X_{\sigma+1}, \ldots, X_n)$.

Let us compute using [2] and Theorem 2 [8] the number $N_{\lambda,X^{(0)}}$. Suppose that $N_{\lambda,X^{(0)}} = 0$. Let τ be an integer. Set $L_{0,j} = L_j$ for $j \in \{0, \ldots, s, \sigma + 1, \ldots, n+1\}$, and $L_{0,j} = L_j + \tau X_j$ for $s+1 \le j \le \sigma$. Put $\lambda_0 = (L_{0,s+1}, \ldots, L_{0,\sigma})$, $\overline{\lambda_0} = (L_{0,0}, \ldots, L_{0,n+1})$. Let us show that there is at most a polynomial in $d^n(d')^{n-\sigma}$ number of τ such that $N_{\lambda_0,X^{(0)}} = 0$. Indeed, let $\mathbb{A}^1(\overline{k})$ has the coordinate function Z. Consider the subvariety

$$(W \times \mathbb{A}^1(\overline{k})) \cap \mathcal{Z}(ZL_{s+1} + X_{s+1}, \ldots, ZL_\sigma + X_\sigma) \cap \mathcal{Z}(e_{\sigma+1}, \ldots, e_n) \tag{37}$$

of $\mathbb{P}^n(\overline{k}) \times \mathbb{A}^1(\overline{k})$. Let C be the union of all the irreducible components C_1 of (37) such that $C_1 \cap (\pi^{-1}(z^{(0)}) \times \{0\}) \ne \emptyset$ (so $C_1 \not\subset \overline{V \setminus W} \times \mathbb{A}^1(\overline{k})$). Then, cf. [2], by (36) every irreducible component C_1 of C is a curve. By the Bézout theorem $\deg C \le d^s 2^{\sigma-s}(d')^{n-\sigma}$, cf. the proof of Lemma 19. Now the required assertion follows from the Bézout theorem and Lemma 9.

Thus, there is an integer τ with length $O(n \log d + (n - \sigma)\log(d'))$ such that $N_{\lambda_0,X^{(0)}} > 0$. Applying Lemma 3 to the line $\mathfrak{l}$ containing $\overline{\lambda}$, $\overline{\lambda_0}$ and the algorithm of reduction to integer coefficients, see Proposition 2 [9], we construct a new element λ_1 similar to λ, satisfying condition (A) for V, and such that $N_{\lambda_1,X^{(0)}} > 0$.

Now replacing λ by λ_1 we shall suppose in what follows that $N_{\lambda,X^{(0)}} > 0$ and condition (A) holds for λ and V. In particular, the intersection $W \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ is transversal and irreducible over k (recall that $n - \sigma > 0$). By Lemma 18 (ii) the morphism q_λ is dominant. We have constructed λ for the base of our main recursion. This is a recursion by λ.

Let us describe the general step of this recursion. The recursive assumption is that λ satisfies condition (A), and the morphism q_λ is dominant (recall here that if $n - \sigma > 0$ and morphism q_λ is dominant then (16) holds, see the Introduction). Applying Theorem 1 [9] to W_λ, we construct L' satisfying (30)–(33). So $N_{\lambda,L'} = \deg q_\lambda \deg W'$. Note that $W' \cap \mathcal{Z}(L'_0, L'_{\sigma+1}, \ldots, L'_n) = \emptyset$. For every $0 \le \beta \le n$ using Theorem 2 [9] (recall that it is based on the results of [5]) we decide whether system (34) has a solution with $x^* \in W(\overline{k_1})$. Suppose that such a solution exists for some β. Set $L^*_j = L_j - z^*_j X_\beta$, $s + 1 \le j \le \sigma$, and $\lambda^* = (L^*_{s+1}, \ldots, L^*_\sigma)$. Put $k^* = k(z^*_{s+1}, \ldots, z^*_\sigma)$. Notice that the algebraic variety W_{λ^*} is not necessarily irreducible over k^*. Let us define the morphism $p_{\lambda^*,L'}$ replacing in the definition of $p_{\lambda,L'}$ the $(\sigma - s)$–tuples λ by λ^* and k by k^*. The morphism $p_{\lambda,L'}$ is dominant by Lemma 18 (i) and (31).

Let $z \in A_{\lambda,L'}$. Let us show that there exists a point $z^* \in A_{\lambda^*,L^*}$ such that $\mathrm{st}_1(z^*) = z$, i.e., $\mathrm{st}_1(A_{\lambda^*,L^*}) \supset A_{\lambda,L'}$. By Lemma 18 (iv) and Lemma 21 every point $z \in A_{\lambda,L'}$ is a smooth point of each algebraic variety W, $\mathcal{Z}(L_{s+1}), \ldots,$ $\mathcal{Z}(L_\sigma)$, $\mathcal{Z}(e'_{\sigma+1}), \ldots, \mathcal{Z}(e'_n)$, and the intersection of tangent spaces of these $n - s + 1$ algebraic varieties at the point z is transversal, i.e., is $\{0\}$. Hence

the required assertion follows from the implicit function theorem. Similarly $\mathrm{st}_1(A_{\lambda^*,L'}) \supset A_{\lambda,L'}$.

Let us show that the morphism $p_{\lambda^*,L'}$ is dominant. By Lemma 5 and Proposition 6 [3] the equality $\mathrm{st}_1(W_{\lambda^*}) = W_\lambda$ holds. We have $\deg W_\lambda = \deg W_{\lambda^*} = \deg W$. Hence again by Lemma 5 for every irreducible over $\overline{k}$ component E of W_λ there is the unique irreducible over $\overline{k^*}$ component E^* of W_{λ^*} such that $\mathrm{st}_1(E^*) \supset E$ (since, if E^* is not unique then $\deg W_{\lambda^*} > \deg W_\lambda$; we leave the details to the reader). The morphism $p_{\lambda,L'}$ is dominant and $N_{\lambda,L'} = \deg p_{\lambda,L'}$. Hence by Lemma 2 [9] there is a point $z \in E \cap A_{\lambda,L'}$. As we have proved there is $z^* \in A_{\lambda^*,L'}$ such that $\mathrm{st}_1(z^*) = z$. Then $z^* \in E^*$ by Lemma 5 and since z is a smooth point of W_λ. We have $A_{\lambda^*,L'} \cap \mathcal{Z}(e'_0) = 0$ since $W' \cap \mathcal{Z}(L'_0, L'_{\sigma+1}, \ldots, L'_n) = \emptyset$ by (31), (33). Thus, the morphism $p_{\lambda^*,L'}$ is dominant by Lemma 9, and the required assertion is proved.

We have proved that the algebraic variety $(W_{\lambda^*} \setminus \mathcal{Z}(e'_0)) \cap \mathcal{Z}(e'_{\sigma+1} - y^*_{\sigma+1}e'_0, \ldots, e'_n - y^*_n e'_0)$ contains $N_{\lambda,L'}$ isolated points such that the coordinates of each of them satisfy the inequality $\sum_{0\le i\le r} |e_i|^2 < \varepsilon_1^{-1}$ (these points are infinitely close to the corresponding points from $A_{\lambda,L'}$). Besides that, there is a solution of system (34) which is different from all these isolated points. Therefore, by Lemma 2 [9] the degree $\deg p_{\lambda^*,L'} > N_{\lambda,L'} = \deg p_{\lambda,L'}$. Let us construct using Theorem 1 [8] and Theorem 2 [8] a representative system of points of the irreducible over k^* components of the algebraic variety V_{λ^*} and choose among them the points from W_{λ^*}. Now applying Theorem 1 [9] for the restrictions of $p_{\lambda^*,L'}$ to irreducible over k^* components E^* of W_{λ^*}, we compute for every E^* the degree $\deg p_{\lambda^*,L'}|_{E^*}$ and a point z' from the statement of assertion (b) of Theorem 1 [9] with $p_{\lambda^*,L'}|_{E^*}$ in place of ρ.

Further, by the algorithm of reduction to integer coefficients, see Proposition 2 [9], considering the lines through the pairs of points z' corresponding to different components E^* and applying assertion (d) of Theorem 1 [9], we can construct a new point which is similar to all z', and is the same for all the components E^*. So we shall suppose without loss of generality that the point z' is the same for all components E^*. Denote $z' = (1 : z'_{\sigma+1} : \ldots : z'_n)$. Hence all z'_j are integers with length $O(n\log d + (n-\sigma)\log(d'))$. Set $L''_j = L'_j - z'_j L'_0$, $\sigma+1 \le j \le n$, and $L'' = (L'_0, L''_{\sigma+1}, \ldots, L''_n)$. Then by assertion (b) of Theorem 1 [9] $N_{\lambda^*,L''} = \deg p_{\lambda^*,L''} > N_{\lambda,L'}$. Using Lemma 19 (a) we replace subsequently the coefficients of the linear forms L^*_j, $s+1 \le j \le \sigma$ by integers with lengths $O(n\log d + (n-\sigma)\log(d'))$ and obtain linear forms L'_j, $s+1 \le j \le \sigma$ (denote $\lambda' = (L'_{s+1}, \ldots, L'_\sigma)$) such that $N_{\lambda',L''} \ge N_{\lambda^*,L''} > N_{\lambda,L'}$.

Applying Lemma 3, the algorithm of reduction to integer coefficients, and Lemma 19 (a) to the line $\mathfrak{l}$ such that $\lambda, \lambda' \in \mathfrak{l}$ we obtain λ'' similar to λ' such that condition (A) holds for λ'' and V, and $N_{\lambda'',L''} \ge N_{\lambda',L''} > N_{\lambda,L'}$. Hence the morphism $q_{\lambda''}$ is dominant by Lemma 18 (ii). We have $N_{\lambda,L'} = \deg q_\lambda \deg W'$ by our construction, and $N_{\lambda'',L''} \le \deg q_{\lambda''} \deg W'$ by Lemma 18 (iii). Therefore, $\deg q_{\lambda''} > \deg q_\lambda$. To avoid the growth of lengths of coefficients by the recursion, again applying Lemma 3, the algorithm of reduction to integer coefficients, and Lemma 19 (a), we replace subsequently all the coefficients of linear forms L''_j, $s+1 \le j \le \sigma$ from λ'' by integers with lengths $O(n\log d + (n-\sigma)\log(d'))$, obtain new linear forms L'''_j, and the element $\lambda''' = (L'''_{s+1}, \ldots, L'''_\sigma)$, which is similar to λ''. Now we replace λ by λ''' and proceed to the next step of the recursion. Note that by Lemma 18 (v) there is at most $\deg q_v$ number of steps with $N_{\lambda'',L''} > N_{\lambda,L'}$ at the end.

Thus we shall come at the end to the case when for every $0 \leq \beta \leq n$ system (34) does not have a solution with $x^* \in W(\overline{k_1})$. Then by Lemma 20 and Lemma 22 we have constructed (λ, L'), satisfying (†). The considered step of the recursion is final. The working time of the described algorithm satisfy the estimate for the working time from Theorem 2. This follows from the bounds for the working times of the applied algorithms.

REMARK 7 *Let $\lambda = (L_{s+1}, \ldots, L_\sigma)$ where all $L_j \in k[X_0, \ldots, X_n]$ are linear forms. Then one can decide whether λ satisfies conditions of Lemma 20, computing* $\deg q_v$ *by the described algorithm and applying the results of [8], [9]. If the conditions of Lemma 20 hold then using the algorithm from [9] one can construct $L' \in \mathcal{U}'_{\lambda,0}$ satisfying (10) for all $\sigma \leq i \leq n-1$ with W_λ, $W_{\lambda,i}(L')$ in place of W, $W_i(L')$ respectively. Suppose additionally that for every irreducible component E of $(W \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i)$ the intersection $E \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma) \neq \emptyset$ for every $\sigma \leq i \leq n-1$. Then (we can assume now without loss of generality that all the coefficients of all linear forms $U_j - L'_j$, $j \in \{0, \sigma+1, \ldots, n\}$, are infinitesimal with respect to the field k) by Lemma 18 (iv),Lemma 5, the implicit function theorem and Lemma 9 condition (10) holds for all $\sigma \leq i \leq n-1$ for W_v, $W_{v,i}(L')$ in place of W, $W_i(L')$. Therefore, original condition (10) is satisfied for every $\sigma \leq i \leq n-1$. We leave the details to the reader here. We shall not use this remark in what follows.*

LEMMA 23 *Let $\nu \geq 0$ and $m_i \geq 1$, $1 \leq i \leq \nu$, be integers. Suppose that $f_1^{(i)}, \ldots, f_{m_i}^{(i)} \in k[X_0, \ldots, X_n]$ are homogeneous polynomials similar to $f_1, \ldots, f_m$, i.e., $f_1^{(i)}, \ldots, f_{m_i}^{(i)}$ satisfy the same estimations for degrees and lengths of coefficients as $f_1, \ldots, f_m$ do. Let $V^{(i)} = \mathcal{Z}(f_1^{(i)}, \ldots, f_{m_i}^{(i)})$ in $\mathbb{P}^n(\overline{k})$. Then one can construct linear forms $L_0, \ldots, L_{n+1}$ with integer coefficients of length $O(n \log d + \log \nu)$ such that the assertion of Theorem 3 holds for $V^{(i)}$ in place of V for all $0 \leq i \leq \nu$ (i.e., linear forms $L_0, \ldots, L_{n+1}$ do not depend on i; other objects from the statement of this theorem for $V^{(i)}$ depend on i). The working time of this algorithm is polynomial in d^n, d_1, d_2, M, M_1, m, ν.*

PROOF This follows from Lemma 3 and the algorithm of reduction to integer coefficients, see Proposition 2 [9] (we leave the details to the reader). The lemma is proved.

For an arbitrary quasiprojective algebraic variety $E \subset \mathbb{P}^n(\overline{k})$ and a point $x \in \mathbb{P}^n(\overline{k})$ denote by $\mu(x, E)$ the multiplicity of the point x at E. For an arbitrary projective algebraic variety $V' \subset \mathbb{P}^n(\overline{k})$ we shall denote by $\mathrm{con}(V')$ the affine algebraic variety in $\mathbb{A}^{n+1}(\overline{k})$ which is the set of zeroes of the homogeneous ideal of V'.

LEMMA 24 *Let $\nu > \nu_0 \geq 0$ be integers. Suppose that $f_1^{(i)}, \ldots, f_{m_i}^{(i)} \in k[X_0, \ldots, X_n]$, $V^{(i)}$, $W^{(i)}$, s_i, $x_i^{(0)}$ for $0 \leq i \leq \nu$ are similar to $f_1, \ldots, f_m$, V, W, s, $x^{(0)}$ from the Introduction. Suppose that $f_1^{(i)}, \ldots, f_{m_i}^{(i)}$, $0 \leq i \leq \nu$, satisfy the same estimations for degrees and lengths of coefficients as $f_1, \ldots, f_m$ do. Then one can construct linear forms $L_0, \ldots, L_{n+1}$ from the statement of Lemma 23 (see also Remark 8 below) satisfying additionally the following conditions. For every $0 \leq j \leq \nu_0$, $\nu_0 + 1 \leq i \leq \nu$ for every $y \in W^{(i)} \cap \mathcal{Z}(L_{s_i+1}, \ldots, L_n)$ the equality $\mu(y, W^{(j)}) = \min\{\mu(x, W^{(j)}) : x \in W^{(i)}\}$ holds. The working time of*

this algorithm is polynomial in d^n, d_1, d_2, M, M_1, m, ν *and the sum of sizes* $\sum_{0\le i\le\nu} \mathrm{L}(x_i^{(0)})$ *of the points* $x_i^{(0)}$.

PROOF We shall describe a recursive algorithm by linear forms $L'_0,\dots,L'_{n+1} \in k[X_0,\dots,X_n]$ and a point $z'=(1:z'_1,\dots,z'_n)\in\mathbb{P}^n(\overline{k})$. Set $\overline{\lambda'}=(L'_0,\dots,L'_{n+1})$.

Let us construct using Lemma 23 linear forms $L'_0,\dots,L'_{n+1}$ (we change here the notation L_w for L'_w). Denote by $\varphi^{(i)}$ the morphism of the linear projection $\mathbb{P}^n(\overline{k})\setminus\mathcal{Z}(L'_0,L'_{s_i+1},\dots,L'_n)\to\mathbb{P}^{n-s_i}(\overline{k})$, $(X_0:\dots:X_n)\mapsto(L'_0:L'_{s_i+1}:\dots:L'_n)$, and by φ_i the restriction of $\varphi^{(i)}$ to $W^{(i)}$. So φ_i is a finite dominant linear projection. Set $P_i(z')=\varphi_i^{-1}((1:z'_{s_i+1}:\dots:z'_n))\subset\mathbb{P}^n(\overline{k})$ and

$$\mu(z')=\sum_{0\le j\le\nu_0}\ \sum_{\nu_0+1\le i\le\nu}\ \sum_{x'\in P_i(z')}\mu(x',W^{(j)})$$

(we shall use this definition also for other points from $\mathbb{P}^n(\overline{k_2})$ in place of $z'\in\mathbb{P}^n(\overline{k})$ replacing k by k_2).

We take the constructed $L'_0,\dots,L'_{n+1}$ and $z'=(1:0:\dots:0)$ as the base of the recursion. The recursive assumption is that linear forms $L'_0, L'_1-z'_1L'_0,\dots,L'_n-z'_nL'_0,L'_{n+1}$ (in place of $L_0,\dots,L_{n+1}$) satisfy assertion of Lemma 23.

Let us describe the general step of this recursion. For every $\nu_0+1\le i\le\nu$ using Theorem 2 [9] and Theorem 2 [8] we compute the set $P_i(z')$.

