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A deterministic polynomial–time algorithm for the first Bertini theorem

Alexander L. Chistov

St. Petersburg Institute for Informatics and Automation of the Academy of Sciences of Russia
14th line 39, St. Petersburg 199178, Russia,
e-mail: labta@iias.spb.su or sliss@iias.spb.su

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Abstract

Consider a projective algebraic variety W which is an irreducible component of a set of all common zeroes of a family of homogeneous polynomials of degrees less than d in $n+1$ variables in zero–characteristic. Consider a linear system on W given by homogeneous polynomials of degree d'. Under the conditions of the first Bertini theorem for W and this linear system we show how to construct an irreducible divisor in general position from the statement of this theorem. This algorithm is deterministic and polynomial in $(dd')^n$ and the size of input.

Introduction

The first Bertini theorem and its different variants were analyzed approximately 50–60 years ago, see the survey in [1]. In the present paper we suggest the deterministic polynomial time algorithm for a general version of this theorem in the case of zero–characteristic. The algorithmic point of view requires to pay more attention to details here than one paid previously. As a result we obtain a simple explicit description of an open in the Zariski topology subset from the statement of the first Bertini theorem. So this theorem ceased to be purely existential (at least in zero–characteristic). More precisely, consider an irreducible hypersurface which is a set of all common zeroes of a homogeneous polynomial Φ in $X_0, \ldots, X_n$ in an n-dimensional projective space in zero–characteristic, $n \geq 2$. Let the degree $\deg_{X_0} \Phi \geq 1$. Consider a linear system on this hypersurface given by $X_1, \ldots, X_n$ (the general case, as we shall see, can be reduced to this one). Let $R = \mathrm{Res}_{X_0}(\Phi, \partial\Phi/\partial X_0)$ be the discriminant of Φ with respect to X_0, see below the for more details. Denote by R_0 the square free part of R with respect to X_0 (i.e., the separable polynomial of the maximal degree dividing the discriminant, which is here and below in this paper by definition the resultant of the polynomial and its partial derivative with respect to the considered variable). Then a linear form L in $X_1, \ldots, X_n$ satisfies the assertion of the first Bertini theorem in the considered case whenever the polynomial in $n-1$ variable $R_0 \bmod L$ is separable.

This paper applies the deterministic polynomial–time algorithms from [11] (the correction of the statement of Lemma 2 [11] see in [12]), [12], [13]. All the cited papers in their turn can be considered as a continuation of our previous ones. They are based on real algebraic geometry or, more precisely, use real structures, see [6], for obtaining polynomial–time algorithms for algebraic varieties over algebraically closed fields of zero–characteristic. Especially we would like to underline here the importance of the fundamental algorithm from [9] for [11], [12], [13] and the present paper. The results of [11] allow to decide whether two points belong to the same irreducible component of an algebraic variety. Besides that, in [11] we suggest an algorithm for the first Bertini theorem in the case of finite morphisms, and show how to construct linear sections (from the linear system of all linear forms) in general position of an algebraic variety. In [12] we investigate rational morphisms from the algorithmic point of view in the case when the dimension of the variety coincides with the one of its image. In [13] we construct the objects in general position related to a rational morphism in the general case. To obtain the algorithms in [13] we suggest a new method for solving systems of polynomial equations and inequalities with squares of absolute values, see [6], in the case when these systems contain generic linear forms. In the present paper we need to strengthen this method for the case of generic homogeneous polynomials (in place of generic linear forms) of the given degree. This requires to use the result of [14], see Section 2.

In [11] and the present paper we attract the transcendental methods of theory of analytical functions, namely, the group of monodromy of an algebraic function to substantiate the algorithms. Theorem 3 Section 4 uses these methods and generalizes Theorem 3 [11] for the case of not necessarily finite morphisms. Basing on this result we get an irreducibility criterion for multivariable polynomials, see below, which generalizes the one from [11].

Now we proceed to more detailed statements. Let k be a field of zero–

characteristic with algebraic closure $\overline{k}$. Let $X_0, X_1, \ldots$ be independent variables over k. Denote by $\mathbb{P}^n(\overline{k})$ the projective space over the field $\overline{k}$ with coordinates $X_0, \ldots, X_n$. We shall suppose that $\mathbb{P}^n(\overline{k})$ is defined over k. For arbitrary homogeneous polynomials $g_1, \ldots, g_m \in \overline{k}[X_0, \ldots, X_n]$ we shall denote by $\mathcal{Z}(g_1, \ldots, g_m)$ the set of all common zeroes of polynomials $g_1, \ldots, g_m$ in $\mathbb{P}^n(\overline{k})$. The similar notations will be used for the sets of zeroes of ideals and polynomials with other fields of coefficients in affine and projective spaces (this will be seen from the context). For a field k and a family of elements $a' = \{a'_i\}_{i \in I}$ from an extension of k we shall denote by $k(a')$ (respectively $k[a']$) the field (respectively the ring) generated over k by all the elements a'_i, $i \in I$. For two (or more) families a', a'' the notations $k(a', a'')$, $k[a', a'']$ have the similar sense. Now suppose that a' is a family of independent variables over k and $f \in k[a']$. Suppose that $b' = \{b'_i\}_{i \in I}$ is a family of elements from k with the same indices as a'. Then we shall denote by $f(b')$ the value of f at $a'_i = b'_i$, $i \in I$, and also we use other similar notations.

In the algorithmic part of the paper we shall suppose that the field k is finitely generated over the field of rational numbers $\mathbb{Q}$. Let $V \subset \mathbb{P}^n(\overline{k})$ be a projective algebraic variety which is a set of all common zeroes of a family of homogeneous polynomials in $X_0, \ldots, X_n$ of degrees less than d with coefficients from a field k. Let W be a defined over k and irreducible over k component of V and W is given by its point which belongs to only one irreducible over k component of V, see Theorem 1 [11] and also below for details. Let $e_0, \ldots, e_r \in k[X_0, \ldots, X_n]$, be homogeneous polynomials such that $\deg e_i = d' \geq 1$ for every nonzero e_i, $0 \leq i \leq r$. We shall suppose in what follows that $\dim W \cap \mathcal{Z}(e_0, \ldots, e_r) < \dim W$. Let

$$p : W \setminus \mathcal{Z}(e_0, \ldots, e_r) \to \mathbb{P}^r, \quad (X_0 : \ldots : X_n) \mapsto (e_0 : \ldots : e_r). \tag{1}$$

be the regular morphism. Set $W' = \overline{p(W \setminus \mathcal{Z}(e_0, \ldots, e_r))}$ (here and below in the similar situations the bar denotes closure with respect to the Zariski topology). Hence W' is irreducible over k closed subvariety of $\mathbb{P}^r(\overline{k})$. We shall suppose in this paper that $\dim W' > 0$. Note that one can consider the rational morphism $W \to \mathbb{P}^r(\overline{k})$ of projective algebraic varieties given by the same rule but to be more accurate in what follows we deal with regular morphisms of quasiprojective algebraic varieties. Denote by

$$p' : W \setminus \mathcal{Z}(e_0, \ldots, e_r) \to W'$$

the dominant morphism induced by p. Suppose that $\dim W = n - s$ and $\dim W' = n - \sigma > 0$ (one can compute s and σ using the algorithm from [7], see [13]). So $0 \leq s \leq \sigma < n$. Let $v_j = \sum_{0 \leq i \leq n} v_{j,i} X_i$, $s + 1 \leq j \leq \sigma$, be generic linear forms in $X_0, \ldots, X_n$, i.e., the family of their coefficients $v_{j,i}$ has transcendency degree $(\sigma - s)(n + 1)$ over k. Denote by k_v the field generated by all the coefficients of linear forms v_j, $s + 1 \leq j \leq \sigma$, over k. Denote by W_v the intersection of W with the linear subspace $\mathcal{Z}(v_{s+1}, \ldots, v_\sigma)$ of codimension $\sigma - s$ (hence W_v is defined over k_v). Denote by k_v the field generated by all the coefficients of linear forms v_j, $s + 1 \leq j \leq \sigma$, over k. Set $v = (v_{s+1}, \ldots, v_\sigma)$ and $\upsilon = \widetilde{v} = \{v_{j,i},\ s + 1 \leq j \leq \sigma, 0 \leq i \leq n\}$ to be the family of all the coefficients of linear forms v_j, $s + 1 \leq j \leq \sigma$ (one should not confuse v and υ).

Let $n - \sigma \geq 2$. Then our aim is to construct within the time polynomial in $(dd')^n$ and the size of input linear forms $L'_{\sigma+1}, \ldots, L'_{n-1} \in k[X_0, \ldots, X_r]$ with

integer coefficients of length $O(n \log d + (n-\sigma)\log d')$ satisfying for all $\sigma \leq i \leq n$ the following conditions.

(i) The intersections $(W \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i)$ and $W' \cap \mathcal{Z}(L'_{\sigma+1}, \ldots, L'_i)$ are transversal (see the definition below); if $\sigma+1 \leq i \leq n-1$ then these intersections are the irreducible over k varieties.

(ii) The morphism $p'_{v,i} : (W_v \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i) \to W'(\overline{k_v}) \cap \mathcal{Z}(L'_{\sigma+1}, \ldots, L'_i)$ induced by p' is dominant. Denote $p'_{v,\sigma} = p'_v$.

(iii) Then the degree of $p'_{v,i}$ coincides with the degree of p'_v.

(iv) The degree of the closure $\overline{(W \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i)}$ is maximal possible (when one varies linear forms L'_j, $\sigma+1 \leq j \leq n-1$).

Even more conditions of general position hold, e.g., $(L'_{\sigma+1}, \ldots, L'_{n-1}) \in \mathcal{U}'''_{n-1}$, where $\mathcal{U}'''_{n-1} \subset \mathbb{A}^{(r+1)(n-\sigma-1)}$ is a nonempty open in the Zariski topology subset introduced in [13], see below. For every $\sigma+1 \leq j \leq n-1$ we shall define an open in the Zariski topology subset $\mathcal{U}_j \subset \mathbb{A}^{(r+1)(n-j)}$ such that if $(L'_{\sigma+1}, \ldots, L'_{n-1}) \in \mathcal{U}_j$ then (i)–(iv) hold for all $\sigma+1 \leq i \leq j$.

In particular, under conditions of the first Bertini theorem for the morphism p', see below, we show how to find an element from the open set $\mathcal{U}$ from the statement of this theorem within the time polynomial in $(dd')^n$ and the size of input. Notice also once more that so far the first Bertini theorem has been mainly existential. There has not been explicit description of an open in the Zariski topology subset $\mathcal{U}$ from the statement of this theorem.