Let us show that using [4] and [8] one can compute the multiplicity $\mu(x,W^{(j)})$ for every x' and every j. We refer to Lemma 18 [4]. Under conditions and in the notation of this lemma $\mu(x,V_{s+1})=\deg V_{s+1}-\#A$, where the finite set $A=(V_{s+1}\setminus\{x\})\cap\mathcal{Z}(L_{s+2},\dots,L_n)$, see formula (21) of [4]. Further, for a defined over k and irreducible over k component W of V_{s+1} the multiplicity $\mu(x,W)=\deg W-\#A(W)$ where $A(W)=(W\setminus\{x\})\cap\mathcal{Z}(L_{s+2},\dots,L_n)$ (there are two obvious misprints here in [4], which are immediately corrected from the context; namely, in formula (21) of [4] defining A one must replace V_{s+1} by $(V_{s+1}\setminus\{x\})$, and below in the similar formula for $A(W)$ one must replace W by $(W\setminus\{x\})$). If a point $z^{(0)}\in W$ is known then one can compute $\deg W$ and $\#A(W)$ using Theorem 2 [8]. Thus, one can compute $\mu(x,W)$. We apply this algorithm for $(V^{(j)},W^{(j)},x_j^{(0)},x')$ in place of $(V,W,z^{(0)},x)$ and compute all the required multiplicities $\mu(x',W^{(j)})$. After that we construct $\mu(z')$. In what follows we compute $\mu(z^*)$ in the similar way for other points $z^*=(1:z_1^*:\dots:z_n^*)$ in the case when elements $z_j^*\in\overline{k}_2$.

At the recursive step we enumerate i,x',j such that $\nu_0+1\le i\le\nu$, $x'\in P_i(z')$ and $0\le j\le\nu_0$. For the considered $x'\in P_i(z')$, j we perform the following construction.

Applying Theorem 2 [9] to the algebraic variety $(\mathrm{con}(V^{(i)})\times\mathrm{con}(V^{(j)}))(\overline{k_2})$ we decide whether there is a pair $(x'',x''')\in(W^{(i)}\times W^{(j)})(\overline{k_2})$ such that

$$\begin{cases}\sum_{0\le w\le n}|(X_w/L'_0)(x'')-(X_w/L'_0)(x')|^2\le\varepsilon_1,\\ \varepsilon_2\le\sum_{0\le w\le n}|(X_w/L'_0)(x'')-(X_w/L'_0)(x''')|^2\le\varepsilon_1,\\ \varphi^{(j)}(x'')=\varphi^{(j)}(x'''),\end{cases}\tag{38}$$

cf. system (2) from [4]. If there does not exist such a pair (x'',x'''), the dimension $\dim W^{(i)}\le\dim W^{(j)}$ (i.e., $s_i\ge s_j$), and not all i,x',j are enumerated then we

proceed to the next i, x', j. Suppose that there exists a pair $(x'', x''') \in (W^{(i)} \times W^{(j)})(\overline{k_2})$ satisfying (38). Then we compute $\varphi_i(x'') = (1 : z''_{s_i+1} : \ldots : z''_n)$ where all $z''_w \in \overline{k_2}$. Put $z''_w = z'_w$ for $1 \le j \le s_i$ and $z'' = (1 : z''_1 : \ldots : z''_n)$. Then $\mu(z'') < \mu(z')$.

Assume that $\dim W^{(i)} > \dim W^{(j)}$, and there does not exist a pair $(x'', x''') \in (W^{(i)} \times W^{(j)})(\overline{k_2})$ satisfying (38). Then we decide whether $x' \in W^{(j)}$ applying Theorem 2 [8]. If $x' \notin W^{(j)}$ and not all i, x', j are enumerated then we proceed to the next i, x', j. Suppose that $x' \in W^{(j)}$. By the recursive assumption, Theorem 3 and Theorem 1 (a) from [8] the point $x' \notin \overline{V^{(j)} \setminus W^{(j)}}$ in this case. Hence there is a point $x'' \in W^{(i)}(\overline{k_2})$ such that

$$\begin{cases} \sum_{0 \le w \le n} |(X_w/L'_0)(x'') - (X_w/L'_0)(x')|^2 \le \varepsilon_1, \\ \sum_{1 \le w \le m_j} |f_w^{(j)}((X_0/L'_0)(x''), \ldots, (X_n/L'_0)(x''))|^2 = \varepsilon_2. \end{cases}$$

We construct such a point x'' applying Theorem 2 [9] to the algebraic variety $(\mathrm{con}(V^{(i)}))(\overline{k_2})$. Now $x'' \in W^{(i)}(\overline{k_2}) \setminus W^{(j)}(\overline{k_2})$ and hence $\mu(x'', W^{(j)}) \le \mu(x', W^{(j)})$. Let us compute $\varphi_i(x'') = (1 : z''_{s_i+1} : \ldots : z''_n)$ where all $z''_w \in \overline{k_2}$. Put $z''_w = z'_w$ for $1 \le j \le s_i$ and $z'' = (1 : z''_1 : \ldots : z''_n)$. Then $\mu(z'') < \mu(z')$.

Let us return to the general case when $\dim W^{(i)}$, $\dim W^{(j)}$ are arbitrary. Suppose that the point z'' is constructed in some of the cases considered above. It is not difficult to see, by e.g., [4], that for an arbitrary line $\mathfrak{l} \subset \mathbb{P}^n(\overline{k_2})$ and any point $x^* \in \mathfrak{l}$ the number of elements $x^{**} \in \mathfrak{l}$ with $\mu(x^{**}, W^{(j)}) > \mu(x^*, W^{(j)})$ is bounded from above by a polynomial in d^n, cf. also the proof of Lemma 1 [7]. Therefore, applying Lemma 3 and the algorithm of reduction to integer coefficients, see Proposition 2 [9], to the points z'' and $(1 : 0 : \ldots : 0)$ one can construct a point $\widetilde{z} = (1 : \widetilde{z}_1 : \ldots : \widetilde{z}_n) \in \mathbb{P}^n(\overline{k})$ with integer $\widetilde{z}_w$ of length $O(n \log d + \log \nu)$ such that $\mu(\widetilde{z}) \le \mu(z'')$ and the linear forms $L'_0, L'_1 - \widetilde{z}_1 L'_0, \ldots, L'_n - \widetilde{z}_n L'_0, L'_{n+1}$ satisfy assertion Lemma 23 (in place of L_w, $0 \le w \le n+1$). Then we replace z' by $\widetilde{z}$ and proceed to the next step of the recursion with the same $\overline{\lambda'}$.

Finally we shall come to the step when for all $\nu_0 + 1 \le i \le \nu$ for all $x' \in P_i(z')$, $0 \le j \le \nu_0$ the point z'' is not constructed by the algorithm. Then obviously $\mu(x', W^{(j)}) = \min\{\mu(x, W^{(j)}) : x \in W^{(i)}\}$ for all $0 \le j \le \nu_0$. This step is final. We set $L_0 = L'_0$, $L_{n+1} = L'_{n+1}$, and $L_j = L'_j - z'_j L'_0$, $1 \le j \le n$. Now $L_0, \ldots, L_{n+1}$ satisfy the required properties by the described construction. The required bound for the working time of the described algorithm follows immediately from the ones for the working times of the applied algorithms. The lemma is proved.

REMARK 8 *Suppose that under the conditions of Lemma 23 and Lemma 24 the sum of the degrees of all the irreducible components of every $V^{(i)}$ is bounded from above by D. Then Lemma 3 and the algorithm of reduction to integer coefficients, see Proposition 2 [9], imply that one can construct linear forms $L_0, \ldots, L_{n+1}$ (from the statements of these lemmas) with integer coefficients of length $O(n + \log D + \log \nu)$.*

LEMMA 25 *Let $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n)$, where L'_j, $j \in \{0, \sigma+1, \ldots, n\}$, are linear forms with coefficients from a finite extension k' of k which is given by a primitive element and its minimal polynomial. Then one can decide whether $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n) \in \mathcal{U}'_0$. If it is true, one can compute $\lambda = (L_{s+1}, \ldots, L_\sigma)$*

satisfying condition (A), see the Introduction, conditions of Lemma 20, and such that $L' \in \mathcal{U}'_{\lambda,0}$*, all* $L_j \in k[X_0, \ldots, X_r]$*,* $s+1 \le j \le \sigma$*, are linear forms with integer coefficients of length* $O(n \log d + (n-\sigma)\log d')$*. The working time of this algorithm is polynomial in* d^n*,* $(d')^n$*,* d_1*,* d_2*,* d'_2*,* M*,* M'*,* M_1*,* m*,* r*, the size* $\mathrm{L}(x^{(0)})$ *of the point* $x^{(0)}$*, and the size* $\mathrm{L}(L')$ *of the element* L' *(including the size of the minimal polynomial of the primitive element of the extension* $k' \supset k$*).*

PROOF At first using the described algorithm one computes $\deg q_v$ and $\deg W'$. After that applying Theorem 1 [8] to the algebraic variety $V \cup \mathcal{Z}(e'_0)$ (respectively $V \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n)$) one constructs linear forms and representative systems of points of irreducible over k components of this algebraic variety.

Let $W^{(i)}$, $0 \le i \le \nu_0$, be all the irreducible over k components of $V \cup \mathcal{Z}(e'_0)$. Let $W^{(i)}$, $\nu_0 + 1 \le i \le \nu$ be all the irreducible over k components of $V \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n)$ such that $\dim W_i = \sigma - s$. Hence ν is bounded from above by a polynomial in $d^n (d')^{n-\sigma}$ by the Bézout theorem. Set $V^{(i)} = V \cup \mathcal{Z}(e'_0)$ for $0 \le i \le \nu_0$, and $V^{(i)} = V \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n)$ for $\nu_0 + 1 \le i \le \nu$. By Theorems 1 [8] and 2 [8] we construct a family of points $x_i^{(0)} \in W^{(i)}$ which are smooth points of $V^{(i)}$. So now one can apply Lemma 24 with the ground field k' in place of k (one can easily give in the considered case the new bounds in place of d, d_1, d_2, M, M_1 from Lemma 24) to the constructed family $(V^{(i)}, W^{(i)}, x_i^{(0)})$, $0 \le i \le \nu$ and the integer ν_0. Thus, we obtain linear forms $L_0, \ldots, L_{n+1}$. Let $0 \le j \le \nu_0$ and $\nu_0 + 1 \le i \le \nu$. Now $W^{(i)} \subset W^{(j)}$ if and only if $W^{(i)} \cap \mathcal{Z}(L_{n-\sigma+s+1}, \ldots, L_n) \subset W^{(j)}$. Hence one can decide whether $W^{(i)} \subset W^{(j)}$ using Theorem 2 [8]. Therefore, one can choose among $W^{(i)}$, $\nu_0 + 1 \le i \le \nu$ all the irreducible components E such that: $E \subset W$, $E \not\subset \mathcal{Z}(e'_0)$, and $E \not\subset W^{(j)}$ whenever $W^{(j)} \ne W$, $0 \le j \le \nu_0$. We have $\deg E = \#(E \cap \mathcal{Z}(L_{n-\sigma+s+1}, \ldots, L_n))$. Therefore, one can compute the degree δ of the union of all such irreducible components E. By Lemma 24 the element $L' \in \mathcal{U}'_0$ if and only if $\delta = \deg q_v \deg W'$. Set $\lambda = (L_{n-\sigma+s+1}, \ldots, L_n)$ for the constructed linear forms L_w. Finally, if $\delta = \deg q_v \deg W'$ then $L' \in \mathcal{U}'_{\lambda,0}$ by the described construction and Lemma 24. The bound for the length of integer coefficients of the linear forms L_j follows from Lemma 23 and Remark 8. The lemma is proved.

5 Proof of Theorem 1

The aim of this section is to prove Theorem 1. The basic construction described here will be used also in the next section. Let V and W be from the Introduction. We compute s and σ similarly to the algorithm from Section 4.

In what follows $\sigma + 1 \le i \le n+1$ be arbitrary but fixed (we effect the construction for all $\sigma + 1 \le i \le n+1$). Let us apply Lemma 23 to $n - \sigma + 1$ algebraic varieties $\widetilde{V}_j$, $\sigma \le j \le n$, and construct linear forms $L_0, \ldots, L_{n+1}$ with integer coefficients of length $O(n \log(dd'))$. Actually by Bézout theorem applying the algorithm of reduction to integer coefficients, see Proposition 2 [9], one can get new linear forms $L_0, \ldots, L_{n+1}$ satisfying the same properties but with integer coefficients of length $O(n \log d + (n-\sigma)\log d')$. So we shall suppose in what follows that the last estimate hold. Now assertion (a) holds for $L_0, \ldots, L_{n+1}$. In what follows the linear forms $L_0, \ldots, L_{n+1}$ will be changed recursively in the algorithm.

The further aim is to construct linear forms $L_0, \ldots, L_{n+1}$ satisfying (a) and

(b). According to Theorem 2 [8] one can decide whether a point $x \in W_i(L')$. Hence (replacing i by $i-1$) one can decide whether $x \in W_{i-1}(L') \cap \mathcal{Z}(e'_i) = W_i(L') \cup W_i^{(1)}(L') \cup W_i^{(2)}(L') \cup W_i^{(3)}(L')$, see the Introduction.

Let us show that the polynomial e'_i does not vanish on any irreducible component $W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$. Indeed, by Theorem 1 [8] for $\widetilde{V}_{i-1}$ all the points of $W_{i-1}(L') \cap \mathcal{Z}(L_\iota, \ldots, L_n)$ are smooth points of $\widetilde{V}_{i-1}$. The dimension $\dim W_{i-1}(L') \cap \mathcal{Z}(e'_i) = n - \iota$, see the Introduction. Hence every irreducible component of $W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$ contains a point from $W_{i-1}(L') \cap \mathcal{Z}(L_\iota, \ldots, L_n) \setminus \mathcal{Z}(e'_i)$. The required assertion is proved. Thus, $\dim W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n) \cap \mathcal{Z}(e'_i) = 0$.

Applying the algorithm from [2] we construct all the defined over k' and irreducible over k' components of dimension one of the algebraic variety $\widetilde{V}_{i-1} \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$, and then using Theorem 2 [8] choose among them components of $W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$. After that we construct all the points from the set $S_i = W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n) \cap \mathcal{Z}(e'_i)$. We shall assume without loss of generality that $S_i \cap \mathcal{Z}(L_0) = \emptyset$ and $L_0(x) \neq 0$, since, otherwise, applying the algorithm of reduction to integer coefficients one can replace the linear form L_0 for a new one and satisfy these conditions and all the conditions from (a).

Denote by $\varphi^{(i)}$ the morphism of linear projection $\mathbb{P}^n(\overline{k}) \setminus \mathcal{Z}(L_0, L_{\iota+1}, \ldots, L_n) \to \mathbb{P}^{n-\iota}(\overline{k})$, $(X_0 : \ldots : X_n) \mapsto (L_0 : L_{\iota+1} : \ldots : L_n)$, and by φ_i the restriction of $\varphi^{(i)}$ to $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$. Hence φ_i is a finite dominant linear projection. Now the projection $\varphi_i : W_{i-1}(L') \cap \mathcal{Z}(e'_i) \to \mathbb{P}^{n-\iota}(\overline{k'})$, $(X_0 : \ldots : X_n) \mapsto (L_0 : L_{\iota+1} : \ldots : L_n)$ is finite dominant. Let $x' \in \mathbb{P}^n(\overline{k'})$. Set $\mu_j(x') = \mu(x', W_{j-1}(L') \cap \mathcal{Z}(e'_j))$, $\sigma + 1 \leq j \leq n+1$. Let $z' \in \mathbb{P}^{n-\iota}(\overline{k'})$. For every $\sigma + 1 \leq i \leq n$ put

$$\mu^{(i)}(z') = \Big(\#\varphi_i^{-1}(z'), \#(\varphi_i^{-1}(z') \setminus \mathcal{Z}(e_0, \ldots, e_r)), \sum_{x' \in \varphi_i^{-1}(z')} \sum_{\sigma+1 \leq j \leq i-1} \mu_j(x')\Big).$$

Therefore, $\mu^{(i)}(z') \in \mathbb{Z}^3$. Let $(a_1, b_1, c_1), (a_2, b_2, c_2) \in \mathbb{Z}^3$. Set $(a_1, b_1, c_1) > (a_2, b_2, c_2)$ if and only if $(a_1 > a_2) \vee ((a_1 = a_2) \& (b_1 > b_2)) \vee ((a_1 = a_2) \& (b_1 = b_2) \& (c_1 < c_2))$. We shall use this definition also with other ground field containing k. This formula defines the linear order on $\mathbb{Z}^3$. In what follows in the proof we shall consider this linear order on $\mathbb{Z}^3$. Let $\mathfrak{l} \subset \mathbb{P}^{n-\iota}(\overline{k'})$ be a line such that $z' \in \mathfrak{l}$. The Bézout theorem implies that for all $z'' \in \mathfrak{l}$, except at most polynomial in $d^n(d')^{n-\sigma}$ number, the inequality $\mu^{(i)}(z'') \geq \mu^{(i)}(z')$ holds (here we leave the details to the reader). The same assertion is valid for any other ground field k^* containing k'.

Let $\nu_{\iota+1}, \ldots, \nu_n \in \overline{k'}$ (or more generally $\nu_{\iota+1}, \ldots, \nu_n$ belong to an extension of $\overline{k'}$) be given. Then one can construct the set $\varphi_i^{-1}((1 : \nu_{\iota+1} : \ldots : \nu_n))$ as follows. Denote by $\mathcal{L}_0$ (respectively $\mathcal{L}_\nu$) the subspace of linear forms generated by $L_{\iota+1}, \ldots, L_n$ (respectively $L_{\iota+1} - \nu_{\iota+1}L_0, \ldots, L_n - \nu_n L_0$) and compute the intersection $\mathcal{L}' = \mathcal{L}_0 \cap \mathcal{L}_\nu$. Let us construct applying Theorems 1 [8] and 2 [8] the set $S''_{i-1} = W_{i-1}(L') \cap \mathcal{Z}(L_\iota, \ldots, L_n)$. If all $\nu_j = 0$ then $\varphi_i^{-1}((1 : 0 : \ldots : 0)) = S_i$. So we shall suppose without loss of generality that not all ν_j are zeroes, say, $\nu_{\iota+1} \neq 0$. Let us construct using the algorithm from [2] the family E_a, $a \in A$, of all the irreducible components of dimension 2 of the algebraic variety $\widetilde{V}_{i-1} \cap \mathcal{Z}(\mathcal{L}')$ such that $E_a \cap S''_{i-1} \neq \emptyset$. Then again using the algorithm from [2] we construct the intersection $E_a \cap \mathcal{Z}(L_{\iota+1} - \nu_{\iota+1}L_0) \cap \mathcal{Z}(e'_i)$. The set $\varphi_i^{-1}((1 : \nu_{\iota+1} : \ldots : \nu_n))$ is the union of the last intersections over all $a \in A$.