REMARK 1 *In Remark 1 from [13] we give a sufficient condition when (i) and (ii) and (iii) imply (iv). It is interesting to find other such conditions.*

Denote by k_u the field generated over k by the algebraically independent over k elements $u_{j,w}$, $j \in \{0, \sigma+1, \ldots, n\}$, $0 \leq w \leq r$. Let K be a field containing k_u and such that K and k_v are linearly disjoint over k. Let $l'_{j,w} \in \overline{k}$, $j \in \{0, \sigma+1, \ldots, n\}$, $0 \leq w \leq r$. Let $L'_j = l'_{j,0}X_0 + \ldots + l'_{j,r}X_r$ (respectively $U_j = u_{j,0}X_0 + \ldots + u_{j,r}X_r$), $j \in \{0, \sigma+1, \ldots, n\}$, be linear forms from $\overline{k}[X_0, \ldots, X_r]$ (respectively $K[X_0, \ldots, X_n]$). Set $e'_j = L'_j(e_0, \ldots, e_r)$ (respectively $u_j = U_j(e_0, \ldots, e_r)$), $j \in \{0, \sigma+1, \ldots, n\}$. Put $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n)$ (respectively $U = (U_0, U_{\sigma+1}, \ldots, U_n)$). Set u (respectively l') to be the family of all $u_{j,i}$, (respectively $l'_{j,i}$) $j \in \{0, \sigma+1, \ldots, n\}$, $0 \leq i \leq r$. In what follows we shall define some objects corresponding to L'. The similar objects corresponding to U are defined by replacing the field k by K. Below we shall consider also the case when linear forms L'_j have coefficients from $\overline{K}$. In this case the objects corresponding to L' are defined by the extension of the field k till $\overline{K}$.

Recall some notations from [13]. Let us identify the affine space $\mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$ (respectively $\mathbb{A}^{(r+1)(n-i)}(\overline{k})$) with the set of all $(n-\sigma+1)$–tuples $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n)$ (respectively $(n-i)$–tuples $(L'_{i+1}, \ldots, L'_n)$) of linear forms $L'_j \in \overline{k}[X_0, \ldots, X_r]$. For convenience of notation for $i = \sigma+1$ we shall identify 1–tuples $(L'_{\sigma+1})$ with the linear form $L'_{\sigma+1}$. Recall that in Theorem 1 [13] we define the open in the Zariski topology subset $\mathcal{U}'_0 \subset \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$ in the case when $s = \sigma$.

In [13] the set $\mathcal{U}'_0$ is defined in the general case. Namely, let $\mathcal{V}$ be the open in the Zariski topology subset from the statement of Theorem 1 [12] for p'_v, k_v in

place of p', k respectively. Then $\mathcal{U}'_0 = \mathcal{V} \cap \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$. In brief, $L' \in \mathcal{U}'_0$ if and only if the numbers of elements of $\#W_v \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n) \setminus (\mathcal{Z}(e'_0) \cup \overline{(V \setminus W)})$ (this is a subset of $\mathbb{P}^n(\overline{k_v})$) is $\deg p'_v \deg W'$ and $\#(W' \cap \mathcal{Z}(L'_{\sigma+1}, \ldots, L'_n)) = \deg W'$, i.e., the last two intersections are finite and have the maximal possible number of elements, see [12]. Hence, if $L' \in \mathcal{U}'_0$ then $W' \cap \mathcal{Z}(L'_0, \ldots, L'_r) = \emptyset$ in $\mathbb{P}^r(\overline{k_v})$ and $W \cap \mathcal{Z}(e'_0, e'_{\sigma+1}, \ldots, e'_n) = W \cap \mathcal{Z}(e_0, \ldots, e_r)$ in $\mathbb{P}^n(\overline{k})$.

REMARK 2 *Obviously if $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n) \in \mathcal{U}'_0$, and γ is a nondegenerate linear transformation of $L'_{\sigma+1}, \ldots, L'_n$ over $\overline{k}$ then $(L'_0, \gamma(L'_{\sigma+1}), \ldots, \gamma(L'_n)) \in \mathcal{U}'_0$.*

Suppose that $L' \in \mathcal{U}'_0$. Consider the morphism

$$p_{L'} : W \setminus \mathcal{Z}(e'_0, e'_{\sigma+1} \ldots, e'_n) \to \mathbb{P}^{n-\sigma}(\overline{k}),$$
$$(X_0 : \ldots : X_n) \mapsto (e'_0 : e'_{\sigma+1} : \ldots : e'_n).$$

In [13] for an arbitrary L' we define algebraic varieties $W_i(L')$, $W_i^{(\beta)}$, $\sigma \le i \le n+1$ (they are subvarieties of W). In the present paper we shall need only the case when $\sigma \le i \le n$ and $L' \in \mathcal{U}'_0$. In what follows unless we state otherwise we assume that $L' \in \mathcal{U}'_0$. Then

- $W_i(L')$, $\sigma \le i \le n$, is the union of all the irreducible components E of $W \cap \mathcal{Z}(e_{\sigma+1}, \ldots, e_i)$ such that $\dim E = \dim \overline{p(E \setminus \mathcal{Z}(e_0, \ldots, e_r)} = n - i + \sigma - s$.

Hence $W_\sigma(L') = W$. Set $W_\sigma^{(\beta)}(L') = \emptyset$, $\beta = 1, 2, 3$. Further, see [13], for $\sigma + 1 \le i \le n$ the dimension $\dim W_{i-1}(L') \cap \mathcal{Z}(e'_i) \le n + \sigma - s - i$ and $\dim W_n(L') \cap \mathcal{Z}(e'_0) \le \sigma - s - 1$. The dimension of each irreducible component of $W_i^{(\beta)}(L')$, $\beta = 1, 2, 3$, is $n - \sigma + s - i$. Namely $W_{i-1}(L') \cap \mathcal{Z}(e'_i) = W_i(L') \cup W_i^{(1)}(L') \cup W_i^{(2)}(L') \cup W_i^{(3)}(L')$, where:

- $W_i^{(1)}(L')$ is the union of all irreducible components E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that E is an irreducible component of $W \cap \mathcal{Z}(e_0, \ldots, e_r)$;

- $W_i^{(2)}(L')$ is the union of all irreducible components E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that $E \subset \mathcal{Z}(e_0, \ldots, e_r)$ but E is not an irreducible component of $W \cap \mathcal{Z}(e_0, \ldots, e_r)$;

- $W_i^{(3)}(L')$ is the union of all irreducible components E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that $E \not\subset \mathcal{Z}(e_0, \ldots, e_r)$ and $\dim \overline{p_{L'}(E \setminus \mathcal{Z}(e'_0, e'_{\sigma+1} \ldots, e'_n))} < n - i + \sigma - s$, see [13].

Since $L' \in \mathcal{U}'_0$ we have, see [13],

- $W_i^{(3)}(L')$ is the union of all irreducible components E of $W_{i-1}(L') \cap \mathcal{Z}(e'_i)$ such that $E \not\subset \mathcal{Z}(e_0, \ldots, e_r)$ and $\dim \overline{p(E \setminus \mathcal{Z}(e_0, \ldots, e_r))} < n - i + \sigma - s$.

Hence if $W_i^{(3)}(L') \ne \emptyset$ then $\dim W_i^{(3)}(L') \ge 1$ (in particular, $W_n^{(3)}(L') = \emptyset$). Also we have $W_{\sigma+1}^{(2)}(L') = \emptyset$. Set $D_i(L') = \deg W_i(L')$ and $D_i^{(\beta)}(L') = \deg W_i^{(\beta)}(L')$, $\beta = 1, 2, 3$.

Let W_j, $j \in J_i(L')$ (respectively $j \in J_i^{(\beta)}(L')$, $\beta = 1, 2, 3$), be the family of all the irreducible over $\overline{k}$ components of $W_i(L')$ (respectively $W_i^{(\beta)}(L')$), and

$\iota_j(L') = i(W_{i-1}(L'), \mathcal{Z}(e'_i); W_j)$ denote the index of intersection of $W_{i-1}(L')$ and $\mathcal{Z}(e'_i)$ at W_j, see [13]. Let $W_i^{(1)} = \bigcup_{j \in J_i^{(1)}} W_j$ be the union of all the irreducible components of dimension $n - \sigma + s - i$ of $W \cap \mathcal{Z}(e_0, \dots, e_r)$. We suppose in what follows without loss of generality that $J_i^{(1)}(L') \subset J_i^{(1)}$ for every $L' \in \mathcal{U}'_0$.

Let $\sigma + 1 \leq \alpha \leq n$ be an integer. By Lemma 11 [12], condition

$$W_i(L') = \overline{W \cap \mathcal{Z}(e'_{\sigma+1}, \dots, e'_i) \setminus \mathcal{Z}(e_0, \dots, e_r)} \tag{2}$$

holds for all $\sigma + 1 \leq i \leq \alpha$ if and only if $W_i^{(3)}(L') = \emptyset$ for all $\sigma + 1 \leq i \leq \alpha$. If (2) holds for every $\sigma + 1 \leq i \leq \alpha$ then, see [13], $W_i^{(1)}(L') = W_i^{(1)}$ for every $\sigma + 1 \leq i \leq \alpha$ does not depend on the choice of L'. Hence in this case that $J_i^{(1)}(L') = J_i^{(1)}$ for every $\sigma + 1 \leq i \leq \alpha$ does not depend on the choice of L'.

We set $D_i = D_i(U)$, $D_i^{(\beta)} = D_i^{(\beta)}(U)$, $\beta = 1, 2, 3$, and $\iota_j = \iota_j(U)$ for every $j \in J^{(1)}$.

In [13] we introduce also the notation $W_i(L') = W(L_{\sigma+1}, \dots, L_i)$ and similarly for $W_i^{(1)}(L')$, $W_i^{(2)}(L')$, $D_i(L')$ (in place of $W_i(L')$) and so on; the considered objects depend actually only on $L_{\sigma+1}, \dots, L_i$.

Let $\sigma \leq \alpha \leq n$ be an integer. *Denote by $\mathcal{U}'_\alpha$, the image of $\mathcal{U}'_0$ under the linear mapping* $\pi_\alpha : \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k}) \to \mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$, $L' \mapsto (L'_{\sigma+1}, \dots, L'_\alpha)$. Then $\mathcal{U}'_\alpha$ is an open with respect to the Zariski topology set since the image under a linear projection of an open set is again open.

Denote by $\mathcal{U}''_\alpha \subset \mathcal{U}'_\alpha$ the subset of all $(L'_{\sigma+1}, \dots, L'_\alpha)$ such that

$$D(L'_{\sigma+1}, \dots, L'_i) = D_i \tag{3}$$

for all $\sigma + 1 \leq i \leq \alpha$. Set also $\mathcal{U}''_0$ to be the set of all $L' \in \mathcal{U}'_0$ such that $(L'_{\sigma+1}, \dots, L'_n) \in \mathcal{U}''_n$

Denote by $\mathcal{U}'''_\alpha \subset \mathcal{U}''_\alpha$ the subset of all $(L'_{\sigma+1}, \dots, L'_\alpha)$ such that

$$D^{(2)}(L'_{\sigma+1}, \dots, L'_i) = D_i^{(2)} \tag{4}$$

for all $\sigma + 1 \leq i \leq \alpha$. Set also $\mathcal{U}'''_0$ to be the set of all $L' \in \mathcal{U}''_0$ such that $(L'_{\sigma+1}, \dots, L'_n) \in \mathcal{U}'''_n$.

If $(L'_{\sigma+1}, \dots, L'_\alpha) \in \mathcal{U}'_\alpha$ then, by Lemma 11 [12], the indices of intersection $\iota_j(L'_{\sigma+1}, \dots, L'_i) = 1$ for all $j \in J(L'_{\sigma+1}, \dots, L'_i)$, $\sigma + 1 \leq i \leq \alpha$. If $(L'_{\sigma+1}, \dots, L'_\alpha) \in \mathcal{U}''_\alpha$ then, see [13], the indices of intersection $\iota_j(L'_{\sigma+1}, \dots, L'_i) = \iota_j$ for all $j \in J_i^{(1)}$, $\sigma + 1 \leq i \leq \alpha$. Further, $\mathcal{U}''_\alpha$, U'''_α are nonempty open in the Zariski topology subsets of $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$, see [13]. By Lemma 14 [12] applied to W_v, p'_v, α in place of W, p', $i + 1$ if $D_i(L') = D_i$ for all $\sigma + 1 \leq i \leq \alpha$ then $W_i^{(3)}(L') = \emptyset$ for all $\sigma + 1 \leq i \leq \alpha$. Therefore, if $L' \in \mathcal{U}''_0$ then condition (2) holds for all $\sigma \leq i \leq n - 1$.

Denote for brevity $\mathcal{U}' = \mathcal{U}'_{\sigma+1}$ (respectively $\mathcal{U}'' = \mathcal{U}''_{\sigma+1}$, $\mathcal{U}''' = \mathcal{U}'''_{\sigma+1}$). Notice that $\mathcal{U}'' = \mathcal{U}'''$.

Notice, see [13], that D_i is the maximal possible value of $D_i(L')$; if $D_i(L') = D_i$ then all the indices of intersection $\iota_j(L') = \iota_j$, $j \in J^{(1)}$ are minimal possible; $D_i^{(2)}$ is the maximal possible value of $D_i^{(2)}(L')$ under conditions $D_j = D_j(L')$ for all $\sigma + 1 \leq j \leq n$, $D_j^{(2)}(L') = D_j^{(2)}$ for all $\sigma + 2 \leq j < i$.

Let $W_1, W_2 \subset \mathbb{P}^n(\overline{k})$ be two quasiprojective algebraic varieties defined over k. Let W_3 be an arbitrary defined over k and irreducible over k component of

$W_1 \cap W_2$. We shall say that the intersection $W_1 \cap W_2$ is transversal at W_3 if and only if there is a point $x \in W_3$ such that x is a smooth point of every W_i, $i = 1, 2$, and the intersection of the tangent spaces $T_{x,W_1} \cap T_{x,W_2}$ of W_1 and W_2 at the point x is transversal, i.e., $\dim(T_{x,W_1} \cap T_{x,W_2}) = n - \dim(T_{x,W_1}) - \dim(T_{x,W_2})$. We shall say that the intersection $W_1 \cap W_2$ is transversal if and only if it is transversal at every its defined over k and irreducible over k component. In the similar way one defines transversality for intersection of a finite number of algebraic varieties.

By, e.g., Lemma 21 [13], if the intersection $W_1 \cap W_2$ is transversal and y is an arbitrary smooth point of $W_1 \cap W_2$ then y is a smooth point of every W_i, $i = 1, 2$ and the intersection of the tangent spaces $T_{y,W_1} \cap T_{y,W_2}$ of W_1 and W_2 at the point y is transversal. Also the following simple result may be useful here (we leave its proof to the reader).

PROPOSITION 1 *Let W_i, $i \in I$, be a finite family of quasiprojective subvarieties of $\mathbb{P}^n(\overline{k})$. Suppose that the intersection $\bigcap_{i\in I} W_i$ is transversal. Assume that for an arbitrary family E_i, $i \in I$, such that E_i is an irreducible component of W_i, $i \in I$, the sum $\sum_{i\in I}(n - \dim E_i) = \sum_{i\in I} \operatorname{codim} E_i \leq n$. Let $I_1 \subset I$. Suppose that for every irreducible component E of $\bigcap_{i\in I_1} W_i$ the intersection $E \cap \bigcap_{i\in I\setminus I_1} W_i \neq \emptyset$ (in particular, this condition holds if all the algebraic varieties W_i, $i \in I$, are projective, i.e., closed in $\mathbb{P}^n(\overline{k})$). Then the intersection $\bigcap_{i\in I_1} W_i$ is also transversal.*

Now we are going to formulate a variant of the first Bertini theorem (mainly for references in what follows). Notice that all the objects introduced above for the varieties V, W and the morphism p can be defined also in the same way for the case when k is a perfect field of nonzero characteristic and the extension $k(W) \supset k(W')$ is separable. In this case similarly to zero–characteristic $\mathcal{U}'_\alpha$, $\mathcal{U}''_\alpha$, $\mathcal{U}'''_\alpha$ are open in the Zariski topology subsets of the corresponding affine spaces (we leave the details to the reader; it is not our aim to consider the case of nonzero characteristic in this paper, except, may be, the Appendix). Recall that if $L' \in \mathcal{U}''$ then $W^{(3)}(L'_{\sigma+1}) = \emptyset$ and $W(L'_{\sigma+1}) = \overline{W \cap \mathcal{Z}(e'_{\sigma+1}) \setminus \mathcal{Z}(e_0, \ldots, e_r)}$. (it is proved for nonzero characteristic similarly to case zero–characteristic, see Lemma 14 [12]). If $V = W$ then the set $\mathcal{U}'$, $\mathcal{U}''$, $\mathcal{U}'''$ does not depend on the choice of $f_1, \ldots, f_m$. From the given here variant of the first Bertini theorem one can deduce the usual one, see e.g., [17], on the language of linear systems. One needs only to put $V = W$ and $k = \overline{k}$.

THEOREM (a version of the Bertini theorem) *Let k be a perfect field (now it is not necessarily of zero–characteristic) and $W \subset \mathbb{P}^n(\overline{k})$ be a defined over k and irreducible over k projective algebraic variety with $\dim W = n - s$. Let the varieties V, W, the polynomials $e_0, \ldots, e_r$, $r \geq 2$, the bound for the degree d' and the morphisms p, p' be as above. Suppose that $\dim W' > 1$ (hence $s \leq n - 2$) and the extension of fields of rational functions $k(W) \supset k(W')$ is separable (may be of the nonzero transcendency degree). Denote by $\mathcal{U}$ the subset of all linear forms $L'_{\sigma+1} \in \mathcal{U}''$ such that the irreducible over $\overline{k}$ components of the algebraic varieties W and $W(L'_{\sigma+1})$ are in the one–to–one correspondence by the rule $W_1 \mapsto W_2$ where W_2 is the unique (the uniqueness is claimed) irreducible over $\overline{k}$ component of $W_1 \cap \mathcal{Z}(e'_{\sigma+1})$ such that $W_2 \subset W(L'_{\sigma+1})$ (hence the intersection of W and $\mathcal{Z}(e'_{\sigma+1})$ is transversal at each irreducible component W_2 of $W(L'_{\sigma+1})$). Then $\mathcal{U}$ is a nonempty open in the Zariski topology defined*

over k subset of $\mathbb{A}^{n+1}(\overline{k})$. For every $L'_{\sigma+1} \in \mathcal{U}(k)$ the algebraic variety $W(L'_{\sigma+1})$ is defined over k and irreducible over k.

One can deduce the formulated theorem from, e.g., [17]. For convenience of the reader we give an independent proof of this fact in the Appendix. The fact that $\mathcal{U}$ is nonempty and open in the Zariski topology follows from Lemma 24 which is based on Theorem 4. Here it is rather difficult to show (in nonzero characteristic) that $\mathcal{U} \neq \emptyset$.

Let $\mathcal{U}_\alpha$, $\sigma+1 \leq \alpha \leq n-1$, be the subset of all $(\alpha-\sigma)$–tuples $(L'_{\sigma+1}, \ldots, L'_\alpha) \in \mathcal{U}'''_\alpha$ such that for every $\sigma+1 \leq i \leq \alpha$ (or, which is equivalent, for $i = \alpha$, see Lemma 11 from [12]) the irreducible over $\overline{k}$ components of the algebraic varieties W and $W(L'_{\sigma+1}, \ldots, L'_i)$ are in the one–to–one correspondence by the rule $W_1 \mapsto W_2$ where W_2 is the unique (the uniqueness is claimed) irreducible over $\overline{k}$ component of $W_1 \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i)$ such that $W_2 \subset W(L'_{\sigma+1}, \ldots, L'_i)$. Hence the intersection of W and $\mathcal{Z}(e'_{\sigma+1}, \ldots, e'_i)$ is transversal at each irreducible component W_2 of $W(L'_{\sigma+1}, \ldots, L'_i)$, $\sigma+1 \leq i \leq \alpha$. Applying the formulated variant of the first Bertini theorem $\alpha - \sigma$ times one gets that $\mathcal{U}_\alpha$, $\sigma+1 \leq \alpha \leq n-1$, is nonempty. By Lemma 24 the set $\mathcal{U}_\alpha$ is a nonempty open in the Zariski topology in $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$ provided that the extension $k(W) \supset k(W')$ is separable.

Denote by $\mathcal{U}_n$ (respectively $\mathcal{U}_0$) the subset of all $(L'_{\sigma+1}, \ldots, L'_n) \in \mathcal{U}'''_n$ (respectively $L' \in \mathcal{U}'''_0$) such that $(L'_{\sigma+1}, \ldots, L'_{n-1}) \in \mathcal{U}_{n-1}$.