Let $\mu^{(i)}(z') = (a(z'), b(z'), c(z'))$. Our aim is to construct a point $z' \in \mathbb{P}^{n-\iota}(\overline{k'})$ with maximal possible $\mu^{(i)}(z')$. At first we shall obtain z' such that $a(z') = \deg W_{i-1}(L') \cap \mathcal{Z}(e'_i)$. Let us describe a recursion by z'. We take $z' = (1 : 0 : \ldots : 0)$ as the base of this recursion. We act similirly to the algorithm for computing the degree from [4]. At the general step we look for a point z'' which is infinitely close to z' and such that $a(z'') > a(z')$ (if z'' exists). Namely, let us construct all the points $x' \in \varphi_i^{-1}(z')$. *Let $i = j$.* For every x' applying Theorem 2 [9] to the algebraic variety $(\mathrm{con}(\widetilde{V}_{i-1}) \times \mathrm{con}(\widetilde{V}_{i-1}))(\overline{k'_2})$ we decide whether there is a pair $(x'', x''') \in (W_{i-1}(L') \times W_{i-1}(L'))(\overline{k'_2})$ such that

$$\begin{cases} \sum_{0\le w\le n} |(X_w/L_0)(x'') - (X_w/L_0)(x')|^2 \le \varepsilon_1, \\ \varepsilon_2 \le \sum_{0\le w\le n} |(X_w/L_0)(x'') - (X_w/L_0)(x''')|^2 \le \varepsilon_1, \\ e'_i(x'') = e'_j(x''') = 0, \\ \varphi^{(j)}(x'') = \varphi^{(j)}(x'''), \end{cases} \tag{39}$$

cf. system (2) from [4]. The equality $a(z') = \deg W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ does not hold if and only if there are x'', x''' for some x'. In this case we choose such x', set $z'' = \varphi_i(x'')$, apply the algorithm of reduction to integer coefficients to $z'' \in \mathbb{P}^{n-\iota}(\overline{k'}) \setminus \mathcal{Z}(X_0)$, see Proposition 2 [9], and obtain a point $\widetilde{z}$ with integer coordinates such that $a(\widetilde{z}) > a(z')$. Hence $\mu^{(i)}(\widetilde{z}) > \mu^{(i)}(z')$. Then we replace z' by $\widetilde{z}$ and proceed to the next step of the recursion.

Finally we shall come to the case when for all x' there is no (x'', x'''). Then $a = \deg W_{i-1}(L') \cap \mathcal{Z}(e'_i)$. Hence, in particular, the differential $d_{x'}\varphi_i$ is an isomorphism for every $x' \in \varphi_i^{-1}(z')$, cf. [4] (we leave the details to the reader). According to the Bézout theorem and the algorithm of reduction to integer coefficients $z' = (1 : z'_{i,i-\sigma+s+1} : \ldots : z'_{i,n})$ where all $z'_{i,j}$ are integers of length $O(n \log d + (n - \sigma) \log d')$.

Now we are going to maximize $\#(\varphi_i^{-1}(z') \setminus \mathcal{Z}(e_0, \ldots, e_r))$. We enumerate the points $x \in \varphi_i^{-1}(z') \cap \mathcal{Z}(e_0, \ldots, e_r)$. Consider the following system of polynomial equations and inequalities with coefficients from $\overline{k'_2}$.

$$\begin{cases} \sum_{0\le j\le r} |e_j|^2 = \varepsilon_2, \\ \sum_{0\le i\le n} |X_i - (X_i/L_0)(x)|^2 \le \varepsilon_1, \\ e'_i = 0, \\ L_0 = 1. \end{cases} \tag{40}$$

Let us apply Theorem 2 [9] to this system and the variety $\mathrm{con}(V \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_{i-1}))$ and construct the set $S'' \subset \mathbb{A}^{n+1}(\overline{k'_2})$ from the statement of this theorem. Denote by $S''' \subset \mathbb{P}^n(\overline{k'_2})$ the set corresponding to S'' by the natural projection $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Further, using Theorem 2 [8] we decide whether there is a point $x^* \in S''' \cap W_{i-1}(L')(\overline{k'_2})$. Obviously such a point exists if and only if $x \in W_i(L') \cap W_i^{(3)}(L')$. Suppose that such a point exists. Then set $z^* = \varphi_i(x^*)$. Hence $x^* \in \varphi_i^{-1}(z^*) \setminus \mathcal{Z}(e_0, \ldots, e_r)$ and for every $x_1 \in \varphi_i^{-1}(z') \setminus \mathcal{Z}(e_0, \ldots, e_r)$ there is $x_1^* \in \varphi_i^{-1}(z^*) \setminus \mathcal{Z}(e_0, \ldots, e_r)$ which is infinitely close to z_1 by, e.g., Proposition 6 [3]. Hence the number of elements $\#(\varphi_i^{-1}(z^*) \setminus \mathcal{Z}(e_0, \ldots, e_r)) > \#(\varphi_i^{-1}(z') \setminus \mathcal{Z}(e_0, \ldots, e_r))$ and, therefore, $\mu^{(i)}(z^*) > \mu^{(i)}(z')$. Using the algorithm of reduction to integer coefficients, see Proposition 2 [9], one can replace subsequently the coordinates of z^* by integers (we find them by the enumeration) of length $O(n \log d + (n - \sigma) \log(d'))$ and obtain a new point z'' such that $a(z'') = a(z^*)$, $b(z'') \ge b(z^*) > b(z')$. Hence,

$\mu^{(i)}(z'') \geq \mu^{(i)}(z^*)$. Now we replace z' by z'' and proceed to the next step of the recursion.

If there is no point $x^* \in W_{i-1}(L')(\overline{k'_2})$ giving a solution of (40) does not exists and not all the points from $\varphi_i^{-1}(z') \setminus \mathcal{Z}(e_0, \ldots, e_r)$ are enumerated then we proceed to the next point $x \in \varphi_i^{-1}(z') \setminus \mathcal{Z}(e_0, \ldots, e_r)$. If all the points from $\varphi_i^{-1}(z') \setminus \mathcal{Z}(e_0, \ldots, e_r)$ are enumerated then we have obtained the required point z' such that $a(z') = \deg W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ and $b(z')$ is maximal possible.

Applying the algorithm of reduction to integer coefficients, see Proposition 2 [9], one can suppose additionally that the elements $z'_{i,j} = z'_{i_1,j}$ for all $\sigma + 1 \leq i_1 < i \leq n$, $i - \sigma + s + 1 \leq j \leq n$, i.e., $z'_{i,j} = z'_j$, $s + 2 \leq j \leq n$, does not depend on i. Then for all $\sigma + 1 \leq i \leq n + 1$

$$W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_0, L_{\iota+1}, \ldots, L_n) = \emptyset, \tag{41}$$

$$\#(W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1} - z'_{\iota+1}L_0, \ldots, L_n - z'_nL_0)) = \deg(W_{i-1}(L') \cap \mathcal{Z}(e'_i)). \tag{42}$$

Further, applying Lemma 3 and the algorithm of reduction to integer coefficients we shall assume without loss of generality that the linear forms $L_0, \ldots, L_{s+1}$, $L_{s+2} - z'_{s+2}L_0, \ldots, L_n - z'_nL_0, L_{n+1}$ satisfies additionally assertion (a) in place of $L_0, \ldots, L_{n+1}$ (we leave the details to the reader). Now replacing L_j by $L_j - z'_jL_0$, $s+2 \leq j \leq n$, we shall suppose in what follows that $z'_j = 0$ for all $s+2 \leq j \leq n$. Put $\overline{\lambda} = (L_0, \ldots, L_{n+1})$. Hence $\overline{\lambda}$ satisfies condition (B) from the beginning of Section 1 for the algebraic variety $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ (in place of V) for every $\sigma + 1 \leq i \leq n + 1$. Besides that, now $a((1 : 0 : \ldots : 0))$ and $b((1 : 0 : \ldots : 0))$ are maximal possible for every $\sigma + 1 \leq i \leq n + 1$.

Recall that the integers $a_\iota(\overline{\lambda}, W_{i-1}(L') \cap \mathcal{Z}(e'_i))$ and $b_\iota(W_{i-1}(L') \cap \mathcal{Z}(e'_i))$ are defined, see Section 1. Till the end of the proof we shall denote for brevity $a_\iota(\overline{\lambda}) = a_\iota(\overline{\lambda}, W_{i-1}(L') \cap \mathcal{Z}(e'_i))$, $b_\iota = b_\iota(W_{i-1}(L') \cap \mathcal{Z}(e'_i))$, and use the similar notations for other elements, say, λ^* in place of $\overline{\lambda}$. Now one can compute the integer $a_\iota(\overline{\lambda})$ as follows. Using the algorithm from [2] for every $\sigma+1 \leq i \leq n+1$ we construct all the irreducible over k' components of dimension 2 of the intersection $\widetilde{V}_{i-1} \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_{n-1})$. By Theorem 1 [8] (one can apply here also Theorem 3 in place of Theorem 1 [8]) and Theorem 2 [8] choose among them all the irreducible over k' components E of $W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_{n-1})$. Again applying the algorithm from [2] we construct all the irreducible over k' components of $E \cap \mathcal{Z}(e'_i)$ for all E. These are all the irreducible over k' components of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$. By [2] we construct also the separable polynomials $\Phi_{\iota,*} \in k'[Z_0, Z_n, Z_{n+1}]$, see Section 1, corresponding to the algebraic variety $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ in place of V from Section 1 (notice here that all the irreducible components of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ have the dimension $n - \iota$, and, therefore, we associate only one polynomial $\Phi_{\iota,*}$ to $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$). After that we compute the discriminant $R_{\iota,*}$ of $\Phi_{\iota,*}$ with respect to Z_{n+1}, its square free part $R_{\iota,*,0}$, and the integer $a_\iota(\overline{\lambda}) = \deg R_{\iota,*,0}$ for every $\sigma + 1 \leq i \leq n + 1$. We shall denote also $\Phi_{\iota,*} = \Phi_{\overline{\lambda},\iota,*}$ and $R_{\iota,*} = R_{\overline{\lambda},\iota,*}$ if the dependence on $\overline{\lambda}$ is important, see Section 1.

Let $1 \leq \tau \leq 4$ be an integer. Let $0 \leq i_1 < \ldots < i_\tau \leq n$ be integers. Assume that the field k' is supplied with a real structure, see [3]. Let the field $k'_j = k'(\varepsilon_1, \ldots, \varepsilon_j)$, $0 \leq j \in \mathbb{Z}$, where $\varepsilon_j > 0$ for $j \geq 1$ is an infinitesimal with respect to the field k'_{j-1}. Let $X_{w,j}$, $1 \leq w \leq 4$, $0 \leq j \leq n$; $Z_{w,j}$, $j \in$

$\{0, \iota+1, \ldots, n+1\} = I_\iota$, $1 \le w \le \tau$, be variables. Denote for brevity $\overline{L}_j = L_j + \sum_{1 \le w \le \tau} Z_{w,j} X_{i_w}$, $j \in I_\iota$ (so all $\overline{L}_j$ depend on the choice of $j_1, \ldots, j_\tau$). Denote $X^{(j)} = (X_{j,0}, \ldots, X_{j,n})$, and for an arbitrary polynomial c in $n+1$ variables $c(X^{(j)}) = c(X_{j,0}, \ldots, X_{j,n})$, $1 \le j \le 4$. Assume that $\iota \le n-2$. Consider the following system of polynomial equations and inequalities with squares of absolute values with coefficients from k_3' with respect to the variables $X_{w,j}$, $1 \le w \le 4$, $0 \le j \le n$; and $Z_{w,j}$, $j \in \{0, \iota+1, \ldots, n+1\} = I_\iota$, $1 \le w \le \tau$.

$$\left\{ \begin{array}{lll} \sum_{j \in I_\iota} \sum_{1 \le w \le \tau} |Z_{j,w}|^2 \le \varepsilon_1, & & \\ \varepsilon_2 \le |\overline{L}_n(X^{(1)}) - \overline{L}_n(X^{(3)})|^2 \le \varepsilon_1, & & \\ |\overline{L}_{n+1}(X^{(j)}) - \overline{L}_{n+1}(X^{(j+1)})|^2 = \varepsilon_3, & j = 1, 3, & \\ \overline{L}_\nu(X^{(j_1)}) = \overline{L}_\nu(X^{(j_2)}) = 0, & 1 \le j_1 < j_2 \le 4, & \iota+1 \le \nu \le n-1, \\ \overline{L}_n(X^{(j)}) - \overline{L}_n(X^{(j+1)}) = 0, & j = 1, 3, & \\ \overline{L}_0(X^{(j)}) - 1 = 0, & 1 \le j \le 4, & \\ e_i'(X^{(j)}) = 0, & 1 \le j \le 4. & \end{array} \right. \tag{43}$$

LEMMA 26 *Let $\iota \le \min\{n-2, n+1-\sigma+s\}$. The inequality $a_\iota(\overline{\lambda}) < b_\iota$ holds if and only if there exist $1 \le \tau \le 4$ and $0 \le i_1 < \ldots < i_\tau \le n$ satisfying the following property. System (43) has a solution $X_{w,i} = x^*_{w,i} \in \overline{k_3'}$, $1 \le w \le 4$, $0 \le i \le n$; $Z_{w,i} = z^*_{w,i} \in \overline{k_3'}$, $i \in \{0, \iota+1, \ldots, n+1\}$, $1 \le w \le \tau$, such that*

$$(x^*_{w,0} : \ldots : x^*_{w,n}) \in W_{i-1}(L')(\overline{k_3'}), \quad 1 \le w \le 4. \tag{44}$$

PROOF The proof of this lemma is similar to the one of Lemma 4. The lemma is proved.

Let us return to the algorithm for assertion (b). We shall describe a recursion on ι. The base of the recursion is $\iota = s+1$. Let us describe the general step of this recursion. Let $s+1 \le \iota \le n-2$ be maximal such that for all $s+1 \le w < \iota$ the integers $b_w(W_{i-1}(L') \cap \mathcal{Z}(e_i'))$ are computed, and $b_w(W_{i-1}(L') \cap \mathcal{Z}(e_i')) = a_w(\overline{\lambda}, W_{i-1}(L') \cap \mathcal{Z}(e_i'))$. Let us construct, see above, the polynomials $\Phi_{\overline{\lambda},\iota,*}$, $R_{\overline{\lambda},\iota,*}$ and $R_{\overline{\lambda},\iota,*,0}$. We shall suppose without loss of generality in what follows that Z_0 does not divide $R_{\overline{\lambda},\iota,*,0}$ replacing if it is necessary Z_0 by $Z_0 - \mu Z_n$ for an appropriate integer μ bounded from above by a polynomial in $d^n (d')^{n-\sigma}$. The linear forms $L_0 - \mu L_n$ satisfies the same properties as L_0 does, see above.

Let us enumerate integers τ and $i_1, \ldots, i_\tau$ such that $1 \le \tau \le 4$ and $0 \le i_1 < \ldots < i_\tau \le n$. Using Theorem 2 [9] to the algebraic variety $\mathrm{con}(\widetilde{V}_{i-1}) \subset \mathbb{A}^{n+1}(\overline{k'})$ and [4] we decide whether system (43) has a solution satisfying (44). If it is true, we construct such a solution. If system (43) does not have a solution satisfying (44) then we proceed to the next τ and $i_1, \ldots, i_\tau$.

Suppose that system (43) does not have a solution satisfying (44) for all $1 \le \tau \le 4$ and $0 \le i_1 < \ldots < i_\tau \le n$. Then $a_\iota(\overline{\lambda}) = b_\iota$ by Lemma 26.

If $\iota = \min\{n-2, n+1-\sigma+s\}$ the element $\overline{\lambda}$ satisfying assertion of Theorem 3 is constructed. Suppose that $\iota < \min\{n-2, n+1-\sigma+s\}$. Then we replace ι by $\iota+1$ and proceed to the next step of the described recursion.

Suppose that (43) has a solution satisfying (44) for some $1 \le \tau \le 4$ and $0 \le i_1 < \ldots < i_\tau \le n$. Let us compute, see [3], $z_{w,j} = \mathrm{st}_{\varepsilon_3}(z^*_{w,j})$, $1 \le w \le \tau$, $j \in I_\iota$, where all $z^*_{w,j}$ are from the statement of Lemma 26. Set $L_j^* = L_j$ for $j \in \{1, \ldots, \iota\}$, and $L_j^* = L_j + \sum_{1 \le w \le \tau} z_{w,j} X_{j_w}$ for $j \in I_\iota$. So L_j^* are

linear forms with coefficients from $\overline{k'_3}$. Put $\lambda^* = (L_0^*, \dots, L_n^*)$. By Lemma 2 the element λ^* satisfies property (B) for all algebraic varieties $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$, $\sigma+1 \le i \le n+1$. From system (43) we get that the polynomial $R_{\lambda^*,\iota,*}(1, Z_n)$ has more different roots than $R_{\overline{\lambda},\iota,*}(1, Z_n)$. So $a_\iota(\overline{\lambda}) < a_\iota(\lambda^*)$ and $a_w(\overline{\lambda}) = a_w(\lambda^*)$ for $1 \le w < \iota$. Applying Lemma 3 and the algorithm of reduction to integer coefficients, see [9] Proposition 2, we replace subsequently all the coefficients of linear forms L_j^* by integers of length $O(n \log d + (n-\sigma) \log d')$ and obtain new linear forms L_j^{**}, $0 \le j \le n+1$, satisfying the following properties. All L_j^{**}, $0 \le j \le n+1$, have integer coefficients of length $O(n \log d + (n-\sigma) \log d')$ and satisfy condition (B) for all algebraic varieties $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$, $\sigma+1 \le i \le n+1$. Further, L_j^{**}, $0 \le j \le n+1$, satisfy assertion (a) of Theorem 1 with L_j^* in place of L_j, $0 \le j \le n$, and (denote $\lambda^{**} = (L_0^{**}, \dots, L_{n+1}^{**})$) $a_w(\lambda^{**}) \ge a_w(\lambda^*)$ for all $1 \le w \le \iota$. In particular, $a_\iota(\lambda^{**}) > a_\iota(\overline{\lambda})$. Now we replace $\overline{\lambda}$ by λ^{**} and proceed to the next step of the described recursion.

Thus, finally we shall construct the element $\overline{\lambda}$ satisfying assertions (a) and (b) of Theorem 1. Notice that by the Bézout theorem the total number of steps of the described recursion is bounded from above by a polynomial in $d^n (d')^{n-\sigma}$.

Now any two points $y', y'' \in W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)$ belong to the same irreducible over k' (respectively irreducible over $\overline{k'}$) component of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ if and only if they belong to the same irreducible over k' (respectively irreducible over $\overline{k'}$) component of the curve $W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_{n-1})$. We have constructed all the irreducible components of the last algebraic variety, see above. Hence one can decide whether y' and y'' belong to the same irreducible over k' (respectively irreducible over $\overline{k'}$) component of the curve $W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_{n-1})$. The details here are similar to the ones from the proof of Theorem 2 [8]. Thus, one can compute the number of points $y'' \in W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)$ such that y' and y'' belong to the same defined over k' and irreducible over k' component of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$.

Let E be an irreducible over $\overline{k'}$ component of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that $y' \in E$. To avoid constructing composites $k'(y', y'')$, we compute at first by [2] the minimal field of definition k'' of $E \cap \mathcal{Z}(L_{\iota+1}, \dots, L_{n-1})$ containing k'. The field k'' coincides with the minimal field of definition of E containing k', cf. [8]. After that we implement all the construction over the field k'', i.e., assume that all the linear forms $L_j \in k''[X_0, \dots, X_n]$, $0 \le j \le n+1$. This allows to compute the number of points $y'' \in W_{i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n)$ such that y' and y'' belong to E. In the similar way in (c) extending the ground field k' till k'', we reduce the case of an irreducible over $\overline{k'}$ component E to the one of a defined over k'' and irreducible over k'' component, i.e., for this reduction in (c) it is sufficient to consider only the case when E is defined over k' and irreducible over k', and after that replace k' by k''. In the similar way in (f) extending the ground field k' till k'', we can suppose that E is defined over the ground field. So in (f) we shall assume without loss of generality that E is defined over k' and irreducible over $\overline{k'}$. The algorithms for assertions (a) and (b) are described completely.

LEMMA 27 *Let $\sigma + 1 \le i \le n + 1$ be an integer. Let k^* be a finitely generated extension of k' given by the minimal polynomial of a primitive element of k^* over a purely trancendental extension of k'. Let $\lambda^* = (L_0^*, \dots, L_{n+1}^*)$, where all $L_j^* \in k'[X_0, \dots, X_n]$ are linear forms. Then one can decide whether the intersection of $\widetilde{V}_{i-1}$ and $\mathcal{Z}(L_\iota^*, \dots, L_n^*)$ is transversal. Suppose that the in-*

tersection $\widetilde{V}_{i-1}$ *and* $\mathcal{Z}(L_\iota^*,\dots,L_n^*)$ *is transversal. Then the number of elements* $\#(W_{i-1}(L')\cap\mathcal{Z}(e_i')\cap\mathcal{Z}(L_\iota^*,\dots,L_n^*))<+\infty$*, and one can construct all the points from* $W_{i-1}(L')\cap\mathcal{Z}(e_i')\cap\mathcal{Z}(L_\iota^*,\dots,L_n^*)$ *in* $\mathbb{P}^n(\overline{k'})$*. Further, one can decide whether the intersection of* $W_{i-1}(L')\cap\mathcal{Z}(e_i')$ *and* $\mathcal{Z}(L_{\iota+1}^*,\dots,L_n^*)$ *is transversal. Hence one can decide whether assertions (a) and (b) hold for* λ^* *in place of* $\overline{\lambda}$*. The working time of the algorithm from this lemma is polynomial in* d^n*,* $(d')^n$*,* d_1*,* d_2*,* d_2'*,* M*,* M'*,* M_1*,* m*,* r*, the size* $\mathrm{L}(x^{(0)})$ *of the point* $x^{(0)}$*, the size* $\mathrm{L}(L')$ *of the element* L'*, and the size* $\mathrm{L}(\lambda^*)$ *of the element* λ^* *(including the sizes of the minimal polynomials of the primitive elements of* k' *and* k^**).*

PROOF By (a) and (b) we know the degrees of all the irreducible components of the algebraic varieties $\widetilde{V}_{i-1}$ and $W_{i-1}(L')\cap\mathcal{Z}(e_i')$. Hence applying Theorem 3 and Theorem 2 [8] one can decide whether the intersection of $\widetilde{V}_{i-1}$ and $\mathcal{Z}(L_\iota^*,\dots,L_n^*)$ is transversal (we leave the details to the reader). Assume that the last intersection is transversal. Then replacing in the proof of (b) L_j by L_j^*, $\iota+1\le j\le n$, we see that $W_{i-1}(L')\cap\mathcal{Z}(e_i')\cap\mathcal{Z}(L_{\iota+1}^*,\dots,L_n^*)=A^*$ is a finite set in $\mathbb{P}^n(\overline{k'})$, and construct this set in the similar way. Now the intersection of $W_{i-1}(L')\cap\mathcal{Z}(e_i')$ and $\mathcal{Z}(L_{\iota+1}^*,\dots,L_n^*)$ is transversal if and only if the number of elements $\#A^*=\deg W_{i-1}(L')\cap\mathcal{Z}(e_i')$. The lemma is proved.

To describe the algorithm for (c) it is sufficient to consider the case when E is defined over k' and irreducible over k', see the end of the proof of (b). Replacing the linear form L_0 for a new one we shall suppose in what follows in the proof that

(i) $L_0(x)\ne 0$,

(ii) the linear forms $L_0,\dots,L_{n+1}$ satisfy assertions (a), (b) of Theorem 1,

(iii) $L_0,X_1,\dots,X_n$ are linearly independent over k'.

Further, applying Lemma 3 and Lemma 27 one can replace the linear forms $L_1,\dots,L_{n+1}$ for new ones such that additionally

(iv) all L_j, $\iota+1\le j\le n$, are linear forms in $L_0,X_1,\dots,X_n$ with integer coefficients of length $O(n\log d+(n-\sigma)\log(d'))$.

Let $\mathcal{L}$ be the subspace of all linear forms from $k'(x)[X_0,\dots,X_n]$ vanishing at x. The basis of $\mathcal{L}$ is $X_i-(X_i/L_0)(x)L_0$, $1\le i\le n$. Set $L_j''=L_j-(L_j/L_0)(x)L_0\in\mathcal{L}$, $\iota+1\le j\le n$. For every $1\le i\le n+1$ let us represent $L_j=\widetilde{L}_j+\mu_jL_0$, where $\widetilde{L}_j\in k[X_1,\dots,X_n]$, and μ_j is an integer. Then obviously $L_j''=\widetilde{L}_j(X_1-(X_1/L_0)(x)L_0,\dots,X_n-(X_n/L_0)(x)L_0)$.

To compute the multiplicities we describe an algorithm recursive by $L_0,\dots,L_{n+1}$. We take the constructed $L_0,\dots,L_{n+1}$ satisfying (i)–(iv) as the base of this recursion. At the end of every step which is not final we get new $L_0,\dots,L_{n+1}$ such that such that the number of elements $W_{i-1}(L')\cap\mathcal{Z}(e_i')\cap\mathcal{Z}(L_{\iota+1}'',\dots,L_n'')$ is greater than at the beginning of the step (for old $L_0,\dots,L_{n+1}$). At the end of every step (and hence, if the considered step is not final, at the beginning of the next step) of this recursion conditions (i)–(iv) hold.

For an arbitrary quasiprojective algebraic variety E and its point a denote by $\mathrm{con}(a,E)$ the tangent cone of E at the point x, see e.g., [4]. Consider the condition

$$\mathrm{con}(x,W_{i-1}(L')\cap\mathcal{Z}(e_i'))\cap\mathcal{Z}(L_{\iota+1}'',\dots,L_n'')=\{x\}. \tag{45}$$

At the end of the recursion condition (45) will be fulfilled.

Let us describe the general step of the recursion. We follow the algorithm for computing the multiplicity from [4]. The only difference here is that the algebraic variety $W_{i-1}(L')\cap\mathcal{Z}(e'_i)$ is given in a slightly different form in comparison with [4]. But still using Theorem 2 [9] we have a certificate to decide whether a point $x\in W_{i-1}(L')$. This allows to describe the analog of the algorithm from [4]. Recall, see the beginning of the section, that for any $\nu_{\iota+1},\dots,\nu_n\in\overline{k'}$ we can compute $\varphi_i^{-1}((1:\nu_{\iota+1}:\ldots:\nu_n))$. Let us compute in this way all the points from $W_{i-1}(L')\cap\mathcal{Z}(e'_i)\cap\mathcal{Z}(L''_{\iota+1},\dots,L''_n)=\varphi_i^{-1}(\varphi_i(x))$. We shall increase recursively the number of elements $W_{i-1}(L')\cap\mathcal{Z}(e'_i)\cap\mathcal{Z}(L''_{\iota+1},\dots,L''_n)$. If it is not possible then $L''_{\iota+1},\dots,L''_n$ satisfies (45).

Namely, let $\mathbb{A}^{2n+1}(\overline{k'})$ has the coordinates functions $X_0,\dots,X_n$, $Y_1,\dots,Y_n$. Let us construct the homogeneous polynomial $\widetilde{e}_i\in k'(x)[X_0,\dots,X_n]$ such that $\widetilde{e}_i(L_0,\ X_1,\dots,X_n)=e'_i$. Now in place of system (15) from [4] consider the system of polynomial equations and inequalities with squares of absolute values in $2n+1$ variables with coefficients from $\overline{k'_2}$

$$\left\{\begin{array}{l}\widetilde{e}_i(1,X_1-(X_1/L_0)(x),\dots,X_n-(X_n/L_0)(x))=0,\\ \sum_{1\le i\le n}|X_i-(X_i/L_0)(x)|^2=\varepsilon_2^2,\\ \widetilde{L}_j(Y_1-(X_1/L_0)(x),\dots,Y_n-(X_n/L_0)(x))=0,\qquad \iota+1\le j\le n,\\ \sum_{1\le i\le n}|X_i-Y_i|^2\le\varepsilon_1\varepsilon_2^2,\\ L_0-1=0.\end{array}\right.\tag{46}$$

Then, cf. Lemma 8 [4], this system has a solution $(x_0^*,\dots,x_n^*,y_1^*,\dots,y_n^*)\in\mathbb{A}^{2n+1}(\overline{k'_2})$ such that

$$(x_0^*:x_1^*:\ldots:x_n^*)\in W_{i-1}(L')\tag{47}$$

if and only if condition (45) does not hold. One can apply Theorem 2 [9] and Theorem 2 [8] to the algebraic variety $(\mathrm{con}(\widetilde{V}_{i-1})\times\mathbb{A}^n)(\overline{k'_2})\subset\mathbb{A}^{2n+1}(\overline{k'})$ and construct a solution of system (46) satisfying (47) if it exists.

Suppose that a solution of (46) satisfying (47) is constructed. Then similarly to [4] one can construct linear forms $L_j^*\in\overline{k'_2}[X_1,\dots,X_n]$, $\iota+1\le j\le n$, such that $L_j^*(X_1-(X_1/L_0)(x),\dots,X_n-(X_n/L_0)(x))=0$, and the coefficients of the linear forms L_j^* are infinitely close with respect to the field k' to the corresponding ones of $\widetilde{L}_j$ for all $\iota+1\le j\le n$. Notice that the number of elements $\#(W_{i-1}(L')\cap\mathcal{Z}(e'_i)\cap\mathcal{Z}(L_{\iota+1}^*,\dots,L_n^*)>\#\varphi_i^{-1}\varphi_i(x)$. Denote by k^* the field generated over k' by the coefficients of linear forms L_j^*, $\iota+1\le j\le n$. Put $\lambda^*=(L_0,L_1\dots,L_\iota,L_{\iota+1}^*+\mu_{\iota+1}L_0,\dots,L_n^*+\mu_nL_0,L_{n+1})$. Hence λ^* is infinitely close to $\overline{\lambda}$. Therefore, by Lemma 2 assertions (a) and (b) of Theorem 1 hold for λ^* and k^* in place of $\overline{\lambda}$ and k'. Hence by Lemma 3, the Bézout theorem and Lemma 27 applying the algorithm of reduction to integer coefficients, one can replace subsequently the coefficients of linear forms $L_j^*+\mu_jL_0$, $\iota+1\le j\le n+1$, in the basis $L_0,X_1,\dots,X_n$, and obtain new linear forms L_j^{**}, $\iota+1\le j\le n+1$, satisfying the following properties. Put $L_j^{**}=L_j$ for all $j\in\{1,\dots,\iota,n+1\}$ and $\lambda^{**}=(L_0^{**},\dots,L_{n+1}^{**})$. Then conditions (i)–(iv) are fulfilled for λ^{**} in place of $\overline{\lambda}$ and the number of elements $W_{i-1}(L')\cap\mathcal{Z}(e'_i)\cap\mathcal{Z}(L_{\iota+1}^{**},\dots,L_n^{**})\ge\#(W_{i-1}(L')\cap\mathcal{Z}(e'_i)\cap\mathcal{Z}(L_{\iota+1}^*,\dots,L_n^*)>\#\varphi_i^{-1}\varphi_i(x)$. Now we replace $\overline{\lambda}$ by λ^{**} and proceed to the next step of the recursion.

Thus, finally we shall come to the case when system (46) does not have a solution satisfying (47). Hence, see above, condition (45) holds. Then, cf. Lemma 18 from [4], the multiplicity $\mu(x, E)$ is the number of the points $x^* \in S^*$ such that $x^* \in E$. Therefore, in this case one can compute the required multiplicity by (b). The algorithm for (c) is described completely.

Now assertion (g) follows from Lemma 3 and Lemma 27 immediately.

At present we are able to compute $\mu^{(i)}(z')$ for a given point $z' \in \mathbb{P}^{n-\iota}(\overline{k})$ (and for any point with coordinates from an extension of $\overline{k}$, when we apply the algorithm of reduction to integer coefficients in the recursion step, see below). Let us return to the construction of the point $z' \in \mathbb{P}^{n-\iota}(\overline{k})$ with maximal possible $\mu^{(i)}(z')$ (recall that we perform the construction for all $\sigma + 1 \le i \le n + 1$). Let us describe a recursion by z'. We take $z' = (1 : 0 : \ldots : 0)$ as the base of this recursion. At the general step we look for a point z'' which is infinitely close to z' and such that $\mu^{(i)}(z'') > \mu^{(i)}(z')$ (if z'' exists). Namely, let us construct all the points $x' \in \varphi_i^{-1}(z')$. For every x' for every $\sigma + 1 \le j < i$ applying Theorem 2 [9] to the algebraic variety $(\mathrm{con}(\widetilde{V}_{i-1}) \times \mathrm{con}(\widetilde{V}_{j-1}))(\overline{k'_2})$ we decide whether there is a pair $(x'', x''') \in (W_{i-1}(L') \times W_{j-1}(L'))(\overline{k'_2})$ such that (39) holds, cf. system (2) from [4]. The element $\mu^{(i)}(z')$ is not maximal possible if and only if there are x'', x''' for some x' and j. In this case we choose such x' and j and set $z'' = \varphi_i(x'')$. It is not difficult to see using e.g., [4] that for any line $\mathfrak{l} \subset \mathbb{P}^{n-\iota}(\overline{k})$ and for any point $z^* \in \mathfrak{l}$ there is only a polynomial in $d^n(d')^{n-\sigma}$ number of points $z^{**} \in \mathfrak{l}$ with $\mu^{(i)}(z^{**}) < \mu^{(i)}(z^*)$, cf. also the proof of Lemma 1 [7]. Hence we apply the algorithm of reduction to integer coefficients to $z'' \in \mathbb{P}^{n-\iota}(\overline{k'}) \setminus \mathcal{Z}(X_0)$, see Proposition 2 [9], and obtain a point $\widetilde{z}$ with integer coordinates such that $\mu^{(i)}(\widetilde{z}) > \mu^{(i)}(z')$. Then we replace z' by $\widetilde{z}$ and proceed to the next step of the recursion.

Finally we shall come to the case when for all x' and j there is no (x'', x'''). Then $\mu^{(i)}(z')$ is maximal possible. According to the Bézout theorem and the algorithm of reduction to integer coefficients $z' = (1 : z'_{i,i-\sigma+s+1} : \ldots : z'_{i,n})$ where all $z'_{i,j}$ are integers of length $O(n \log d + (n - \sigma) \log d')$.