REMARK 3 *Recall that Remark 2 [13] related to the extensions of ground field holds for all $\mathcal{U}'_\alpha$, $\mathcal{U}''_\alpha$, $\mathcal{U}'''_\alpha$. The similar remark is true now for all $\mathcal{U}_\alpha$ and $\mathcal{U}^{(\mathrm{v})}_\alpha$ defined below.*

Now let $\lambda = (L_{s+1}, \ldots, L_\sigma)$ and all $L_j = \sum_{0 \leq i \leq n} l_{j,i} X_i \in \overline{k}[X_0, \ldots, X_n]$, $s+1 \leq j \leq \sigma$. Denote by $\widetilde{\lambda}$ the family of coefficients $l_{j,i}$, $s+1 \leq i \leq \sigma$, $0 \leq i \leq n$ of linear forms L_j. Unless we state otherwise we shall suppose in what follows (the other assumptions, see at the beginning of Section 1) that the following properties holds for λ.

(C) All the coefficients $l_{j,i} \in k$, $s+1 \leq j \leq \sigma$, $0 \leq i \leq n$. The dimension $\dim W \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma) = n - \sigma > 0$; the last intersection is transversal. The algebraic variety $W \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ is irreducible over k. Denote $W_\lambda = W \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$, $V_\lambda = V \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$. The algebraic variety $W_\lambda \not\subset \overline{V_\lambda \setminus W_\lambda}$.

Recall that if condition (A) from [13] holds then (C) is fulfilled. Denote by

$$p_\lambda : W_\lambda \setminus \mathcal{Z}(e_0, \ldots, e_r) \to \mathbb{P}^r(\overline{k}), \tag{5}$$

$$p'_\lambda : W_\lambda \setminus \mathcal{Z}(e_0, \ldots, e_r) \to W' \tag{6}$$

the morphisms induced by p and p'. Suppose that the morphism p'_λ is dominant. Let us replace (V, W, p, p') by $(V_\lambda, W_\lambda, p_\lambda, p'_\lambda)$ in the definitions of all $W_i(L')$, $W(L'_{\sigma+1}, \ldots, L'_n)$, $\mathcal{U}'_\alpha$, $\mathcal{U}''_\alpha$, $\mathcal{U}'''_\alpha$ and so on. We shall denote the obtained objects by $W_{\lambda,i}(L')$, $W_\lambda(L'_{\sigma+1}, \ldots, L'_n)$, $\mathcal{U}'_{\lambda,\alpha}$, $\mathcal{U}''_{\lambda,\alpha}$, $\mathcal{U}'''_{\lambda,\alpha}$ and so on. Notice that the field $k(v_\lambda) = k$ since $s_\lambda = \sigma_\lambda = \sigma$. It will be convenient for us to consider the morphism p'_λ together with p'_v, see Theorem 2 below. Let us define V_v, W_v and so on extending the ground field k till k_v.

Consider the basis over k of the linear space of all homogeneous polynomials of degree d' consisting of all different monomials $X_0^{i_0}\cdots X_n^{i_n}$, $i_0+\ldots+i_n=d'$, $i_0,\ldots,i_n\geq 0$. Denote by I the set of the all considered multiindices $(i_0,\ldots,i_n)$. So the number of elements $\#I=\binom{n+d'}{n}$. Let $\widetilde{e}_{n+1}$ be a generic homogeneous polynomial of degree d' in $X_0,\ldots,X_n$, i.e., the family $c=\{c_i\}_{i\in I}$ of its coefficients in the considered basis has transcendency degree $\binom{n+d'}{n}$ over K_v. Set $K'=K(v,c)\supset k_v(c,u)=k(c,v,u)$.

Let e'_{n+1} be a homogeneous polynomial with coefficients from k. Denote by $c'=\{c'_i\}_{i\in I}$ the family of coefficients of e'_{n+1} in the considered basis.

Let λ be as above and $L'\in\mathcal{U}'_{\lambda,0}$ (unless we state otherwise we shall suppose that these assumptions hold when we consider the morphisms (7)–(11); but sometimes the assumptions will be other). Let us define the morphisms

$$p_{\lambda,L'}\,:\,W_\lambda\setminus\mathcal{Z}(e'_0,e'_{\sigma+1},\ldots,e'_n)\to\mathbb{P}^{n-\sigma}(\overline{k}), \tag{7}$$
$$(X_0:\ldots:X_n)\mapsto(e'_0:e'_{\sigma+1}:\ldots:e'_n),$$
$$p_{c',\lambda}\,:\,W_\lambda\setminus\mathcal{Z}(e_0,\ldots,e_r,e'_{n+1})\to\mathbb{P}^{r+1}(\overline{k}), \tag{8}$$
$$(X_0:\ldots:X_n)\mapsto(e_0:\ldots:e_r:e'_{n+1});$$
$$p_{c',\lambda,L'}\,:\,W_\lambda\setminus\mathcal{Z}(e'_0,e'_{\sigma+1}\ldots,e'_{n+1})\to\mathbb{P}^{n-\sigma+1}(\overline{k}), \tag{9}$$
$$(X_0:\ldots:X_n)\mapsto(e'_0:e'_{\sigma+1}:\ldots:e'_{n+1});$$
$$p_{\lambda,L',*}\,:\,W_\lambda\cap\mathcal{Z}(e'_{\sigma+1},\ldots,e'_{n-1})\setminus\mathcal{Z}(e'_0,e'_n)\to\mathbb{P}^1(\overline{k}); \tag{10}$$
$$(X_0:\ldots:X_n)\mapsto(e'_0:e'_n),$$
$$p_{c',\lambda,L',*}\,:\,W_\lambda\cap\mathcal{Z}(e'_{\sigma+1},\ldots,e'_{n-1})\setminus\mathcal{Z}(e'_0,e'_n,e'_{n+1})\to\mathbb{P}^2(\overline{k}), \tag{11}$$
$$(X_0:\ldots:X_n)\mapsto(e'_0:e'_n:e'_{n+1}).$$

Put

$$W'(c',\lambda)=\overline{p_{c',\lambda}(W_\lambda\setminus\mathcal{Z}(e_0,\ldots,e_r,e'_{n+1}))}\subset\mathbb{P}^{r+1}(\overline{k}), \tag{12}$$
$$W'(c',\lambda,L')=\overline{p_{c',\lambda,L'}(W_\lambda\setminus\mathcal{Z}(e'_0,e'_{\sigma+1}\ldots,e'_{n+1}))}\subset\mathbb{P}^{n-\sigma+1}(\overline{k}), \tag{13}$$
$$W'(c',\lambda,L',*)=$$
$$\overline{p_{c',\lambda,L',*}(W_\lambda\cap\mathcal{Z}(e'_{\sigma+1},\ldots,e'_{n-1})\setminus\mathcal{Z}(e'_0,e'_n,e'_{n+1}))}. \tag{14}$$

Hence $W'(c',\lambda,L',*)\subset\mathbb{P}^2(\overline{k})$. Since $L'\in\mathcal{U}'_{\lambda,0}$ the morphism $p_{\lambda,L'}$ is dominant. If the morphism $p_{\lambda,L'}$ is dominant and $\dim W_\lambda=n-\sigma$ (without any additional assumptions for $\lambda\in\mathbb{A}^{(\sigma-s)(n+1)}(\overline{k})$ and $L'\in\mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$) then $\dim W'(c',\lambda,L')=\dim W'(c',\lambda)=n-\sigma$. By [13] if assertions (c) and (d) of Theorem 2 [13] hold for λ and L' then the dimension of every irreducible component of $W_\lambda\cap\mathcal{Z}(e'_{\sigma+1},\ldots,e'_{n-1})\setminus\mathcal{Z}(e'_0,e'_n,e'_{n+1})$ is 1, and the morphism $p_{\lambda,L',*}$ is dominant. If the morphism $p_{\lambda,L',*}$ is dominant and $\dim W_\lambda\cap\mathcal{Z}(e'_{\sigma+1},\ldots,e'_{n-1})\setminus\mathcal{Z}(e'_0,e'_n,e'_{n+1})=1$ (without any additional assumptions for $\lambda\in\mathbb{A}^{(\sigma-s)(n+1)}(\overline{k})$ and $L'\in\mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$) then $\dim W'(c',\lambda,L',*)=\dim W'(c',\lambda,*)=1$.

For an arbitrary polynomial ψ we shall denote by ψ_0 its square free part, i.e., the separable (that is without multiple factors) polynomial of the maximal degree dividing ψ. This polynomial ψ_0 is defined up to a factor from k; we choose and fix it. Now let $\psi\in\overline{k}[a',Z_0,Z_{\sigma+1},\ldots,Z_{n+1}]$ where a' are some other variables (or variables and families of variables). Then we put $\psi_*=\psi(a',Z_0,0,\ldots,0,Z_n,Z_{n+1})\in\overline{k}[a',Z_0,Z_n,Z_{n+1}]$. If $\psi\in\overline{k}[a',Z_0,Z_{\sigma+1},\ldots,Z_n]$ then we shall assume that $\psi_*\in\overline{k}[a',Z_0,Z_n]$.

Let c', $\lambda = (L_{s+1}, \dots, L_\sigma)$, $L' = (L'_0, L'_{\sigma+1}, \dots, L'_n) \in \mathcal{U}'_{\lambda,0}$ be as above. Recall that we suppose that all linear forms $L_j \in k[X_0, \dots, X_n]$, $s+1 \le j \le \sigma$; $L'_w \in \overline{k}[X_0, \dots, X_r]$, $\sigma+1 \le w \le n$. Denote for brevity by $A_{\lambda,L'}$ the set of isolated points of the intersection

$$(W \setminus \mathcal{Z}(e_0, \dots, e_r)) \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma) \cap \mathcal{Z}(e'_{\sigma+1}, \dots, e'_n). \tag{15}$$

Hence $\# A_{\lambda,L'} = \deg p'_\lambda \deg W' \le \deg p'_v \deg W'$, see [13].