Applying the algorithm of reduction to integer coefficients, see Proposition 2 [9], one can suppose additionally that the elements $z'_{i,j} = z'_{i_1,j}$ for all $\sigma + 1 \le i_1 < i \le n$, $i - \sigma + s + 1 \le j \le n$, i.e., $z'_{i,j} = z'_j$, $s + 2 \le j \le n$, do not depend on i. Further, applying assertion (g) (proved above) and the algorithm of reduction to integer coefficients we shall assume without loss of generality that the linear forms $L_0, \ldots, L_{s+1}, L_{s+2} - z'_{s+2}L_0, \ldots, L_n - z'_nL_0, L_{n+1}$ satisfies additionally assertions from (a) and (b) (in place of $L_0, \ldots, L_{n+1}$; we leave the details to the reader). Replacing L_j by $L_j - z'_jL_0$, $s + 2 \le j \le n$, we shall suppose in what follows that $z'_j = 0$ for all $s + 2 \le j \le n$. Put $\overline{\lambda} = (L_0, \ldots, L_{n+1})$. Hence now $\overline{\lambda}$ satisfies assertions (a) and (b), and $\mu^{(i)}((1 : 0 : \ldots : 0))$ is maximal possible for every $\sigma + 1 \le i \le n + 1$.

Now we are going to prove (d). Assume that a point $x \in \mathbb{P}^n(\overline{k'})$ is given. Recall that $z' = (1 : 0 : \ldots : 0) \in \mathbb{P}^{n-\iota}(\overline{k'})$. Let us construct a line $\mathfrak{l}_x \subset \mathbb{P}^{n-\iota}(\overline{k'})$ such that the points $z', \varphi_i(x) \in \mathfrak{l}_x$. Let us construct a point $z^* \in \mathfrak{l}_x(\overline{k'(\varepsilon_1)})$ (here ε_1 is an infinitesimal with respect to the field k) that is infinitely close to $\varphi_i(x)$. Further let us construct using the algorithms from [8], [9] the subset $S^* \subset \varphi_i^{-1}(z^*)$ of all the points that are infinitely close to x.

The number $\mu^{(i)}(z^*) = \mu^{(i)}(z')$ is maximal possible. This immediately implies the following assertions. The equality $\varphi_i^{-1}(z^*) \setminus \mathcal{Z}(e_0, \ldots, e_r) = \varphi_i^{-1}(z^*) \cap (W_i(L') \cup W_i^{(3)}(L'))$ holds. For an arbitrary irreducible over $\overline{k'}$ component E of $W_{i-1}(L') \cap \mathcal{Z}(e_i')$ the point $x \in E$ if and only if there is $x^* \in S^*$ such that $x^* \in E$. Besides that, if $E \not\subset \mathcal{Z}(e_0, \ldots, e_r)$ then $x^* \in E \setminus \mathcal{Z}(e_0, \ldots, e_r)$. Further, every point $x^* \in S^*$ belongs to only one irreducible component E of $W_{i-1}(L') \cap \mathcal{Z}(e_i')$. If $x^* \in E$ and $x^* \in E_1$ for some irreducible component of $W_{j-1} \cap \mathcal{Z}(e_j')$ with $\sigma \le j < i$ then $E \subset E_1$.

Now obviously

- $x^* \in W_i^{(3)}(L')$ if and only if $x^* \notin W_i(L')$ and $x^* \notin \mathcal{Z}(e_0, \ldots, e_r)$.

Therefore, one can decide whether $x \in W_i^{(3)}(L')$. Further,

- $x^* \in W_i^{(1)}(L') \cup W_i^{(2)}(L')$ if and only if $x^* \in \mathcal{Z}(e_0, \ldots, e_r)$.

Hence, one can decide whether $x \in W_i^{(1)}(L') \cup W_i^{(2)}(L')$. Since i is arbitrary one can also decide for every $\sigma \le j < i$ whether $x \in W_j^{(1)}(L') \cup W_j^{(2)}(L')$. Extending the ground field and replacing x by x^* one can decide for every $x^* \in S^*$ for every $\sigma \le j < i$ whether $x^* \in W_j^{(1)}(L') \cup W_j^{(2)}(L')$. Now notice that

- $x^* \in W_i^{(2)}(L')$ if and only if $x^* \in W_i^{(1)}(L') \cup W_i^{(2)}(L')$ and $x^* \in W_j^{(1)}(L') \cup W_j^{(2)}(L')$ for some $\sigma \le j < i$,
- $x^* \in W_i^{(1)}(L')$ if and only if $x^* \in W_i^{(1)}(L') \cup W_i^{(2)}(L')$ and $x^* \notin W_j^{(1)}(L') \cup W_j^{(2)}(L')$ for every $\sigma \le j < i$.

Thus, one can decide whether x is an element of $W_i(L')$, $W_i^{(1)}(L')$, $W_i^{(2)}(L')$, $W_i^{(3)}(L')$, $\sigma \le i \le n+1$.

REMARK 9 *Let E be an arbitrary irreducible component of $W_i^{(3)}(L')$ such that $x^* \in E$. Actually the described construction allows to decide also whether there is an irreducible component E_1 of $W_j^{(3)}(L')$ with $\sigma + 1 \le j < i$ such that $E \subset E_1$.*

Applying the described algorithm to the points from $\varphi_i^{-1}(z')$ we compute $\deg W_i(L') = \#(\varphi_i^{-1}(z') \cap W_i(L'))$ and $\deg W_i^{(j)}(L') = \#(\varphi_i^{-1}(z') \cap W_i^{(j)}(L'))$ for $j = 1, 2, 3$. The algorithm for (d) is described.

Assertion (e) follows from (b), (c) and (d) immediately.

Let us prove (f). It is sufficient to consider only the case when E is defined over k' and irreducible over $\overline{k'}$, see the end of the proof of assertion (b). So we shall assume that E is defined over k'. Let $\overline{\lambda}$ be the element constructed in (b). Let us construct homogeneous polynomials $e_i'', e_0'' \in k[X_0, \ldots, X_n]$ such that $\deg e_i'' = \deg e_0'' = d'$ and the number of elements

$$\#(W_{i-1}(L') \cap \mathcal{Z}(e_i'') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n) \setminus (\mathcal{Z}(e_{i+1}' \cup \overline{V \setminus W}) =$$
$$\#(e_i''/e_0'')(W_{i-1}(L') \cap \mathcal{Z}(e_i'') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)) = D_{i-1}(L')d'.$$

This can be done directly, but we use Theorem 1 [9]. Namely, let $m_0, \ldots, m_\nu$, $\nu = \binom{n+d'}{n} - 1$, be all the monomials in $X_0, \ldots, X_n$ of degree d'. Consider the

morphism $\omega : W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n) \to \mathbb{P}^{\nu}(\overline{k})$, $(X_0, \ldots, X_n) \to (m_0 : \ldots : m_{\nu})$, i.e., ω is the restriction of the Veronese morphism of degree d' to $W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$. We construct e_i'', e_0'' applying Theorem 1 [9] to $\widetilde{V}_{i-1} \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$, $W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$ and ω in place of V, W and p respectively.

Set $\widetilde{e}_i = e_i' + \varepsilon e_i''$, where $\varepsilon > 0$ is an infinitesimal with respect to the field k'. Let us extend the ground field k' till $k'(\varepsilon)$. Now the mapping of the standard part $\mathrm{st}_{\varepsilon} : \mathbb{P}^n(\overline{k(\varepsilon)}) \to \mathbb{P}^n(\overline{k})$ is defined. y Lemma 7 the set $W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n) \cap \mathcal{Z}(\widetilde{e}_i)$ is finite in $\mathbb{P}^n(\overline{k(\varepsilon)})$. Applying (b) over the ground field $k'(\varepsilon)$ we construct all the points (up to conjugation over $k'(\varepsilon)$; one can compute the number of elements of each class of conjugated over $k'(\varepsilon)$ points according to (b), see [8] for details) of the finite set $\widetilde{I} = W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \ldots, L_n) \cap \mathcal{Z}(\widetilde{e}_i) \subset \mathbb{P}^n(\overline{k(\varepsilon)})$. Denote by $\widetilde{I}_{y'}$ the subset of all the points $y \in \widetilde{I}$ such that $\mathrm{st}_{\varepsilon}(y) = y'$, i.e., all the points y which are infinitely close to y' with respect to the field k'. Computing the standard parts of the points from $\widetilde{I}$ we construct the set $\widetilde{I}_{y'}$ (again the points from $\widetilde{I}_{y'}$ are given up to a conjugation over $k'(\varepsilon)$; recall also that the standard parts are computed using Newton–Puiseux expansions, see [3]), and the number of elements $\#\widetilde{I}_{y'}$. Now assertion (f) follows from the following lemma.

LEMMA 28 *The index of intersection* $i(W_{i-1}(L'), \mathcal{Z}(e_i'); E) = \#\widetilde{I}_{y'}$.

PROOF Denote for brevity $\mathcal{L} = \mathcal{Z}(L_{\iota+1}, \ldots, L_n) \subset \mathbb{P}^n(\overline{k})$. The intersection $W_{i-1}(L') \cap \mathcal{Z}(e_i')$ and $\mathcal{L}$ is transversal. The point $y' \in E \cap \mathcal{L}$ is a smooth point of $W_{i-1}(L') \cap \mathcal{L}$. Hence there is only one irreducible over $\overline{k'}$ component E' of $W_{i-1}(L') \cap \mathcal{L}$ such that $y' \in E'$. Now by the property of indices of intersection corresponding to the associativity of intersection

$$i(W_{i-1}(L'), \mathcal{Z}(e_i'); E) = i(W_{i-1}(L'), \mathcal{Z}(e_i'); E) i(E, \mathcal{L}; y') =$$
$$i(E', \mathcal{Z}(e_i'); y') i(W_{i-1}, \mathcal{L}; E') = i(E', \mathcal{Z}(e_i'); y') = i(W_{i-1}(L') \cap \mathcal{L}, \mathcal{Z}(e_i'); y').$$

But $\widetilde{e}_i$ is infinitely close to to e_i', and the number of elements $\#(W_{i-1}(L') \cap \mathcal{L} \cap \mathcal{Z}(\widetilde{e}_i)) = D_{i-1}(L')d'$. It is known that in this case $i(W_{i-1}(L') \cap \mathcal{L}, \mathcal{Z}(e_i'); y') = \#\widetilde{I}_{y'}$, see e.g., [12]. The lemma is proved.

Assertion (f) is proved. The required estimations for the working times of the algorithms constructed in this section follow immediately from the ones of the applied algorithms. Theorem 1 is proved completely.

6 Construction of an element from $\mathcal{U}_{n+1}'''$

We shall use the notations from the Introduction. Let $L' = (L_0', L_{\sigma+1}', \ldots, L_n') \in \mathcal{U}_0'$ be given, e.g., the element construct in Section 4. We shall assume that all linear forms L_j' have coefficients from k. The element L' will be changed recursively in the algorithm. Our aim now is to get at the end of the recursion an element $L' \in \mathcal{U}_0''$.

Recall that $e_j' = L_j'(e_0, \ldots, e_r)$, $j \in \{0, \sigma+1, \ldots, n\}$. Let an integer $\sigma+1 \leq i \leq n-1$ be given. This integer i will be changed recursively (it is a second recursion). We take $i = \sigma + 1$ as the base of this recursion. The recursive assumption here is that additionally $(L_{\sigma+1}', \ldots, L_{i-1}') \in \mathcal{U}_{i-1}''$. Let the algebraic

varieties $\widetilde{V}_j$, $\sigma \le j \le n$, and linear forms $L_0, \dots, L_{n+1}$ be from the statement of Theorem 1. Put $S_i = W_{i-1}(L') \cap \mathcal{Z}(L_{\iota+1}, \dots, L_n) \cap \mathcal{Z}(e'_i) = \varphi_i^{-1}(z')$, where $z' = (1 : 0 : \dots : 0) \in \mathbb{P}^{n-\iota}(\overline{k})$, $\iota = i - \sigma + s$, see Section 5. Let us construct $L_0, \dots, L_{n+1}$ and S_i, see Section 5. Similarly to Section 5 we suppose without loss of generality that $L_0(x^{(0)}) \ne 0$. We have $S_i \cap \mathcal{Z}(L_0) = \emptyset$, $\deg(W_{i-1} \cap \mathcal{Z}(e'_i)) = \#\varphi_i^{-1}(z')$. Set $S_i^{(0)} = \varphi_i^{-1}(z')$ and $S_i^{(1)} = \varphi_i^{-1}(z') \cap \mathcal{Z}(e_0, \dots, e_r)$. Therefore, by assertion (d) of Theorem 1 the sets $S_i^{(1)} = \varphi_i^{-1}(z') \cap (W_i^{(1)}(L') \cup W_i^{(2)}(L'))$ and $\varphi_i^{-1}(z') \setminus S_i^{(1)} = \varphi_i^{-1}(z') \cap (W_i(L') \cup W_i^{(3)}(L'))$.

As previously we shall assume that the field $\overline{k}$ is supplied with a real structure. Let the field $k_j = k(\varepsilon_1, \dots, \varepsilon_j)$ be as above, see Section 4. Recall that the mapping of the standard part $\mathrm{st}_j : \mathbb{P}^n(\overline{k_j}) \to \mathbb{P}^n(\overline{k})$ is defined. We assume that k_j is linearly disjoint with k_v over k. Let $z = (z_0 : \dots : z_n) \in S_i^{(1)}$ where all $z_j \in \overline{k}$. Let $0 \le \alpha \le r$ be an integer. Consider the system of equations and inequalities with respect to the variables $X_0, \dots, X_n, Y_{\sigma+1}, \dots, Y_i$ with coefficients from the field $\overline{k_2}$

$$\begin{cases} e'_j - Y_j e_\alpha = 0, & \sigma + 1 \le j \le i, \\ \varepsilon_2 \le \sum_{0 \le j \le r} |e_j|^2 \le \varepsilon_1, & \\ \sum_{\sigma+1 \le j \le i} |Y_j|^2 \le \varepsilon_1 & \\ \sum_{0 \le i \le n} |X_i - (X_i/L_0)(z)|^2 \le \varepsilon_1, & \\ L_0 = 1. & \end{cases} \tag{48}$$

LEMMA 29 *Let $L' \in \mathcal{U}'_0$ and $(L'_{\sigma+1}, \dots, L'_{i-1}) \in \mathcal{U}''_{i-1}$. Suppose that $W_i^{(3)}(L') = \emptyset$. Then $D_i > D(L'_{\sigma+1}, \dots, L'_i)$ if and only if there are $z = (z_0 : \dots : z_n) \in S_i^{(1)}$ and an integer $0 \le \alpha \le r$ such that system (48) has a solution $(x_0^*, \dots, x_n^*, y_{\sigma+1}^*, \dots, y_i^*) \in \mathbb{A}^{n-\sigma+i+1}(\overline{k}_2)$ with $x^* = (x_0^* : \dots : x_n^*) \in W(\overline{k}_2)$. More precisely, for the considered solution put*

$$L_j^* = L'_j - y_j^* X_\alpha, \quad e_j^* = L_j^*(e_0, \dots, e_r), \quad \sigma + 1 \le j \le i, \tag{49}$$

$$L_j^* = L'_j, \quad j \in \{0, i+1, \dots, n\}, \quad L^* = (L_0^*, L_{\sigma+1}^*, \dots, L_n^*). \tag{50}$$

Then $L^* \in \mathcal{U}'_0$, $(L_{\sigma+1}^*, \dots, L_{i-1}^*) \in \mathcal{U}'_{i-1}$, $x^* \in W_i(L^*) \cup W_i^{(3)}(L^*)$ and $D_i(L^*) > D_i(L')$, see Remark 2.

PROOF Let $D_i > D(L'_{\sigma+1}, \dots, L'_i)$. There exists $\varepsilon > 0$ such that ε is an infinitesimal with respect to the field k_1 and ε_2 is an infinitesimal with respect to the field $k_1(\varepsilon)$. Let us extend the ground field k till $k_2(\varepsilon)$. Since $\mathcal{U}''_i$ is open in the Zariski topology there are linear forms $\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_i \in \overline{k(\varepsilon)}[X_0, \dots, X_n]$ such that $(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_i) \in \mathcal{U}''_i$, and all the nonzero coefficients of the linear forms $\widetilde{L}_j - L'_j$, $\sigma + 1 \le j \le i$ are infinitesimal with respect to the field k.

Set $\widetilde{e}_j = \widetilde{L}_j(e_0, \dots, e_r)$, $\sigma + 1 \le j \le i$. The mapping of the standard part $\mathrm{st}_\varepsilon : \mathbb{P}^n(\overline{k(\varepsilon)}) \to \mathbb{P}^n(\overline{k})$ is defined similarly to st_j. By Lemma 15 the equality $\mathrm{st}_\varepsilon(W(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_{i-1})) = W(L'_{\sigma+1}, \dots, L'_{i-1})$ holds. Hence by Lemma 5 every irreducible component of $\mathrm{st}_\varepsilon(W(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_{i-1}) \cap \mathcal{Z}(\widetilde{e}_i))$ is an irreducible component of $W(L'_{\sigma+1}, \dots, L'_{i-1}) \cap \mathcal{Z}(e'_i)$.

Let us show that that every $z \in \varphi_i^{-1}(z') \cap W_i(L')$ is an smooth point of the intersection $W \cap \mathcal{Z}(e'_{\sigma+1}, \dots, e'_i)$. Indeed, $z \notin \mathcal{Z}(e_0, \dots, e_r)$ by assertion (d) of Theorem 1, and $W_{i-1}(L') \setminus \mathcal{Z}(e_0, \dots, e_r) = W \cap \mathcal{Z}(e'_{\sigma+1}, \dots, e'_{i-1}) \setminus \mathcal{Z}(e_0, \dots, e_r)$

since $(L'_{\sigma+1},\dots,L'_{i-1})\in\mathcal{U}''_{i-1}$, see the Introduction. Hence it is sufficient to prove that z is an smooth point of $W_{i-1}(L')\cap\mathcal{Z}(e'_i)$. But this follows from Theorem 1 (b). The required assertion is proved.