Denote by $\Phi \in \overline{k}[Z_0, Z_{\sigma+1}, \dots, Z_{n+1}]$ the separable polynomial of the minimal degree with respect to all the variables such that $\Phi(e'_0, e'_{\sigma+1}, \dots, e'_{n+1})$ is vanishing on W_λ. We choose and fix Φ (it is uniquely defined up to a nonzero factor from $\overline{k}$ even under less restrictive assumptions, namely, that the morphism $p_{\lambda,L'}$ is dominant, $\dim W_\lambda = n - \sigma$, $\lambda \in \mathbb{A}^{(\sigma-s)(n+1)}(\overline{k})$ and $L' \in \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$). *Denote by*

$$R = \mathrm{Res}_{Z_{n+1}}(\Phi, \partial\Phi/\partial Z_{n+1}) \in \overline{k}[Z_0, Z_{\sigma+1}, \dots, Z_n]$$

the discriminant of the polynomial Φ with respect to Z_{n+1} (this definition of the discriminant as the resultant of the polynomial and its partial derivative is different from the generally accepted by a factor $\mathrm{lc}_{Z_{n+1}}\Phi$, but it is more convenient for us). *Denote by $R_0 \in \overline{k}[Z_0, Z_{\sigma+1}, \dots, Z_n]$ the square free part of R.* If all linear forms L'_j, $\sigma+1 \le j \le n$, have coefficients from k then we choose Φ to be a polynomial with coefficients from k. Hence R, R_0 also have coefficients from k in the considered case. We shall denote also $\Phi = \Phi_{c',\lambda,L'}$, $R = R_{c',\lambda,L'}$, $R_0 = R_{c',\lambda,L',0}$ when the dependence on e'_{n+1}, λ, L' will be important. Obviously $W'(c', \lambda, L') = \mathcal{Z}(\Phi_{c',\lambda,L'}(X_0, \dots, X_{n-\sigma+1}))$, the degrees $\deg_{Z_0, Z_{\sigma+1}, \dots, Z_{n+1}} \Phi_{c',\lambda,L'} = \deg W'(c', \lambda, L')$ and $\deg_{Z_{n+1}} \Phi_{c',\lambda,L'} = \deg r_{c',\lambda,L'}$, see (23) below. *Hence if all linear forms $L'_j \in k[X_0, \dots, X_r]$, $j \in \{0, \sigma+1, \dots, n\}$ then one can compute $\deg_{Z_{n+1}} \Phi_{c',\lambda,L'}$ and $\deg_{Z_0, Z_{\sigma+1}, \dots, Z_{n+1}} \Phi_{c',\lambda,L'}$ using Theorem 1 [12].* We have $W'(c', \lambda) \cap \mathcal{Z}(U_0, U_{\sigma+1}, \dots, U_n, X_{n+1}) = \emptyset$ in $\mathbb{P}^{n-\sigma+1}(\overline{k_u})$. Hence $\deg_{Z_0, Z_{\sigma+1}, \dots, Z_{n+1}} \Phi_{c',\lambda} = \deg W'(c', \lambda)$. If 1), 2) and 3) holds, see below Definition 1, then by Lemma 2 the polynomials $\Phi_{c,\lambda,L',*}$, $R_{c,\lambda,L',*}$ and $R_{c,\lambda,L',*,0}$ are nonzero. Set

$$a(c', \lambda, L') = \deg_{Z_0, Z_n} R_{c',\lambda,L',*,0}, \quad b(c', \lambda, L') = \deg_{Z_0, Z_n} R_{c',\lambda,L',0,*}.$$

Obviously $a(c', \lambda, L') \le b(c', \lambda, L')$. In Theorem 3 Section 4 the case is considered when this inequality becomes the equality, $d' = 1$, and condition 4), see below, holds.

DEFINITION 1 *We shall say that (c', λ, L') satisfies (*) (or (*) holds for (c', λ, L')) if and only if the following conditions are fulfilled.*

1) *All the linear forms $L_j \in k[X_0, \dots, X_n]$, $s+1 \le j \le \sigma$; $L'_w \in \overline{k}[X_0, \dots, X_r]$, $w \in \{0, \sigma+1, \dots, n\}$.*

2) *Further, L' and λ satisfy assertions (c) and (d) of Theorem 2 [13] (now we do not suppose that L_j and L'_j have integer coefficients as in assertions (a) and (b) of this theorem). In the other words, λ satisfies condition (A) from [13]; the morphism p'_λ, is dominant; the degree $\deg p'_\lambda = \deg p'_v$; $D_{\lambda,i} = D_i$, $D^{(1)}_{\lambda,i} = D^{(1)}_i$, $D^{(2)}_{\lambda,i} = D^{(2)}_i$ for all $\sigma \le i \le n$; and $L' \in \mathcal{U}'''_{\lambda,0} \subset \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$.*

3) The polynomial $e'_{n+1} \in k[X_0, \ldots, X_n]$ *is homogeneous of degree* d'*, and number of elements*

$$\#(e'_{n+1}/e'_0)(W_\lambda \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n) \setminus \mathcal{Z}(e_0, \ldots, e_r)) = \deg p'_v \deg W', \quad (16)$$

4) Finally, the polynomial $\Phi_{c',\lambda,L',*}$ *does not have irreducible factors from* $\overline{k}[Z_0, Z_n]$*, i.e., for every irreducible factor* Ψ *of* $\Phi_{c',\lambda,L',*}$ *the degree* $\deg_{Z_{n+1}} \Psi \geq 1$.

Let $c_0, \lambda_0, L^{(0)}$ *be similar to* c', λ, L'*. We shall say that* $(c_0, \lambda_0, L^{(0)})$ *satisfies (*) (respectively 1), 2), 3), 4)) if and only if it satisfies (*) (respectively 1), 2), 3), 4)) in place of* (c', λ, L')*. Let us extend the ground field* k *till* k_u*. We shall say that* (c', λ) *satisfies property (*) (respectively 1), 2), 3), 4)) if and only if* (c, λ, U) *satisfies (*) (respectively 1), 2), 3), 4)).*

By Lemma 10 (c', λ) satisfies (*) if and only if (c', λ, U) satisfies 1)–3) from (*) (for the ground field k_u).

REMARK 4 *One can modify 2) replacing (d) by* $L' \in \mathcal{U}'_0$*, and suggest new versions of the results given below. Then (iv), see above, will not necessarily hold. One can also weaken (c). The reader can suggest other such modifications.*

Under conditions 1), 2) and 3) the number of elements $\#(e'_{n+1}/e'_0)(W_\lambda \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_n))$ is maximal possible (when it is finite, see [13]), and $\deg_{Z_{n+1}} \Phi_{c',\lambda,L'} = \deg p'_v \deg W'$. By assertion (e) of Theorem 2 [13] (c) and (d) imply $L' \in \mathcal{U}'''_0$. If 1), 2), and 3) hold then $\deg_{Z_{n+1}} \Phi = \deg p'_v \deg W'$. Below we shall see how to construct at least one element (c', λ, L') satisfying (*).

Recall that under conditions 1), 2) and 3) the polynomial $\Phi_{c',\lambda,L',*} \neq 0$. Then $\deg_{Z_0,Z_n,Z_{n+1}} \Phi_{c',\lambda,L',*} = \deg_{Z_0,Z_{\sigma+1},\ldots,Z_{n+1}} \Phi_{c',\lambda,L'} = \deg W'(c',\ \lambda, L')$. *Assume that 1), 2) and 3) hold. Then by Lemma 3 property 4) is satisfied for* (c', λ, L') *if and only if* $\deg W'(c', \lambda, L') = \deg W'(c', \lambda, L', *)$.

REMARK 5 *Suppose that* (c', λ, L') *satisfies 1) and, more than that, all the linear forms* L'_j *have coefficients from* k*. Then one can decide whether* (c', λ, L') *satisfies (*) by the algorithms from [12] and [13] with the time bounds similar to the ones from Theorem 1 [12] and Theorem 2 [13]. More precisely, under conditions 1) and 2) to decide whether* (c', λ, L') *satisfies 4) it is sufficient to compute* $\deg W'(c', \lambda, L')$ *and* $\deg W'(c', \lambda, L', *)$ *by Theorem 1 [12].*

We shall consider (c', λ, L') as a point of $\mathbb{A}^\nu(\overline{k})$ where $\nu = \binom{n+d'}{n} + (\sigma - s)(n+1) + (n-\sigma)(r+1)$. The coordinates of (c', λ, L') are c'_i, $i \in I$; $l_{j,i}$, $s+1 \leq j \leq \sigma$, $0 \leq i \leq n$; $l'_{j,w}$, $\sigma \leq j \leq n$, $0 \leq w \leq r$. Similarly (c', λ) is a point of $\mathbb{A}^{\nu_1}(\overline{k})$ where $\nu_1 = \binom{n+d'}{n} + (\sigma - s)(n+1)$.

Let $(c_1, \lambda_1, L^{(1)})$ (respectively (c_1, λ_1)) be a generic point of an affine linear subspace $\mathcal{L}_{c_1,\lambda_1,L^{(1)}}$ (respectively $\mathcal{L}_{c_1,\lambda_1}$) of $\mathbb{A}^\nu(\overline{k})$ (respectively $\mathbb{A}^{\nu_1}(\overline{k})$) such that c_1 and λ_1 are defined over k.

More explicitly, the generic point (c_1, λ_1) is given as follows. A point $(c_0, \lambda_0, L^{(0)}) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}}$ is given. Let $c_{0,i}$, $i \in I$; $l_{0,j,i}$, $s+1 \leq j \leq \sigma$, $0 \leq i \leq n$; $l^{(0)}_{j,i}$, $j \in \{0, \sigma+1, \ldots, n\}$, $0 \leq i \leq r$ for $(c_0, \lambda_0, L^{(0)})$ be similar to the corresponding coefficients c'_i; $l_{j,i}$; $l'_{j,i}$ defined for (c', λ, L'). Further, suppose that $c_1 = \{c_{1,i}\}_{i \in I}$; $\lambda_1 = (L_{1,s+1}, \ldots, L_{1,\sigma})$, $L_{1,j} = \sum_{0 \leq i \leq n} l_{1,j,i} X_i$, $s+1 \leq j \leq \sigma$;

$L^{(1)} = (L^{(1)}_{\sigma+1}, \dots, L^{(1)}_n)$, $L^{(1)}_j = \sum_{0\le i\le r} l^{(1)}_{j,i} X_i$, $j \in \{0, \sigma+1, \dots, n\}$;

$$c_{1,i} = c_{0,i} + \sum_{1\le w\le n_1} c_{1,i,w}\tau_w, \quad i \in I, \tag{17}$$

$$l_{1,j,i} = l_{0,j,i} + \sum_{1\le w\le n_1} l_{1,j,i,w}\tau_w, \quad s+1 \le j \le \sigma,\ 0 \le i \le n, \tag{18}$$

$$l^{(1)}_{j,i} = l^{(0)}_{j,i} + \sum_{1\le w\le n_1} l^{(1)}_{j,i,w}\tau_w, \quad j \in \{0, \sigma+1, \dots, n\},\ 0 \le i \le r, \tag{19}$$

where $\nu_1 \ge 0$ is bounded from above by a polynomial in $(d')^n$ and r; $\tau_1, \dots, \tau_{n_1}$ is a family of algebraically independent elements over k; and all $c_{1,i,w} \in k$, $l_{1,j,i,w} \in k$, $l^{(1)}_{j,i,w} \in \overline{k}$. The point (c_1, λ_1) is given in the similar way, one should only delete (19). Set

$$e_{j,n+1} = \sum_{i=(i_0,\dots,i_n)\in I} c_{1,j,i} X_0^{i_0} \cdots X_n^{i_n}, \quad j = 0, 1. \tag{20}$$

We do not assume that n_1 is minimal possible, but when it is necessary one can always construct solving a linear system new representations (17), (18), (19) giving a generic point of the same linear space and such that n_1 is minimal possible. In this case $n_1 \le \nu$ (respectively $n_1 \le \nu_1$).