Recall that the index of intersection $\iota_j(L')=1$ for every $j\in J_i(L')$, $\sigma+1\le i\le n$, see the Introduction. Hence the intersection of the algebraic varieties W, $\mathcal{Z}(e'_{\sigma+1}),\dots,\mathcal{Z}(e'_i)$ is transversal at every irreducible component of $W_i(L')$. By Theorem 1 every point $z\in\varphi_i^{-1}(z')\cap W_i(L')$ is a smooth point of $W_i(L')$, and the intersection of $W_i(L')$ and $\mathcal{Z}(L_{\iota+1},\dots,L_n)$ is transversal at z. Now by Lemma 21 *the intersection of* $n-s+1$ *algebraic varieties* W, $\mathcal{Z}(e'_j)$, $\sigma+1\le j\le i$; $\mathcal{Z}(L_w)$, $\iota+1\le w\le n$, *is transversal at each point from* $\varphi_i^{-1}(z')\cap W_i(L')$.

We claim that there is an irreducible component E' of $W(\widetilde{L}_{\sigma+1},\dots,\widetilde{L}_i)$ such that $\mathrm{st}_\varepsilon(E')\not\subset W(L'_{\sigma+1},\dots,L'_i)$. Indeed, otherwise, $\mathrm{st}(W(\widetilde{L}_{\sigma+1},\dots,\widetilde{L}_i)\cap\mathcal{Z}(L_{\iota+1},\dots,L_n))\subset W_i(L')\cap\mathcal{Z}(L_{\iota+1},\dots,L_n)=\varphi_i^{-1}(z')\cap W_i(L')$. Hence by the implicit function theorem for every $z\in\varphi_i^{-1}(z')\cap W_i(L')$ there is the unique $\widetilde{z}$ from the intersection of $n-s+1$ algebraic varieties W, $\mathcal{Z}(\widetilde{e}_j)$, $\sigma+1\le j\le i$; $\mathcal{Z}(L_w)$, $\iota+1\le w\le n$, such that $\mathrm{st}_\varepsilon(\widetilde{z})=z$. Besides that, the considered intersection is transversal at $\widetilde{z}$. Therefore, the degree $\deg W(\widetilde{L}_{\sigma+1},\dots,\widetilde{L}_i)=\#(W(\widetilde{L}_{\sigma+1},\dots,\widetilde{L}_i)\cap\mathcal{Z}(L_{\iota+1},\dots,L_n))=D_i(L'_{\sigma+1},\dots,L'_i)<D_i$. This contradiction proves the required assertion is proved.

Now $\mathrm{st}(E')$ contains an irreducible component E'' of $W_i^{(1)}(L')\cup W_i^{(2)}(L')$, and $E'\ne E''$. There is a point $z\in E''\cap S_i^{(1)}$. Hence there are $x_0^*,\dots,x_n^*\in\overline{k(\varepsilon)}$ such that $x^*=(x_0^*:\dots:x_n^*)\in E'\setminus\mathcal{Z}(e_0,\dots,e_r)$ is infinitely close to z with respect to the field k, and $L_0(x_0^*,\dots,x_n^*)=1$. There is $0\le\alpha\le r$ such that $e_\alpha(x^*)\ne0$ and $(e_j/e_\alpha)(x^*)$ is not infinitely large with respect to the field k for every $0\le j\le r$. Set $y_j^*=((\widetilde{e}_j-e'_j)/e_\alpha)(x^*)$, $\sigma+1\le j\le i$. Then y_j^* is infinitesimal with respect to the field k for every $\sigma+1\le j\le i$. Obviously $(x_0^*,\dots,x_n^*,\,y_1^*,\dots,y_i^*)\in\mathbb{A}^{n-\sigma+i+1}(\overline{k(\varepsilon)})$ is a solution of system (48) such that $(x_0^*:\dots:x_n^*)\in W(\overline{k(\varepsilon)})$. Hence there is also a similar solution of this system but with the field $\overline{k}_2$ in place of $\overline{k(\varepsilon)}$.

Conversely, let $(x_0^*,\dots,x_n^*,\,y_{\sigma+1}^*,\dots,y_i^*)\in\mathbb{A}^{n-\sigma+i+1}(\overline{k}_2)$ be a solution of system (48) such that $x^*=(x_0^*:\dots:x_n^*)\in W(\overline{k}_2)$. Put $\overline{L}_j=L_j-(L_j/L_0)(x^*)L_0$ for $\iota+1\le j\le n$. We have $L^*\in\mathcal{U}'_0$, $(L^*_{\sigma+1},\dots,L^*_{i-1})\in\mathcal{U}''_{i-1}$ by Lemma 15. Hence condition (10) for $i-1$, L^* and k_2 in place of i, L' and k holds. Therefore, see the Introduction, $x^*\in(W_i(L^*)\cup W_i^{(3)}(L^*))\cap\mathcal{Z}(\overline{L}_{\iota+1},\dots,\overline{L}_n)$. The point $\mathrm{st}_2x^*\notin W_i(L')$ since, otherwise, $\mathrm{st}_2x^*\in W_i(L')\cap\mathcal{Z}(L_{\iota+1},\dots,L_n)\cap\mathcal{Z}(e_0,\dots,e_r)$ which is impossible by the construction from Section 5. On the other hand by the implicit function theorem, see above, there are $D_i(L')$ isolated point from $W_i(L^*)\cap\mathcal{Z}(\overline{L}_{\iota+1},\dots,\overline{L}_n)$ that are infinitely close to the corresponding points from $W_i(L')\cap\mathcal{Z}(L_{\iota+1},\dots,L_n))$. Hence $D_i(L^*)\ge D_i(L')$. Suppose that $x^*\in W_i(L^*)$. Then $D_i\ge D_i(L^*)>D_i(L')$. Assume $x^*\in W_i^{(3)}(L^*)$. Denote by k^* the field generated over k by the coefficients of all linear forms L_j^*, $j\in\{0,\sigma+1,\dots,n\}$. Let $k_v^*=k_v\otimes_k k^*$ be the composite of the fields k^* and k_v over k. Then by Remark 6 from [9] with the field $k'=k_v^*$ and W_v in place of W we have $D_i>D_i(L^*)\ge D_i(L')$. The lemma is proved.

Let us return to the description of the algorithm. Using the algorithm from Section 5 we decide whether $W_i^{(3)}(L')=\emptyset$. Assume that $W_i^{(3)}(L')=\emptyset$. Let us enumerate $z\in S_i^{(1)}$ and integers $0\le\alpha\le r$. For the considered z and α

let us apply Theorem 2 [9] to the algebraic variety $\mathrm{con}(V) \times \mathbb{A}^{i-\sigma}(\overline{k})$ (with the coordinate functions $X_0, \ldots, X_n$, $Y_{\sigma+1}, \ldots, Y_i$) and the semialgebraic set U'' given by system (48). By this theorem one constructs the finite set S'' of points of $\mathrm{con}(V)(\overline{k}_2) \times \mathbb{A}^i(\overline{k}_2)$. Denote by π the natural rational through projection $\mathrm{con}(V) \times \mathbb{A}^i \to \mathrm{con}(V) \to V$. Using Theorem 2 [8] we decide whether $\pi(S'') \cap W(\overline{k}_2) \neq \emptyset$. By Theorem 2 [9] and Lemma 29 the intersection $\pi(S'') \cap W(\overline{k}_2) = \emptyset$ for all $z \in S_i^{(1)}$ and integers $0 \leq \alpha \leq r$ if and only if $D_i = D_i(L')$.

Suppose that $\pi(S'') \cap W(\overline{k}_2) \neq \emptyset$ for the current $z \in S_i^{(1)}$ and integer $0 \leq \alpha \leq r$. Let us construct $(x_0^*, \ldots, x_n^*, y_{\sigma+1}^*, \ldots, y_i^*) \in S''$ such that $(x_0^* : \ldots : x_n^*) \in W(\overline{k}_2)$. Let us define $L_{\sigma+1}^*, \ldots, L_i^*$ by (49), (50). Let us construct the field k^* defined above by the algorithm from [2]. Using the algorithm from [4] we decide whether the point $x^* \in W_i(L^*)$.

Suppose that $x^* \in W_i(L^*)$. As we have seen in the proof of Lemma 29 the degree $D_i(L^*) > D_i(L')$. By Lemma 14 (with the field k_2 in place of k) using the algorithm of reduction to integer coefficients one can replace subsequently all the coefficients of linear forms $L_0^*, L_{\sigma+1}^*, \ldots, L_n^*$ by integers with lengths $O(n \log d + (n-\sigma)\log d')$ and obtain linear forms $L_0'', L_{\sigma+1}'', \ldots, L_n''$ similar to $L_0', L_{\sigma+1}', \ldots, L_n'$ such that $D(L_{\sigma+1}'', \ldots, L_i'') \geq D(L_{\sigma+1}^*, \ldots, L_i^*) > D_i(L')$. Set $L'' = (L_0'', L_{\sigma+1}'', \ldots, L_n'')$. Now we replace L' by L'' and return proceed to the next step of the described recursion with the same i. By the Bézout theorem for fixed i there might be at most $d^s(d')^{i-\sigma}$ steps when $x^* \in W_i(L^*)$.

Suppose that $x^* \in W_i^{(3)}(L^*)$ or $W_i^{(3)}(L') \neq \emptyset$. If $W_i^{(3)}(L') \neq \emptyset$ we set $L^* = L'$, $k^* = k$ (the described below construction is slightly simplified in this case). Let us define the set $S_i^{(w)}$, $w = 0, 1$, replacing in the definition of $S_i^{(w)}$ the linear forms L_j' by L_j^*, $j \in \{0, \sigma+1, \ldots, n\}$, and the field k by k^*. Let us construct the sets $S_i^{(w)}(L^*)$, $w = 0, 1$, replacing in the construction $S_i^{(w)}$ the linear forms L_j' by L_j^* and the field k by k^*. Let us compute a linear form $L_i''' \in k[X_0, \ldots, X_r]$ with integer coefficients of length $O(n \log d + (n-\sigma)\log d')$ such that $(S_i^{(0)}(L^*) \setminus S_i^{(1)}(L^*)) \cap \mathcal{Z}(L_i'''(e_0, \ldots, e_r)) = \emptyset$. Set $\widetilde{L}_j = L_j^*$ for $j \in \{0, \sigma+1, \ldots, n\} \setminus \{i\}$ and $\widetilde{L}_i = L_i^* + \varepsilon_3 L_i'''$. Set $\widetilde{e}_j = \widetilde{L}_j(e_0, \ldots, e_r)$ for every j. Hence all the linear forms $\widetilde{L}_j$ have coefficients from the field $k^*(\varepsilon_3) \subset \overline{k_3}$. We have $W_i^{(3)}(\widetilde{L}) \subset W_{i-1}(L^*) \cap W^{(4)}$. Hence $W_i^{(3)}(\widetilde{L})$ is a union of some irreducible components of dimension $n-i+\sigma-s$ of the variety $W_{i-1}(L^*) \cap W^{(4)}$. In particular, $W_i^{(3)}(L^*)$ is defined over $\overline{k^*}$. Therefore, $W_i^{(3)}(L^*) \supset W_i^{(3)}(\widetilde{L})$. Since $\widetilde{L}_i$ does not vanish on any irreducible component E of $W_i^{(3)}(L^*)$, the algebraic variety $W_i^{(3)}(\widetilde{L}) = \emptyset$. Hence $W_i(\widetilde{L}) \setminus \mathcal{Z}(e_0, \ldots, e_r) = W_{i-1}(L^*) \cap \mathcal{Z}(\widetilde{e}_i) \setminus \mathcal{Z}(e_0, \ldots, e_r)$ over the field $\overline{k_1}$.

Let us replace in the statement of Lemma 14 [9] W, $\widetilde{L}$, L', k, ε, $i+1$, s by W_v, $\widetilde{L}$, L^*, k_v^*, ε_3, i, σ respectively and apply this lemma. Then we get $\deg W_i(\widetilde{L}) = \deg W_{v,i}(\widetilde{L}) > \deg W_{v,i}(\widetilde{L}) = \deg W_i(L^*)$.

By Lemma 14 (with the field $k^*(\varepsilon_3)$ in place of k) using the algorithm of reduction to integer coefficients one can replace subsequently all the coefficients of linear forms $\widetilde{L}_0, \widetilde{L}_{\sigma+1}, \ldots, \widetilde{L}_n$ by integers with lengths $O(n \log d + (n-\sigma)\log d')$ and obtain linear forms $L_0'', L_{\sigma+1}'', \ldots, L_n''$ similar to $L_0', L_{\sigma+1}', \ldots, L_n'$ and such that $D(L_{\sigma+1}'', \ldots, L_i'') \geq D(\widetilde{L}_{\sigma+1}, \ldots, \widetilde{L}_i) > D_i(L')$. Now we replace L' by L'' and proceed to the next step of the described recursion with the same i. By the Bézout theorem for fixed i there might be at most $d^s(d')^{i-\sigma}$ steps when

$x^* \in W_i^{(3)}(L^*)$ or $W_i^{(3)}(L') \neq \emptyset$.

Suppose that $\pi(S'') \cap W(\overline{k}_2) = \emptyset$. Then we proceed to the next pair z, α. If all these pairs are enumerated then $D_i = D_i(L')$ as we have seen. In this case if $i < n$ we replace i by $i+1$ and proceed to the next step of the recursion with the same L'. Finally if in the considered case $i = n-1$ then $L' \in \mathcal{U}_0''$.

We have constructed $L' \in \mathcal{U}_0''$. Therefore, see the Introduction, conditions (8) and (10) hold for all $\sigma+1 \leq i \leq n-1$. Set this L' as a base of a new recursion. Our aim is to obtain at the end of this recursion an element $L' \in \mathcal{U}_0'''$ such that $\pi_{n+1}(L') \in \mathcal{U}_{n+1}'''$, see the Introduction. Let an integer $\sigma+1 \leq i \leq n+1$. Recall that $L'_{n+1} = L'_0$. This integer i will be changed recursively (it is a second recursion). The recursive assumption here is that additionally $(L'_{\sigma+1}, \ldots, L_{i-1}) \in \mathcal{U}_{i-1}'''$. We take $i = \sigma+2$ as the base of this recursion (recall that $\mathcal{U}_{\sigma+1}'' = \mathcal{U}_{\sigma+1}'''$, see the Introduction). Let $L_0, \ldots, L_{n+1}$, the morphism φ_i, the point z' be as above. Put $S_i^{(2)} = \varphi_i^{-1}(z') \cap W_i^{(2)}(L')$. Let us compute $S_i^{(2)}$ using the algorithms from Section 5. Let $0 \leq \alpha_1 < \alpha_2 \leq r$ be integers and $z \in S_i^{(2)}$. Denote for brevity $X^{(w)} = (X_{w,0}, \ldots, X_{w,n})$, $w = 1, 2$, and for an arbitrary polynomial c in $n+1$ variable set $c(X^{(w)}) = c(X_{w,0}, \ldots, X_{w,n})$. Consider the following system of polynomial equations and inequalities with respect to the variables $X_{w,0}, \ldots, X_{w,n}$, $w = 1, 2$, $Y_{\sigma+1}, \ldots, Y_i$, $Z_{\sigma+1}, \ldots, Z_i$ with coefficients from the field $\overline{k_4}$

$$\left\{\begin{array}{ll} (e'_j - Y_j e_{\alpha_1} - Z_j e_{\alpha_2})(X^{(w)}) = 0, & \sigma+1 \leq j \leq i,\ w = 1, 2, \\ \sum_{\sigma+1 \leq j \leq i} (|Y_j|^2 + |Z_j|^2) \leq \varepsilon_1, & \\ \sum_{0 \leq j \leq n} |X_{w,j} - (X_j/L_0)(z)|^2 \leq \varepsilon_1, & w = 1, 2 \\ \varepsilon_4 \leq \sum_{0 \leq j \leq r} |e'_j(X^{(w)})|^2 \leq \varepsilon_3, & w = 1, 2, \\ \sum_{0 \leq j \leq n} |X_{1,j} - X_{2,j}|^2 \geq \varepsilon_2, & \\ \sum_{\iota+1 \leq j \leq n} |L_j(X^{(1)}) - L_j(X^{(2)})|^2 \leq \varepsilon_3, & \\ L_0(X^{(w)}) = 1, & w = 1, 2. \end{array}\right. \tag{51}$$

LEMMA 30 *Let $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n) \in \mathcal{U}_0''$ and $(L'_{\sigma+1}, \ldots, L'_{i-1}) \in \mathcal{U}_{i-1}'''$. Then $D_i^{(2)} > D_i^{(2)}(L'_{\sigma+1}, \ldots, L'_i)$ if and only if there are $z \in S_i^{(2)}$ and integers $0 \leq \alpha_1 < \alpha_2 \leq r$ such that system (51) has a solution*

$$(x^*_{1,0}, \ldots, x^*_{1,n}, x^*_{2,0}, \ldots, x^*_{2,n}, y^*_{\sigma+1}, \ldots, y^*_i, z^*_{\sigma+1}, \ldots, z^*_i) \in \mathbb{A}^{2n-2\sigma+2i+2}(\overline{k}_4) \tag{52}$$

with $x^*_w = (x^*_{w,0} : \ldots : x^*_{w,n}) \in W(\overline{k}_4)$ for $w = 1, 2$.

PROOF Suppose that $D_i^{(2)} > D_i^{(2)}(L'_{\sigma+1}, \ldots, L'_i)$. There exists $\varepsilon' > 0$ such that ε' is an infinitesimal with respect to the field $k(\varepsilon_1)$ and ε_2 is an infinitesimal with respect to the field $k(\varepsilon_1, \varepsilon')$. There exists $\varepsilon'' > 0$ such that ε'' is an infinitesimal with respect to the field $k_3(\varepsilon')$ and ε_4 is an infinitesimal with respect to the field $k_3(\varepsilon', \varepsilon'')$. Let us extend the ground field k till $k_4(\varepsilon', \varepsilon'')$.

Since $\mathcal{U}_i'''$ is open in the Zariski topology there are linear forms $\widetilde{L}_{\sigma+1}, \ldots, \widetilde{L}_n \in \overline{k(\varepsilon')}[X_0, \ldots, X_n]$ such that $(\widetilde{L}_{\sigma+1}, \ldots, \widetilde{L}_i) \in \mathcal{U}_i'''$ and all the nonzero coefficients of the linear forms $\widetilde{L}_j - L'_j$, $\sigma+1 \leq j \leq i$ are infinitesimal with respect to the field k.