If $(0, \dots, 0) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}}$ (respectively $\mathcal{L}_{c_1,\lambda_1}$) we shall omit sometimes $c_{0,i}$, $l'_{j,i}$, $l^{(0)}_{j,i}$ from (17)–(19) but remember that they exist.

Let us extend the field k till $k(\tau_1, \dots, \tau_{n_1})$. We shall say that $(c_1, \lambda_1, L^{(1)})$ (respectively (c_1, λ_1)) satisfies (*) (or this property holds for $(c_1, \lambda_1, L^{(1)})$) if and only if (*) holds for $(c_1, \lambda_1, L^{(1)})$ (respectively (c_1, λ_1)) and the field $k(\tau_1, \dots, \tau_{n_1})$ in place of (c', λ, L') (respectively (c', λ)) and k. *Consider the following property of* $(c_0, \lambda_0, L^{(0)})$.

5) $\deg W'(c_0, \lambda_0, L^{(0)}) = \deg W'(c_1, \lambda_1, L^{(1)})$.

We shall say that $(c_4, \lambda_4, L^{(4)})$ *(which is similar to* $(c_0, \lambda_0, L^{(0)})$*) satisfies 5) for* $(c_1, \lambda_1, L^{(1)})$ *if and only if this property holds with* $(c_4, \lambda_4, L^{(4)})$ *in place of* $(c_0, \lambda_0, L^{(0)})$. We shall omit "for $(c_1, \lambda_1, L^{(1)})$" when this will not lead to an ambiguity.

Suppose that $(c_0, \lambda_0, L^{(0)})$ satisfies conditions 1) and 2) from (*). Let $\overline{\lambda_0} = (L_{0,0}, \dots, L_{0,n+1})$, see condition (A) from the Introduction of [13]. Set $\overline{\lambda_1} = (L_{0,0}, \dots, L_{0,s}, L_{1,s+1}, \ \dots, \ L_{1,\sigma}, L_{0,\sigma+1}, \dots, L_{0,n+1})$. Then condition (A) [13] holds for λ_1, see [13] Section 1 for details. Suppose that $(c_0, \lambda_0, L^{(0)})$ satisfies (*) and 5). Then by Lemma 9 the element $L^{(1)} \in \mathcal{U}_0'''$ and (*) holds for $(c_1, \lambda_1, L^{(1)})$. We shall always choose the polynomial $\Phi_{c_1,\lambda_1,L^{(1)}}$ to be a separable polynomial from the ring $\overline{k}[\tau_1, \dots, \tau_{n_1}, Z_0, Z_{\sigma+1}, \dots, Z_{n+1}]$ of the minimal degree with respect to all the variables. If all $l^{(1)}_{j,i,w} \in k$ we choose $\Phi_{c_1,\lambda_1,L^{(1)}}$ to be a polynomial with coefficients from k. The polynomials $R_{c_1,\lambda_1,L^{(1)}}, R_{c_1,\lambda_1,L^{(1)},0} \in \overline{k}[\tau_1, \dots, \tau_{n_1}, Z_0, Z_{\sigma+1}, \dots, Z_n]$; $R_{c_1,\lambda_1,L^{(1)},*}$, $R_{c_1,\lambda_1,L^{(1)},0,*}$, $R_{c_1,\lambda_1,L^{(1)},*,0} \in \overline{k}[\tau_1, \dots, \tau_{n_1}, Z_0, Z_n]$.

One can consider particular cases when some of τ_j coincide with elements of the families c, υ, u. More precisely, it is convenient to suppose in the case, when one (or more) element from $c_1, \lambda_1, L^{(1)}$ coincides with c, υ, u respectively, that in

(17)–(19) there are $\tau_1, \ldots, \tau_{n_1}$ together with algebraically independent of them elements from families c, υ, u in place of $\tau_1, \ldots, \tau_{n_1}$ from the initial case. Further, if we consider, e.g., the points of the type $(c_1, v, L^{(1)})$ (respectively (c_1, v)) we suppose that all the coefficients of $v_{j,i}$ in (17), (19) are zeroes. Similarly for the point (c_1, v, U) (respectively (c_1, λ_1, U)) all the coefficients of $v_{j,i}$ and u_{j_1,i_1} (respectively of u_{j_1,i_1}) in (17) (respectively (17), (18)) are zeroes.

For example, we may consider one of the eight cases (c', λ, L'), (c', v, L'), (c, λ, U), (c', v, U), (c, λ, L'), (c, v, L'), (c, λ, U), (c, v, U) as $(c_1, \lambda_1, L^{(1)})$. Set $\Phi_{c,v,U} = F$, $R_{c,v,U} = \Delta$, $b(c, v, U) = b$; $a(c, v, U) = a$, $R_{c,v,U,0} = \Delta_0$. In the other cases for convenience sometimes we omit generic elements c, v, U from the notations. For example, $R_{c_1,\lambda_1,U} = R_{c_1,\lambda_1}$, $\Phi_{c_1,v,L'} = \Phi_{c_1,L'}$, $\Phi_{c,v,L'} = \Phi_{L'}$, $R_{c,\lambda,U} = R_\lambda$, $b(c, \lambda_1, U) = b(\lambda_1)$, $W'(c, \lambda_1) = W'(\lambda_1)$ and so on (in the case of (c, v, U) this operation may lead to an ambiguity, so we have introduced the new notations).

Obviously for (c, v, U) property (*) holds. For (c, λ, L') one can omit 3) from (*) (and obtain the equivalent property). *Consider the following property of* $(c_1, \lambda_1, L^{(1)})$.

(**) *All* $l^{(0)}_{j,i}$ *and* $l^{(1)}_{j,i,w}$ *from (19) are elements from* k. *Further, a point* $(c_3, \lambda_3, L^{(3)}) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}}$ *is given such that* $L^{(3)}$ *and* λ_3 *satisfy 1) and 2) from (*) (in place of* L' *and* λ*), and all the coefficients* $l^{(3)}_{j,i}$ *corresponding to* $L^{(3)}$ *are elements from* k.

In Section 3 we shall construct *an algorithm to decide whether (*) holds for* $(c_1, \lambda_1, L^{(1)})$ *satisfying (**). If (*) is fulfilled for* $(c_1, \lambda_1, L^{(1)})$ *we have at the output of this decision algorithm a point* $(c_4, \lambda_4, L^{(4)}) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}}$ *such that (*) holds for* $(c_4, \lambda_4, L^{(4)})$, *the degree* $\deg W'(c_4, \lambda_4, L^{(4)}) = \deg W'(c_1, \lambda_1, L^{(1)})$, *and all the coefficients* $c_{4,i}$, $l_{4,j,i}$, $l^{(4)}_{j,i}$ *corresponding to* $(c_4, \lambda_4, L^{(4)})$ *are obtained from (17)–(19) by the substitution* τ'_i *for* τ_i, $1 \le i \le n_1$, *where all* τ'_i *are integers of length* $O(n \log d + (n - \sigma) \log d')$. *The working time of this algorithm is the same as for the one from assertion (e) of Theorem 2.*

Suppose that we have constructed by this decision algorithm a point $(c_4, \lambda_4, L^{(4)})$. Then replacing in (17)–(19) $(c_0, \lambda_0, L^{(0)})$ by $(c_4, \lambda_4, L^{(4)})$ we can suppose without loss of generality that $(c_0, \lambda_0, L^{(0)})$ satisfies (*) and 5). In what follows, unless it is not state otherwise, we shall suppose that if (*) holds for $(c_1, \lambda_1, L^{(1)})$ then (*) and 5) holds for $(c_0, \lambda_0, L^{(0)})$.

Actually we need this decision algorithm only for partial cases such as, e.g., $(c_1, \lambda_1, L^{(1)}) = (c, v, L')$. In the last case one can always construct the required $(c_3, \lambda_3, L^{(3)})$ using [13].

The partial case of this decision algorithm is $(c_1, \lambda_1, L^{(1)}) = (c', \lambda, U)$. In this case one can also construct $(c_3, \lambda_3, L^{(3)}) = (c', \lambda, L^{(3)})$ using [13]. *Therefore, one can decide whether* (c', λ) *satisfies (*).*

Note also that extending the field k till $k^{(1)}$ (where the degree $[k^{(1)} : k]$ is finite) one can generalize the property (**) and this decision algorithm for the case when $l^{(0)}_{j,i}, l^{(1)}_{j,i,w}, l^{(3)}_{j,i}, l^{(4)}_{j,i} \in k^{(1)}$.

A decision algorithm without knowing a point $(c_3, \lambda_3, L^{(3)})$ (if it is possible to describe it) requires probably a revision of the results from [12] and [13].

By Lemma 10 the polynomial $R_{c_1,\lambda_1,0,*}$ is nonzero separable, and $a(c_1, \lambda_1) = b(c_1, \lambda_1)$. In particular, $\Delta_{0,*}$ is nonzero, do not have multiple factors with

respect to any variables, and $a = b$.

The definition of the open set $\mathcal{U}$ (and more generally, $\mathcal{U}_\alpha$, $\alpha \in \{0, \sigma + 1, \dots, n\}$) is an existential one. So far there has not been a simple description of any open subset of the set $\mathcal{U}$. Now we are going to define explicitly a nonempty defined over k open in the Zariski topology subset $\mathcal{U}^{(\mathrm{v})} \subset \mathcal{U}$ (and more generally, $\mathcal{U}^{(\mathrm{v})}_\alpha \subset \mathcal{U}_\alpha$, $\alpha \in \{0, \sigma + 1, \dots, n\}$) in the case of zero–characteristic. This subset has a simple structure. Actually we shall define more other open in the Zariski topology subsets of $\mathcal{U}$.

DEFINITION 2 *Let (c_1, λ_1) be as above and satisfy (*). Then $\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$ is the set of all $L' \in \mathcal{U}'''_{\lambda_1,0}(\overline{k}) = \mathcal{U}'''_{\lambda_1,0} \cap \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$ (recall here that $\mathcal{U}'''_{\lambda_1,0}$ is defined for the ground field $k(\tau_1, \dots, \tau_{n_1})$, see above and [13]; also $L' \in \mathcal{U}'''_{\lambda_1,0}$ implies $L' \in \mathcal{U}'''_0$) such that*

(i) (c_1, λ_1, L') satisfies property (*),

(ii) the degree $\deg W'(c_1, \lambda_1) = \deg_{Z_0, Z_{\sigma+1}, \dots, Z_n} \Phi_{c_1,\lambda_1,L'}$,

(iii) $b(c_1, \lambda_1) = a(c_1, \lambda_1, L')$.

For every $\sigma + 1 \le \alpha \le n$ denote by $\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,\alpha}$ the image of $\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$ under the linear projection $\pi_{0,\alpha} : \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k}) \to \mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$, $L' \mapsto (L'_{\sigma+1}, \dots, L'_\alpha)$. Set $\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1}$ to be $\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,\sigma+1}$.

REMARK 6 *Obviously $\pi_{0,\alpha}(\mathcal{U}''_0) \subset \mathcal{U}''_\alpha$, $\pi_{0,\alpha}(\mathcal{U}'''_0) \subset \mathcal{U}'''_\alpha$, $\pi_{0,\alpha}(\mathcal{U}_0) \subset \mathcal{U}_\alpha$ for all α but we do not define $\mathcal{U}''_\alpha$, $\mathcal{U}'''_\alpha$ and $\mathcal{U}_\alpha$ just as the projections of $\mathcal{U}''_0$, $\mathcal{U}'''_0$ and $\mathcal{U}_0$; our definitions are more fine. In the case of $\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,\alpha}$ one can also give the similar more fine definitions separately for every α (it requires more other definitions above). In this way one can obtain the objects which might be different from $\pi_{0,\alpha}(\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0})$. But here to simplify the presentation we bound ourselves by the definitions given above.*

Definition 2 is especially simple for the particular case $(c_1, \lambda_1) = (c, v)$. Then one can replace $\mathcal{U}'''_{\lambda_1,0}$ by $\mathcal{U}'''_0$ and omit (i).

By Lemma 11 the set $\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$ is nonempty open with respect to the Zariski topology in $\mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$. Further, see the proof of the same lemma, $L' \in \mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$ if and only if $L' \in \mathcal{U}'''_{\lambda_1,0}(\overline{k})$, the polynomials $\Phi_{c_1,\lambda_1,L'}$ and $\Phi_{c_1,\lambda_1}|_{u=l'}$ coincide up to a nonzero factor from $\overline{k}[\tau_1, \dots, \tau_{n_1}]$, the polynomial $\Phi_{c_1,\lambda_1,*}|_{u=l'}$ is nonzero and does not have irreducible factors from $\overline{k}[\tau_1, \dots, \tau_{n_1}, Z_0, Z_n]$, and $R_{c_1,\lambda_1,L',*,0}$ coincides with $R_{c_1,\lambda_1,0,*}|_{u=l'}$ up to a nonzero factor from $\overline{k}[\tau_1, \dots, \tau_{n_1}]$.

REMARK 7 *Suppose that (i) is fulfilled, and all linear forms $L'_j \in k[X_0, \dots, X_r]$, $j \in \{0, \sigma + 1, \dots, n\}$. Then one can decide whether (ii) (respectively (iii)) holds by Theorem 2, see below.*

For convenience sometimes we omit generic elements c, v from the notations, cf. above. For example $\mathcal{U}^{(\mathrm{v})}_{c,\lambda,0} = \mathcal{U}^{(\mathrm{v})}_{\lambda,0}$, $\mathcal{U}^{(\mathrm{v})} = \mathcal{U}^{(\mathrm{v})}_{c,v}$, $\mathcal{U}^{(\mathrm{v})}_\alpha = \mathcal{U}^{(\mathrm{v})}_{c,v,\alpha}$ for $\alpha \in \{0, \sigma + 1, \dots, n\}$. We get four nonempty sets $\mathcal{U}^{(\mathrm{v})}_{c',\lambda,0}, \mathcal{U}^{(\mathrm{v})}_{c',0}, \mathcal{U}^{(\mathrm{v})}_{\lambda,0}, \mathcal{U}^{(\mathrm{v})}_0$.

Let $(c_1, \lambda_1, L^{(1)})$, n_1 be as above, and $(c_2, \lambda_2, L^{(2)})$, n_2 be similar to $(c_1, \lambda_1, L^{(1)})$, n_1. *Set $(c_2, \lambda_2, L^{(2)}) \le (c_1, \lambda_1, L^{(1)})$ if and only if $\mathcal{L}_{c_2,\lambda_2,L^{(2)}} \subset \mathcal{L}_{c_1,\lambda_1,L^{(1)}}$.*

For example $(c_1, \lambda_1,\ L^{(1)}) \le (c, v, U)$ for any $(c_1, \lambda_1, L^{(1)})$. If $(c_2, \lambda_2, L^{(2)}) \le (c_1, \lambda_1, L^{(1)})$ we shall suppose in what follows without loss of generality for convenience of notation that the family $\tau_1, \dots, \tau_{n_2}$ defined for $(c_2, \lambda_2, L^{(2)})$ is a part of the family $\tau_1, \dots, \tau_{n_1}$ defined for $(c_1, \lambda_1, L^{(1)})$, and (17)–(19) for $(c_2, \lambda_2, L^{(2)})$ are obtained from (17)–(19) for $(c_1, \lambda_1, L^{(1)})$ by the substitution of $\tau'_{n_2+1}, \dots,$ $\tau'_{n_1} \in k$ for $\tau_{n_2+1}, \dots, \tau_{n_1}$. So we do not suppose that $(c_0, \lambda_0, L^{(0)})$ is the same for $(c_1, \lambda_1, L^{(1)})$ and $(c_2, \lambda_2, L^{(2)})$. In the similar way we define the relation $(c_1, \lambda_1) \le (c_2, \lambda_2)$ (with the analogous assumptions on the families $\tau_1, \dots, \tau_{n_i}$, $i = 1, 2$); one need only to omit above everywhere $L^{(1)}$, $L^{(2)}$ and (19).

THEOREM 1 *Let λ, L' and a homogeneous polynomial e'_{n+1} of degree d' with the corresponding family of coefficients c' be as above, and (c', λ) satisfy (*). Let $(c_2, \lambda_2, L^{(2)}) \le (c_1, \lambda_1, L^{(1)})$ (respectively $(c_2, \lambda_2) \le (c_1, \lambda_1)$) be as above, and the both satisfy (*). Then the following assertions hold.*

(a) $\mathcal{U}^{(\mathrm{v})}_{c',\lambda,0}$, $\mathcal{U}^{(\mathrm{v})}_{c',0}$, $\mathcal{U}^{(\mathrm{v})}_{\lambda,0}$, and $\mathcal{U}^{(\mathrm{v})}_{0}$ are nonempty open in the Zariski topology subsets of $\mathcal{U}_0$. More generally, the sets $\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,\alpha}$ defined above are nonempty open in the Zariski topology subsets of $\mathcal{U}_\alpha$ for all $\alpha \in \{0, \sigma+1, \dots, n\}$. In particular, $\mathcal{U}^{(\mathrm{v})} \subset \mathcal{U}$.

(b) For every (c_1, λ_1) for an arbitrary $L' \in \mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$ there exists (c', λ) such that (c', λ) satisfies (*), $(c', \lambda) \le (c_1, \lambda_1)$, $\deg W'(c', \lambda, L') = \deg W'(c', \lambda) = \deg W'(c_1, \lambda_1)$, $a(c', \lambda, L') = b(c', \lambda) = b(c_1, \lambda_1)$, and, therefore, $L' \in \mathcal{U}^{(\mathrm{v})}_{c',\lambda,0} \subset \mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$.

(c) Besides that, in assertion (b) the pair (c', λ) always can be obtained from (c_1, λ_1) by the substitution in (17), (18) values τ'_i for τ_i, $1 \le i \le n_1$, where all τ'_i are integers of length $O(n \log d + (n-\sigma)\log d')$. In particular, if $c_1 = c$ (respectively $\lambda_1 = v$) then c' is a family of integers of lengths $O(n \log d + (n-\sigma)\log d')$ (respectively all linear forms L_j, $s \le j \le \sigma$, have integer coefficients of length $O(n \log d + (n-\sigma)\log d')$).

(d) In the notation from (c) let $\mathfrak{l} \subset \mathbb{A}^{\nu_1}(\overline{k})$, $\nu_1 = \binom{n+d'}{n} + (\sigma - s)(n+1)$ see above, be a line defined over k and such that $(c', \lambda) \in \mathfrak{l} \subset \mathcal{L}_{c_1,\lambda_1}$, where $\mathcal{L}_{c_1,\lambda_1}$ is a affine linear subspace corresponding to (c_1, λ_1), see above. Then for all $(c_3, \lambda_3) \in \mathfrak{l}(k)$, apart of at most a polynomial in $d^n (d')^{n-\sigma}$ number, assertion (b) holds for (c_3, λ_3) in place of (c', λ).

(e) The integers $a(c_2, \lambda_2, L^{(2)}) \le a(c_1, \lambda_1, L^{(1)})$, $b(c_2, \lambda_2, L^{(2)}) \le b(c_1, \lambda_1, L^{(1)})$, $\deg_{Z_0, Z_{\sigma+1}, \dots, Z_{n+1}} \Phi_{c_2,\lambda_2,L^{(2)}} \le \deg_{Z_0, Z_{\sigma+1}, \dots, Z_{n+1}} \Phi_{c_1,\lambda_1,L^{(1)}}$.

(f) The integers $b(c_2, \lambda_2) \le b(c_1, \lambda_1)$, the degrees $\deg W'(c_2, \lambda_2) \le \deg W'(c_1,$ $\lambda_1)$. If $\deg W'(c_2, \lambda_2) = \deg W'(c_1,\ \lambda_1)$ and $b(c_2, \lambda_2) = b(c_1, \lambda_1)$ then $\mathcal{U}^{(\mathrm{v})}_{c_2,\lambda_2,0} \subset \mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$.

(g) For any (c_1, λ_1) there is $L' = (L'_0, L'_{\sigma+1}, \dots, L'_n) \in \mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$ such that all linear forms L'_j, $j \in \{0, \sigma+1, \dots, n\}$, have integer coefficients of length $O(n \log d + (n-\sigma)\log d')$.

Theorem 2 uses in its proof Theorem 3, Section 4 which generalizes Theorem 3 from [11] for the case of rational morphisms. From Theorem 3 and its Corollary 1 we get immediately the following result.