Set $\widetilde{e}_j = \widetilde{L}_j(e_0, \ldots, e_r)$, $\sigma+1 \leq j \leq i$. By Lemma 15 the equality $\mathrm{st}(W^{(2)}(\widetilde{L}_{\sigma+1}, \ldots, \widetilde{L}_i)) = W^{(2)}(L'_{\sigma+1}, \ldots, L'_i)$ holds. By Lemma 6 the intersection $W^{(2)}(\widetilde{L}_{\sigma+1}, \ldots, \widetilde{L}_i) \cap \mathcal{Z}(L_0, L_{\iota+1}, \ldots, L_n) = \emptyset$ in $\mathbb{P}^n(\overline{k(\varepsilon')})$. Hence, cf.

[4], there exist $z \in S_i^{(2)}$ and $\widetilde{x}_{w,0}, \dots, \widetilde{x}_{w,n} \in \overline{k(\varepsilon')}$ such that $\widetilde{x}_w = (\widetilde{x}_{w,0} : \dots : \widetilde{x}_{w,n}) \in W^{(2)}(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_i)(\overline{k(\varepsilon')})$, $L_0(\widetilde{x}_{w,0}, \dots, \widetilde{x}_{w,n}) = 1$, $\sum_{0 \le j \le n} |(X_j/L_0)(\widetilde{x}_w) - (X_j/L_0)(z)|^2 < \varepsilon_1$ for $w = 1, 2$; $\sum_{0 \le j \le n} |\widetilde{x}_{1,j} - \widetilde{x}_{2,j}|^2 > \varepsilon_2$; $(\widetilde{L}_j/L_0)(\widetilde{x}_1) = (L_j/L_0)(\widetilde{x}_2)$ for $\sigma + 1 \le j \le n$.

Further, for $w = 1, 2$ there exist $x^*_{w,0}, \dots, x^*_{w,n} \in \overline{k(\varepsilon', \varepsilon'')}$ such that $x^*_w = (x^*_{1,0} : \dots : x^*_{1,n}) \in W(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_{i-1})(\overline{k(\varepsilon', \varepsilon'')})$, $L_0(x^*_{w,0}, \dots, x^*_{w,n}) = 1$, $\mathrm{st}_{\varepsilon''}(x^*_w) = \widetilde{x}_w$; and $X_{w,j} = x^*_{w,j}$, $0 \le j \le n$, $w = 1, 2$, is a solution of the inequalities $\varepsilon_4 \le \sum_{0 \le j \le r} |e'_j(X^{(w)})|^2 \le \varepsilon_3$. Therefore, by our construction $X_{w,j} = x^*_{w,j}$ is a solution of all the equations and inequalities from system (51) which do not depend on Y_j and Z_j.

For all $0 \le w_1 < w_2 \le r$ denote $\Delta(w_1, w_2) = e_{w_1}(x^*_{1,0}, \dots, x^*_{1,n}) e_{w_2}(x^*_{2,0}, \dots, x^*_{2,n}) - e_{w_2}(x^*_{1,0}, \dots, x^*_{1,n}) e_{w_1}(x^*_{2,0}, \dots, x^*_{2,n})$. Suppose that not all $\Delta(w_1, w_2) = 0$. Then there are $0 \le \alpha_1 < \alpha_2 \le r$ such that $\Delta(\alpha_1, \alpha_2) \ne 0$ and $\Delta(w_1, w_2)/\Delta(\alpha_1, \alpha_2)$ are not infinitely large with respect to the field k for all $0 \le w_1 < w_2 \le r$. By Cramer's rule one can represent

$$(e_q - \mu_{q,1} e_{\alpha_1} - \mu_{q,2} e_{\alpha_2})(x^*_w) = 0, \tag{53}$$

where $\mu_{q,w} \in \overline{k(\varepsilon', \varepsilon'')}$ is not infinitely large with respect to the field k for every $0 \le q \le r$ and $w = 1, 2$.

Suppose that $\Delta(\alpha_1, \alpha_2) = 0$ for all $0 \le \alpha_1 < \alpha_2 \le r$. Let us choose α_1 such that $(e_q/e_{\alpha_1})(x^*_w)$ is not infinitely large with respect to k for every $0 \le q \le r$. Represent

$$e_q(x^*_w) = \mu_q e_{\alpha_1}(x^*_w), \tag{54}$$

for $w = 1, 2$. Permuting indices we can assume without loss of generality that $\alpha_1 \le r - 1$. In this case put $\alpha_2 = \alpha_1 + 1$ and all $z^*_j = 0$, $\sigma + 1 \le j \le i$.

Let $\widetilde{e}_j = e'_j + \sum_{0 \le q \le r} \nu_{j,q} e_q$ where all $\nu_{q,j} \in \overline{k(\varepsilon')}$ are infinitesimals with respect to the field k. By (53) or (54) there is a solution (52) of system (51) with $\overline{k(\varepsilon', \varepsilon'')}$ in place of $\overline{k}_4$ such that $x^*_w \in W(\overline{k(\varepsilon', \varepsilon'')})$, $w = 1, 2$. Now by the transfer principle, see [1], or directly we deduce that there is also solution (52) with $x^*_w \in W(\overline{k}_4)$, $w = 1, 2$.

Conversely, suppose that for some $0 \le \alpha_1 < \alpha_2 \le r$ there is a solution (52) such that $x^*_w \in W(\overline{k}_4)$, $w = 1, 2$. Then set $\overline{x}_{w,j} = \mathrm{st}_{\varepsilon_3, \varepsilon_4} x^*_{w,j} \in \overline{k}_2$, $0 \le j \le n$, $w = 1, 2$; $\overline{x}_w = (\overline{x}_{w,0} : \dots : \overline{x}_{w,n}) \in \mathbb{P}^n(\overline{k}_2)$, $w = 1, 2$; $\overline{y}_j = \mathrm{st}_{\varepsilon_3, \varepsilon_4} y^*_j \in \overline{k}_2$, $\overline{z}_j = \mathrm{st}_{\varepsilon_3, \varepsilon_4} z^*_j \in \overline{k}_2$, $\sigma + 1 \le j \le i$;

$$\overline{L}_j = L'_j - \overline{y}_j X_{\alpha_1} - \overline{z}_j X_{\alpha_2}, \quad \overline{e}_j = \overline{L}_j(e_0, \dots, e_r), \quad \sigma + 1 \le j \le i; \tag{55}$$

$\widetilde{L}_j = L'_j - y^*_j X_{\alpha_1} - z^*_j X_{\alpha_2}$, $\widetilde{e}_j = \widetilde{L}_j(e_0, \dots, e_r)$, $\sigma + 1 \le j \le i$. Notice that $\overline{x}_1 \ne \overline{x}_2$ and $(L_j/L_0)(\overline{x}_1) = (L_j/L_0)(\overline{x}_2)$, $\iota + 1 \le j \le n$, since $\sum_{0 \le j \le n} |x^*_{1,j} - x^*_{2,j}|^2 \ge \varepsilon_2$ and $\sum_{\iota+1 \le j \le n} |(L_j/L_0)(x^*_1) - (L_j/L_0)(x^*_2)|^2 \le \varepsilon_3$. Further, from system (51) and Lemma 15 we deduce that the point $\overline{x}_w \in \mathrm{st}_{\varepsilon_3, \varepsilon_4}(W(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_{i-1}) \cap \mathrm{st}_{\varepsilon_3, \varepsilon_4} \mathcal{Z}(\widetilde{e}_i) \subset W(\overline{L}_{\sigma+1}, \dots, \overline{L}_{i-1}) \cap \mathcal{Z}(\overline{e}_i)$, $w = 1, 2$. The point $\mathrm{st}_{\varepsilon_1, \varepsilon_2} \overline{x}_w = z \notin W_i(L') \cup W_i^{(1)}(L')$ by our construction. Recall that $W_i^{(1)}(L') = W_i^{(1)}(\overline{L}) = W_i^{(1)}$ since $L', \overline{L} \in \mathcal{U}''_0$, see the Introduction. Hence $\overline{x}_w \notin W(\overline{L}_{\sigma+1}, \dots, \overline{L}_i) \cup W^{(1)}(\overline{L}_{\sigma+1}, \dots, \overline{L}_i)$ again by Lemma 15. Therefore, $\overline{x}_w \in W^{(2)}(\overline{L}_{\sigma+1}, \dots, \overline{L}_i)$, $w = 1, 2$. On the other hand, we have $\mathrm{st}_2(W^{(2)}(\overline{L}_{\sigma+1}, \dots, \overline{L}_i) = W_i^{(2)}(L')$ by Lemma 15 and obviously $\mathrm{st}_2(\mathcal{Z}(L_{\iota+1} - (L_{\iota+1}/L_0)(\overline{z}) L_0, \dots, L_n - (L_n/L_0)(\overline{z}) L_0))$

$= \mathcal{Z}(L_{\iota+1}, \ldots, L_n)$. Assume that $\deg W^{(2)}(\overline{L}_{\sigma+1}, \ldots, \overline{L}_i) = \deg W_i^{(2)}(L')$. Then by Lemma 7 the points of $W^{(2)}(\overline{L}_{\sigma+1}, \ldots, \overline{L}_i) \cap \mathcal{Z}(L_{\iota+1} - (L_{\iota+1}/L_0)(\overline{z})L_0, \ldots, L_n - (L_n/L_0)(\overline{z})L_0)$ and $S_i^{(2)}$ are in the one–to–one correspondence by the rule $x \mapsto \mathrm{st}_2(x)$. This contradicts to the existence of $\overline{x}_1 \neq \overline{x}_2$ such that $\mathrm{st}_2(\overline{x}_1) = \mathrm{st}_2)(\overline{x}_2) = z$. Hence $\deg W^{(2)}(\overline{L}_{\sigma+1}, \ldots, \overline{L}_i) > \deg W_i^{(2)}(L')$ by Lemma 5. The lemma is proved.

Let us return to the description of the algorithm. Let us enumerate $z \in S_i^{(2)}$ and pairs of integers $0 \leq \alpha_1 < \alpha_2 \leq r$. For the considered z and α_1, α_2 let us apply Theorem 2 [9] to the algebraic variety $\mathrm{con}(V)^2 \times \mathbb{A}^{2i-2\sigma}(\overline{k})$ and the semialgebraic set U'' given by system (51). One constructs by this theorem the finite set S'' of points of $\mathrm{con}(V)(\overline{k}_4)^2 \times \mathbb{A}^{2i-2\sigma}(\overline{k}_4)$. Denote by π the natural rational through projection $\mathrm{con}(V)^2 \times \mathbb{A}^{2i-2\sigma} \to \mathrm{con}(V)^2 \to V \times V$. Using Theorem 2 [8] we decide whether $\pi(S'') \cap W(\overline{k}_4)^2 \neq \emptyset$. By Theorem 2 [9] and Lemma 29 the intersection $\pi(S'') \cap W(\overline{k}_4)^2 = \emptyset$ for all $z \in S_i^{(2)}$ and pairs of integers $0 \leq \alpha_1 < \alpha_2 \leq r$ if and only if $D_i^{(2)} = D_i^{(2)}(L')$.

Suppose that $\pi(S'') \cap W(\overline{k}_4)^2 \neq \emptyset$ for the current $z \in S_i^{(2)}$ and integers α_1, α_2. Then we construct a solution (52) from S'' such that $x_w^* \in W(\overline{k}_2)$ for $w = 1, 2$. Let us define $\overline{L}_j$ and $\overline{e}_j$ by (55). Set $\overline{L}_j = L'_j$ for $j = 0, i+1, \ldots, n$. Let us construct all $\overline{L}_j$ and $\overline{e}_j$, see [3]. As we have seen in the proof of Lemma 30 the degree $D^{(2)}(\overline{L}_{\sigma+1}, \ldots, \overline{L}_i) > D_i^{(2)}(L')$. By Lemma 16 (with the field k_2 in place of k) using the algorithm of reduction to integer coefficients one can replace subsequently all the coefficients of linear forms $\overline{L}_0, \overline{L}_{\sigma+1}, \ldots, \overline{L}_i$ by integers with lengths $O(n \log d + (n - \sigma) \log d')$ and obtain linear forms $L''_0, L''_{\sigma+1}, \ldots, L''_n$ similar to $L'_0, L'_{\sigma+1}, \ldots, L'_n$ and such that $D^{(2)}(L''_{\sigma+1}, \ldots, L''_i) \geq D^{(2)}(\widetilde{L}_{\sigma+1}, \ldots, \widetilde{L}_i) > D_i(L')$. Set $L'' = (L''_0, L''_{\sigma+1}, \ldots, L''_n)$. Now we replace L' by L'' and return to the beginning of the described recursion.

Suppose that $\pi(S'') \cap W(\overline{k}_2) = \emptyset$. Then we proceed to the next z, α_1, α_2. If all z, α_1, α_2 are enumerated then $D_i^{(2)} = D_i^{(2)}(L')$ as we have seen it. In this case if $i < n$ we replace i by $i + 1$ and return to the beginning of the recursion with the same L'. Finally if in the considered case $i = n$ then $L' \in \mathcal{U}_0'''$ and $\pi_{n+1}(L') \in \mathcal{U}_{n+1}'''$.

The working time of the algorithms from this section is polynomial in d^n, $(d')^n$, d_1, d_2, d'_2, M, M', M_1, m, r, the size $\mathrm{L}(x^{(0)})$ of the point $x^{(0)}$, and the sum of sizes of linear forms L'_j, $\sigma \leq j \leq n$. In particular, if all the linear forms L'_j have integer coefficients of length $O(n \log d + (n - \sigma) \log(d'))$ (as in the statement of Theorem 1 [9]) then this working time is the same as in Theorem 2. This follows immediately from the estimations for the working times of the applied algorithms.

7 Proof of Theorem 2

Now everything is ready for the proof of Theorem 2. We have proved assertion (a) of this theorem and constructed $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n) \in \mathcal{U}_0'''$ such that $\pi_{n+1}(L') \in \mathcal{U}_{n+1}'''$ in the previous section. Let us prove (b) (and hence also (c) and (d)). Let us replace the notation $L_0, \ldots, L_{n+1}$ by $\widetilde{L}_0, \ldots, \widetilde{L}_{n+1}$ in the statement of Theorem 1 (to avoid an ambiguity in what follows). Let us construct

these linear forms $\widetilde{L}_0, \dots, \widetilde{L}_{n+1}$ from Theorem 1 for the considered L'.

Recall some properties of linear forms $\widetilde{L}_0, \dots, \widetilde{L}_{n+1}$, which we shall need, see the statement of Theorem 1 and its proof in Section 5.

(i) $\widetilde{L}_0, \dots, \widetilde{L}_{n+1}$ have integer coefficients of length $O(n \log d + (n-\sigma)\log d')$; satisfy (in place of $L_0, \dots, L_{n+1}$) assertion (a) of Theorem 1.

(ii) The intersection of the algebraic variety $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ and the linear subspace $\mathcal{Z}(\widetilde{L}_{\iota+1}, \dots, \widetilde{L}_n)$ is transversal for every $\sigma + 1 \le i \le n$.

(iii) Let us replace in the definitions of φ^i and $\mu^{(i)}$ in Section 4 linear forms $L_0, \dots, L_{n+1}$ by $\widetilde{L}_0, \dots, \widetilde{L}_{n+1}$. Then $\mu^{(i)}((1 : 0 : \dots : 0))$ is maximal possible, see the notation in Section 4 ,for every $\sigma + 1 \le i \le n$.

Let us apply Lemma 25 to the algebraic varieties $\widetilde{V}_j$, $\sigma \le j \le n$, in place of $V^{(i)}$, $0 \le i \le \nu$, and construct

(iv) $\lambda = (L_{s+1}, \dots, L_\sigma) \in k[X_0, \dots, X_n]$ satisfying the assertion of Lemma 25 for the considered $L' \in \mathcal{U}'''_0$.

Now we identify the affine space $\mathbb{A}^{(n+2)(n+1)}(\overline{k})$ with the set of all $(n+2)$–tuples of linear forms from $\overline{k}[X_0, \dots, X_n]$. Put $\lambda_0 = (0, \dots, 0, L_{s+1}, \dots, L_\sigma, 0) \in \mathbb{A}^{(n+2)(n+1)}(\overline{k})$ and $\widetilde{\lambda} = (\widetilde{L}_0, \dots, \widetilde{L}_{n+1})$. Applying Lemma 3 and the algorithm of reduction to integer coefficients (it is possible to apply them by Lemma 19 and Theorem 1 (d) [9]) to the line $\mathfrak{l}$ containing $\widetilde{\lambda}$ and λ_0 we construct an element $\lambda^* = (L^*_0, \dots, L^*_{n+1}) \in \mathfrak{l}$ satisfying (i)–(iii) in place of $\widetilde{\lambda}$ and such that (iv) holds for $(L^*_{n-\sigma+s+1}, \dots, L^*_n)$ in place of λ. Then we replace $\widetilde{\lambda}$ by λ^*. Thus, we shall assume in what follows without loss of generality that $L_j = \widetilde{L}_{j+n-\sigma}$, $s+1 \le j \le \sigma$, and (i)–(iv) hold.

For any irreducible component E of $W_{\lambda,i}(L')$ there is an irreducible component E_1 of $W \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma) \cap \mathcal{Z}(e'_{\sigma+1}, \dots, e'_i)$ such that $E_1 \supset E$. Obviously $E_1 \subset W_i(L')$. Hence

$$W_i(L') \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma) \supset W_{\lambda,i}(L'), \quad \sigma \le i \le n \tag{56}$$

(and, more than that, these inclusions hold for any λ, L' such that $W_{\lambda,i}(L')$ and $W_i(L')$ are defined).

We have $L' \in \mathcal{U}'''_0$ and $L' \in \mathcal{U}'_{\lambda,0}$. By Lemma 11 [9] an irreducible component E of $W_i(L') \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma)$ is an irreducible component of $W_{\lambda,i}(L')$ if and only if E contains a point from $W_{\lambda,n}(L')$.

The degree $\deg W_n(L') = \deg p_v \deg W' = \deg W_{\lambda,n}(L')$. Hence by the Bézout theorem $W_n(L') \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma) = W_{\lambda,n}(L')$. By Lemma 11 [9] for every irreducible component E_0 of $W_{v,i}(L')$ the intersection $E_0 \cap W_{v,n}(L') \ne \emptyset$. Hence every irreducible component E_1 of $W_i(L')$ contains an irreducible component E_2 of $W_n(L')$. By (iv) the variety $E_1 \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma)$ is irreducible for $i < n$. Therefore, every irreducible component E of $W_i(L') \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma)$ contains a point z from $W_{\lambda,n}(L')$. Thus,

$$W_{i-1}(L') \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma) = W_{\lambda,i-1}(L'), \quad \sigma + 1 \le i \le n+1. \tag{57}$$

Since $L' \in \mathcal{U}''_0$ condition (10) holds for all $\sigma + 1 \le i \le n-1$. Hence by (57) condition (10) is satisfied for W_λ, p_λ, L' in place of W, p, L' for all $\sigma + 1 \le i \le$

$n-1$. Intersections in the left part of (57) are transversal by (i). Therefore, $D_i = D_i(L') = D_{\lambda,i}(L') \le D_{\lambda,i}$ by (57) and $D_{\lambda,i} \le D_i$ by (56) with U in place of L'. Hence $D_{\lambda,i} = D_i$.

Intersecting the both sides of (57) with $\mathcal{Z}(e'_i)$ we get for every $\sigma+1 \le i \le n$

$$(W_i(L') \cup W_i^{(1)} \cup W_i^{(2)}(L')) \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma) = W_{\lambda,i}(L') \cup W_{\lambda,i}^{(1)} \cup W_{\lambda,i}^{(2)}(L'). \quad (58)$$

The codimension of every irreducible component of the algebraic variety in the right part of (58) is the sum of codimension of two intersected varieties from the left part for every $\sigma+1 \le i \le n$. Hence

$$W_i^{(2)}(L') \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma) \subset W_{\lambda,i}^{(2)}(L'), \quad (59)$$

$$W_i^{(1)} \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma) \subset W_{\lambda,i}^{(1)} \cup W_{\lambda,i}^{(2)}(L'), \quad (60)$$

By (iii) every irreducible component E_3 of $W_i^{(1)} \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ is not contained in any irreducible component of $W_j^{(1)}$ for every $\sigma+1 \le j < i$. Hence also E_3 is not contained in any irreducible component of $W_j^{(2)}(L')$ for every $\sigma+1 \le j \le i$. Therefore, E_3 is not contained in any irreducible component of $W_j^{(1)} \cup W_j^{(2)}(L')$, $\sigma+1 \le j \le i$. Intersecting the last algebraic variety with $\mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ we get that E_3 is not contained in any irreducible component of $W_{\lambda,j}^{(1)} \cup W_{\lambda,j}^{(2)}(L')$ for every $\sigma+1 \le j \le i$. Thus, E_3 is not an irreducible component of $W_{\lambda,i}^{(2)}(L')$. Therefore, $W_i^{(1)} \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma) = W_{\lambda,i}^{(1)}$, $W_i^{(2)}(L') \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma) = W_{\lambda,i}^{(2)}(L')$. The intersection of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ and the linear subspace $\mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ is transversal. Hence $D_i^{(1)} = D_i^{(1)}(L') = D_{\lambda,i}^{(1)}(L') = D_{\lambda,i}^{(1)}$ since condition (10) holds for every $\sigma+1 \le i \le n-1$, and also the similar condition is satisfied for W_λ, p_λ, L' in place of W, p, L' for every $\sigma+1 \le i \le n-1$. Finally, $D_i^{(2)} = D_i^{(2)}(L') = D_{\lambda,i}^{(2)}(L') \le D_{\lambda,i}^{(2)}$.

LEMMA 31 *Let λ satisfies condition (A), see the Introduction. Suppose that $D_i = D_{\lambda,i}$ for every $\sigma \le i \le n$. Then $D_i^{(1)} \ge D_{\lambda,i}^{(1)}$, $D_i^{(2)} \ge D_{\lambda,i}^{(2)}$, for all $\sigma+1 \le i \le n$*

PROOF Indeed, (56) is fulfilled for U in place of L'. We have $U \in \mathcal{U}_0''$. Hence in the similar way (57)–(60) hold with U in place of L'. Every irreducible component E_3 of $W_i^{(1)} \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ is not an irreducible component of $W_{\lambda,i}^{(2)}(U)$ since the last algebraic variety does not have irreducible components defined over $\overline{k}$. Now replacing L' by U we deduce as above $W_i^{(1)} \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma) = W_{\lambda,i}^{(1)}$, $W_i^{(2)}(U) \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma) = W_{\lambda,i}^{(2)}(U)$. Hence the required inequalities follows from the Bézout theorem. The lemma is proved.

Now $D_i^{(2)} = D_i^{(2)}(L') = D_{\lambda,i}^{(2)}(L') = D_{\lambda,i}^{(2)}$ by Lemma 31. Using the algorithms from Section 5 we prove that one can compute D_i, $D_i^{(1)}$ and $D_i^{(2)}$, $\sigma \le i \le n$. Thus, $L' \in \mathcal{U}_{\lambda,0}'''$ and assertions (b)–(d) hold.

Let us prove (e). We have $L' \in \mathcal{U}_{\lambda,0}'''$ and (c), (d) hold. Therefore, see Section 4, $L' \in \mathcal{U}_0'$. As previously the left part of (57) contains its right part. Since $D_{\lambda,i-1} = D_{i-1}$ the equality (57) holds and the intersection in the left part of (57) is transversal. Further, $D_i(L') = D_{\lambda,i}(L') = D_{\lambda,i} = D_i$, $\sigma \le i \le n$. Hence $L' \in \mathcal{U}_0''$. Hence (58) is fulfilled. Therefore, again (59) and (60) are

fulfilled. Now the equalities $D^{(2)}_{\lambda,i} = D^{(2)}_i$, $D^{(1)}_{\lambda,i} = D^{(1)}_i$ imply all the assertions of (e).

Let us prove (f), cf. the proof of Lemma 13 [9] and Lemma 19 (a). Let $(\lambda', L'') \in \mathfrak{l}$, $(\lambda', L'') \neq (\lambda, L')$, $\lambda' = (L'_{s+1}, \dots, L'_\sigma)$, $L'' = (L''_{\sigma+1}, \dots, L''_n)$, where all $L'_j \in k[X_0, \dots, X_n]$, $s+1 \leq j \leq \sigma$; $L''_w \in \overline{k}[X_0, \dots, X_r]$, $w \in \{0, \sigma+1, \dots, n\}$, are linear forms. Let ε be the coordinate function of $\mathbb{A}^1(\overline{k})$. Set $L_{\varepsilon,j} = L_j + \varepsilon L'_j \in k(\varepsilon)[X_0, \dots, X_n]$, $s+1 \leq j \leq \sigma$; $L^{(\varepsilon)}_w = L'_w + \varepsilon L''_w \in \overline{k}(\varepsilon)[X_0, \dots, X_r]$, $e^{(\varepsilon)}_w = L^{(\varepsilon)}_w(e_0, \dots, e_r)$, $e''_w = L''_w(e_0, \dots, e_r)$, $w \in \{0, \sigma+1, \dots, n\}$, $\lambda_\varepsilon = (L_{\varepsilon,s+1}, \dots, L_{\varepsilon,\sigma})$, $L^{(\varepsilon)} = (L^{(\varepsilon)}_{\sigma+1}, \dots, L^{(\varepsilon)}_n)$. For every $a \in k$ put $\lambda_a = (L_{a,s+1}, \dots, L_{a,\sigma})$, $L^{(a)} = (L^{(a)}_{\sigma+1}, \dots, L^{(a)}_n)$, where $L_{\varepsilon,j} = L_j + aL'_j \in k[X_0, \dots, X_n]$, $s+1 \leq j \leq \sigma$; $L^{(a)}_w = L'_w + aL''_w \in \overline{k}[X_0, \dots, X_r]$, $e^{(a)}_w = L^{(\varepsilon)}_w(e_0, \dots, e_r)$, $w \in \{0, \sigma+1, \dots, n\}$.

By Lemma 3 for all $a \in k$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, λ_a satisfies condition (A), see the Introduction. In what follows, we shall assume without loss of generality condition (A) holds for the element a.

For every $\sigma \leq i \leq n$ consider the subvariety

$$(W \times \mathbb{A}^1(\overline{k})) \cap \mathcal{Z}(L_{\varepsilon,s+1}, \dots, L_{\varepsilon,\sigma}) \cap \mathcal{Z}(e^{(\varepsilon)}_{\sigma+1}, \dots, e^{(\varepsilon)}_i) \tag{61}$$

of $\mathbb{P}^n(\overline{k}) \times \mathbb{A}^1(\overline{k})$. Denote by $\mathcal{W}_i$ the union of all the irreducible over $\overline{k}$ components E of variety (61) such that $E \cap \mathcal{Z}(\varepsilon)$ contains an irreducible component of $W_{\lambda,i}(L') \times \{0\}$. By the definition of $\mathcal{W}_i$ and $\mathcal{W}_n$ every irreducible component of $\mathcal{W}_i$ contains an irreducible component of the curve $\mathcal{W}_n$ and $\mathcal{W}_n \subset \mathcal{W}_i$.

Generalizing the proof of Lemma 13 [9] (it is straightforward and we leave the details to the reader) we ascertain that for all $a \in k$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, $\mathcal{W}_i \cap \mathcal{Z}(\varepsilon - a) = W_{\lambda_a,j}(L^{(a)}) \times \{a\}$, the degree $\deg W_{\lambda_a,j}(L^{(a)}) \geq \deg W_{\lambda,L'} = D_i$ for all $\sigma \leq i \leq n$, and $W_{\lambda_a,n}(L^{(a)}) \cap \overline{V \setminus W} = \emptyset$. But obviously $D_{\lambda_a,i}(L^{(a)}) \leq D_{\lambda_a,i}$, and $D_{\lambda_a,i} \leq D_i$ by (56) with U in place of L'. Hence $\deg W_{\lambda_a,i}(L^{(a)}) = D_{\lambda_a,i}(L^{(a)}) = D_{\lambda_a,i} = D_i$ and $L^{(a)} \in \mathcal{U}''_{\lambda_a,0} \cap \mathcal{U}''_0$ by (56) for these a (in what follows in the proof of (f) we shall always assume that these properties are fulfilled for the considered a). In particular, $\deg W_{\lambda_a,n}(L^{(a)}) = D_n = \deg q_v \deg W'$. Hence $\deg p'_\lambda = \deg p'_v$ in this case. Besides that, for these a conditions (57)–(60) hold with $(\lambda_a, L^{(a)})$ in place of (λ, L'). Finally, in this case by Lemma 31 the inequalities $D^{(2)}_i \geq D^{(2)}_{\lambda_a,i}$, $D^{(1)}_i \geq D^{(1)}_{\lambda,i}$ are satisfied.

Let us choose linear forms $\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_n \in k[X_0, \dots, X_n]$ such that the intersection of the algebraic varieties $W_{\lambda,i-1}(L') \cap \mathcal{Z}(e'_i)$ and $\mathcal{Z}(\widetilde{L}_{i+1}, \dots, \widetilde{L}_n)$ is transversal for every $\sigma+1 \leq i \leq n$ (this is a condition of general position for the considered linear forms). For every $\sigma+1 \leq i \leq n$ denote by $\mathcal{W}'_i$ the union of all the irreducible over $\overline{k}$ components E of the intersection $\mathcal{W}_{i-1} \cap \mathcal{Z}(e^{(\varepsilon)}_i)$ such that E contains an irreducible component of $(W_{\lambda,i-1}(L') \cap \mathcal{Z}(e'_i)) \times \{0\}$. Then for all $a \in k$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, $\mathcal{W}^{(1)}_i \cap \mathcal{Z}(\varepsilon - a) = (W_{\lambda_a,i-1}(L^{(a)}) \cap \mathcal{Z}(e^{(a)}_i)) \times \{a\}$. Denote by C''_i the union of all the irreducible over $\overline{k}$ components C'' of $\mathcal{W}'_i \cap \mathcal{Z}(\widetilde{L}_{i+1}, \dots, \widetilde{L}_n)$ such that C'' contains a point from $(W_{\lambda,i-1}(L') \cap \mathcal{Z}(e'_i) \cap \mathcal{Z}(\widetilde{L}_{i+1}, \dots, \widetilde{L}_n)) \times \{0\}$. Hence C''_i is a curve. The natural projection $C''_i \to \mathbb{A}^1(\overline{k})$, $((X_0 : \dots : X_n), \varepsilon) \mapsto \varepsilon$ is dominant. Hence

by the Bézout for all $a \in k$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, $+\infty > \#(C_i'' \cap \mathcal{Z}(\varepsilon - a)) \geq \#(C_i'' \cap \mathcal{Z}(\varepsilon)) = D_i + D_i^{(1)} + D_i^{(2)}$. Hence for all $a \in k$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, the intersection $W_{\lambda_a,i-1}(L') \cap \mathcal{Z}(e_i^{(a)}) \cap \mathcal{Z}(\widetilde{L}_{i+1}, \ldots, \widetilde{L}_n)$ contains at least $D_i + D_i^{(1)} + D_i^{(2)}$ isolated points, and, therefore, $\deg(W_{\lambda_a,i-1}(L') \cap \mathcal{Z}(e_i^{(a)})) \geq D_i + D_i^{(1)} + D_i^{(2)}$.

We have, see above, $D_i(L^{(a)}) = D_{\lambda_a,i}(L^{(a)}) = D_{\lambda_a,i} = D_i^{(j)}$ and $D_{\lambda_a,i}^{(j)} \leq D_i^{(j)}$, $j = 1, 2$. Hence, see the Introduction, $D_{\lambda_a,i}^{(j)}(L^{(a)}) \leq D_{\lambda_a,i}^{(j)} \leq D_i^{(j)}$, $D_i^{(j)}(L^{(a)}) \leq D_i^{(j)}$ $j = 1, 2$. Therefore, $D_{\lambda_a,i}^{(j)}(L^{(a)}) = D_{\lambda_a,i}^{(j)} = D_i^{(j)}$ and $D_i^{(j)}(L^{(a)}) = D_i^{(j)}$, $j = 1, 2$. This implies that for the considered a intersection (58) with $\lambda_a, L^{(a)}$ in place of λ, L' is transversal. Hence (59) with $\lambda_a, L^{(a)}$ in place of λ, L' implies $W_i^{(2)}(L^{(a)}) \cap \mathcal{Z}(L_{a,s+1}, \ldots, L_{a,\sigma}) = W_{\lambda_a,i}^{(2)}(L^{(a)})$. Finally from (60) with $\lambda_a, L^{(a)}$ in place of λ, L' we get $W_i^{(1)}(L^{(a)}) \cap \mathcal{Z}(L_{a,s+1}, \ldots, L_{a,\sigma}) = W_{\lambda_a,i}^{(1)}(L^{(a)})$. This implies all the assertions of (f).

The required bounds for the working time of the algorithms from Theorem 2 follow immediately from the ones for the applied algorithms. The theorem is proved.

References

[1] **Bochnak J., Coste M., Roy M.–F.:** *"Géométrie algébrique réelle"*, Springer–Verlag, Berlin, Heidelberg, New York, 1987.

[2] **Chistov A. L.:** *"Polynomial complexity algorithm for factoring polynomials and constructing components of a variety in subexponential time"*, Zap. Nauchn. Semin. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 137 (1984), p. 124–188 (in Russian) [English transl.: J. Sov. Math. 34, (4), (1986) p. 1838–1882].

[3] **Chistov A. L.:** *"Polynomial–time computation of the dimensions of components of algebraic varieties in zero–characteristic"*, Journal of Pure and Applied Algebra, 117 & 118 (1997) 145–175.

[4] **Chistov A. L.:** *"Polynomial–time computation of the degrees of algebraic varieties in zero–characteristic and its applications"*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) v. 258 (1999), p. 7–59 (in Russian) [1]

[5] **Chistov A. L.:** *"Strong version of the basic deciding algorithm for the existential theory of real fields"*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 256 (1999) p. 168–211 (Russian). [English transl.: J. of Mathematical Sciences v.107 No.5 p.4265–4295]

[6] **Chistov A. L.:** *"Efficient Construction of Local Parameters of Irreducible Components of an Algebraic Variety"*, Proc. of the St.–Petersburg Mathematical Society, 1999, v. 7, p. 230–266 (Russian).

[7] **Chistov A. L.:** *"Efficient Smooth Stratification of an Algebraic Variety in Zero–Characteristic and its Applications"*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) v. 266 (2000) p.254–311 (in Russian) [English transl.: to appear in J. of Mathematical Sciences].

[8] **Chistov A. L.:** *"Monodromy and irreducibility criteria with algorithmic applications in zero–characteristic"*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 292 (2002) p. 130–152 (in Russian) [in English, see Preprint of St.Petersburg Mathematical Society (2004)].

[9] **Chistov A. L.:** *"Polynomial–time computation of the degree of a dominant morphism in zero–characteristic I"*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 307 (2004) p. 189–235 (in Russian) [in English, see Preprint of St.Petersburg Mathematical Society (2004)].

[10] **Hartshorne R.:** *"Algebraic geometry"*, Springer–Verlag, New York, Heidelberg, Berlin, 1977.

[11] **Hodge W. V. D., Pedoe D.:** *"Methods of algebraic geometry"*, v. 1, Cambridge, 1947.

[12] **Mumford D.:** *"Algebraic geometry I. Complex projective varieties"*, Springer–Verlag, Berlin, Heidelberg, New York, 1976.

[1] Preliminary versions of [4], [5], [6] and [7] in English can be found as Preprints (1999) , http://www.MathSoc.spb.ru