CRITERION FOR IRREDUCIBILITY *Let* $\Phi = \sum_{0 \le i \le D} \Phi_i X_0^i \in k[X_0, \ldots, X_n]$, $n \ge 1$, *be a homogeneous polynomial over a field of zero–characteristic, and all* $\Phi_i \in k[X_1, \ldots, X_n]$. *Let* $\deg_{X_0} \Phi = D \ge 1$. *Suppose that the discriminant* R *of* Φ *with respect to* X_0 *is nonzero. Let* R_0 *be square free part of* R *(i.e.,* R_0 *is separable and* $\mathcal{Z}(R_0) = \mathcal{Z}(R)$*). Let* L *be a nonzero linear form in* $X_1, \ldots, X_n$ *with coefficients from* k, *and* $A = k[X_0, \ldots, X_n]/(L)$ *(hence* A *is isomorphic to the ring of polynomials in* n *variables and* $\deg_{X_0} a$ *is defined for every* $a \in A$*).*

1) *Suppose that the polynomial* $R_0 \bmod L \in A$ *is separable. Then the irreducible over* k *factors of degree with respect to* X_0 *at least* 1 *of the polynomials* Φ *and* $\Phi \bmod L \in A$ *are in the one–to–one correspondence by the rule* $\Psi \mapsto \Psi \bmod L$.

2) *Suppose additionally that the greatest common divisor* $\mathrm{G\,C\,D}\,\{\Phi_i \bmod L : 0 \le i \le D\} = 1$ *in* A. *Then every irreducible factor of* Φ *(respectively* $\Phi \bmod L$*) is of degree with respect to* X_0 *at least* 1.

Besides that, for every linear subspace $\mathcal{L}$ *of linear forms in* $X_1, \ldots, X_n$ *with* $\dim \mathcal{L} \ge 3$ *there is* $L \in \mathcal{L}$ *satisfying all the formulated conditions from 1) (respectively 1) and 2) whenever* $\mathrm{G\,C\,D}\,\{\Phi_i : 0 \le i \le D\} = 1$ *in* $k[X_0, \ldots, X_n]$*). In particular, under conditions of 1) and 2) if* Φ *is irreducible over* k *then* $\Phi \bmod L$ *is irreducible over* k.

Now we proceed to the precise statements required for Theorem 2. Let homogeneous polynomials $f_1, \ldots, f_m \in k[X_0, \ldots, X_n]$ be given, $m \ge 1$. Consider the closed algebraic set or algebraic variety (in this paper we do not distinguish these two concepts)

$$V = \{(x_0 : \ldots : x_n) : \ f_i(x_0, \ldots, x_n) = 0, \ x_i \in \overline{k}, \ \forall 1 \le i \le m\} \subset \mathbb{P}^n(\overline{k})\,.$$

This is a set of all common zeros of polynomials $f_1, \ldots, f_m$ in $\mathbb{P}^n(\overline{k})$, where $\overline{k}$ is an algebraic closure of k. Hence $V = \mathcal{Z}(f_1, \ldots, f_m)$. Let $e_0, \ldots, e_r \in k[X_0, \ldots, X_n]$ be homogeneous polynomials such that the degree $\deg_{X_0,\ldots,X_n} e_j = d'$ if $e_j \ne 0$ for every $0 \le j \le r$.

Now let the field $k = \mathbb{Q}(t_1, \ldots, t_l, \theta)$ where $t_1, \ldots, t_l$ are algebraically independent over the field $\mathbb{Q}$ and θ is algebraic over $\mathbb{Q}(t_1, \ldots, t_l)$ with the minimal polynomial $\psi_1 \in \mathbb{Q}[t_1, \ldots, t_l, Z]$ and leading coefficient $\mathrm{lc}_Z \psi_1$ of ψ_1 is equal to 1. We shall represent each polynomial $f = f_i$ (and similarly e_j) in the form

$$f = \frac{1}{a_0} \sum_{i_0,\ldots,i_n} \sum_{0 \le j < \deg_Z \psi_1} a_{i_0,\ldots,i_n,j} \theta^j X_0^{i_0} \cdots X_n^{i_n}\,,$$

where $a_0, a_{i_0,\ldots,i_n,j} \in \mathbb{Z}[t_1, \ldots, t_l]$, $\mathrm{G\,C\,D}_{\,i_0,\ldots,i_n,j}(a_0, a_{i_0,\ldots,i_n,j}) = 1$. Define the length $\mathrm{l}(a)$ of an integer a by the formula $\mathrm{l}(a) = \min\{s \in \mathbb{Z} : \ |a| < 2^{s-1}\}$. The length of coefficients $\mathrm{l}(f)$ of the polynomial f is defined to be the maximum of lengths of coefficients from $\mathbb{Z}$ of polynomials $a_0, a_{i_0,\ldots,i_n,j}$ and the degree

$$\deg_{t_\gamma}(f) = \max_{i_0,\ldots,i_n,j} \{\deg_{t_\gamma}(a_0), \deg_{t_\gamma}(a_{i_0,\ldots,i_n,j})\}\,,$$

where $1 \le \gamma \le l$. In the similar way we shall define degrees and lengths of integer coefficients of other polynomials, in particular $\deg_{t_\gamma} F$ and $\mathrm{l}(F)$ are defined.

We shall suppose that we have the following bounds

$$\deg_{X_0,\ldots,X_n}(f_i) < d,\ \deg_{t_\gamma}(f_i) < d_2,\ \mathrm{l}(f_i) < M,$$
$$\deg_{X_0,\ldots,X_n}(e_j) = d' \text{ or } e_j = 0,\ \deg_{t_\gamma}(e_j) < d_2',\ \mathrm{l}(e_j) < M',$$
$$\deg_Z(\psi_1) < d_1,\ \deg_{t_\gamma}(\psi_1) < d_1,\ \mathrm{l}(\psi_1) < M_1\ .$$

for all $1 \le i \le m$, $0 \le j \le r$, $1 \le \gamma \le l$. The size $\mathrm{L}(f)$ of the polynomial f is defined to be the product of $\mathrm{l}(f)$ to the number of all the coefficients from $\mathbb{Z}$ of f in the dense representation. We have

$$\mathrm{L}(f_i) < (\binom{d+n}{n} d_1 + 1) d_2^l M$$

Similarly $\mathrm{L}(\psi_1) < d_1^{l+1} M_1$.

REMARK 8 *Unless we state otherwise, in what follows we suppose l to be fixed. The working time of the algorithm from Theorem 2, see below, is essentially the same as for solving systems of polynomial equations with a finite set of solutions in the projective space. So this theorem can be formulated also in the case when l is not fixed, see [5]. Notice that the constants $O(\ldots)$, see Theorem 2 and lemmas below, in the estimate of the lengths of integer coefficients of linear forms L_j', L_j are absolute; they does not depend on l.*

We shall represent a point $z \in V$ with coordinates from a finite extension of k as follows. An index $0 \le i_0 \le n$ is known such that $X_{i_0}(z) \ne 0$ and an isomorphism of fields

$$k(z) = k((X_1/X_{i_0})(z), \ldots, (X_n/X_{i_0})(z)) = k[\eta] \simeq k[Z]/(\psi_2) \qquad (21)$$

is given where $\eta = \sum_{0 \le i \le n} \nu_i (X_i/X_{i_0})(z)$, the coefficients $\nu_i \in \mathbb{Z}$ are given and $\psi_2 \in k[Z]$ is minimal polynomial of η over k with leading coefficient $\mathrm{lc}_Z \psi_2 = 1$ (so the point z is defined up to a conjugation over k).

Let $g \in k[\eta]$ be an arbitrary element. Then $g = G(\eta)$ for the uniquely defined polynomial $A \in k[Z]$ such that $\deg_Z G < \deg_Z \Phi$. The length of integer coefficients $\mathrm{l}(g)$, the size $\mathrm{L}(g)$ and the degrees $\deg_{t_\alpha} g$, $1 \le \alpha \le l$, of g are defined by the formulas

$$\mathrm{l}(g) = \mathrm{l}(G), \quad \mathrm{L}(g) = \mathrm{L}(G), \quad \deg_{t_\alpha} g = \deg_{t_\alpha} G.$$

Let us define the size of the point z to be $\mathrm{L}(\psi_2) + \sum_{0 \le i \le n} \mathrm{L}(X_i/X_{i_0})$.

THEOREM 2 *Let the field k of zero–characteristic, the algebraic variety V, the polynomials F, $f_1, \ldots, f_m$ and $e_0, \ldots, e_r$, $r \ge 2$, be given as above. Let a point $x^{(0)} \in V$ be given and $x^{(0)}$ belongs to only one irreducible over k component of V. Denote this component by W. Let W' be as above, see (1). Then one can compute the dimension* $\dim W'$. *Suppose that* $\dim W' = n - \sigma > 1$ *(so $s \le \sigma \le n-2$).*

(a) Let (c_1, λ_1) be given by (17), (18). Suppose that (c_0, λ_0) from (17), (18) satisfies (*) and 5) (and hence (c_1, λ_1) also satisfies (*) by Lemma 9). Then one can compute the integer $b(c_1, \lambda_1)$ and the degree $\deg W'(c_1, \lambda_1)$. *Further, one can construct $L' = (L_0', L_{\sigma+1}', \ldots, L_n') \in \mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$ satisfying assertion (b) of Theorem 1, i.e., all L_j' are linear from $k[X_0, \ldots, X_r]$ with integer coefficients of length $O(n \log d + (n-\sigma) \log d')$.*

(b) *In particular, one can construct an element* L' *from an open in the Zariski topology subset* $\mathcal{U}_0^{(\mathrm{v})}$ *of* $\mathcal{U}_0$ *such that all* L'_j *have integer coefficients satisfying the estimate from (a). Hence one can compute a linear form* $L'_{\sigma+1} \in \mathcal{U}^{(\mathrm{v})}$ *with integer coefficients satisfying the estimate from (a).*

(c) *Under conditions from (a) let* $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n)$ *be given, and all* $L'_j \in k[X_0, \ldots, X_r]$, $j \in \{0, \sigma+1, \ldots, n\}$. *Let* (c_1, λ_1, L') *satisfy (**), see above. Then one can decide whether* $L' \in \mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$. *If* $L' \in \mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$ *one can construct* (c', λ) *such that assertions from (b) and (c) of Theorem 1 hold.*

(d) *Let* $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n)$ *be given, and all* $L'_j \in k[X_0, \ldots, X_r]$, $j \in \{0, \sigma+1, \ldots, n\}$. *Then, in particular, one can decide whether* $L' \in \mathcal{U}_0^{(\mathrm{v})}$. *If* $L' \in \mathcal{U}_0^{(\mathrm{v})}$ *one can construct* (c', λ, L') *such that* $\deg W'(c', \lambda, L') = \deg W'(c, v)$, $a(c', \lambda, L') = b$, *assertion (c) of Theorem 1 holds for* (c', λ, L') *and* $(c_1, \lambda_1) = (c, v)$.

(e) *Let* $(c_1, \lambda_1, L^{(1)})$ *be given by (17)–(19) and satisfy (**). Then one can decide whether* $(c_1, \lambda_1, L^{(1)})$ *satisfies (*). If (*) holds for* $(c_1, \lambda_1, L^{(1)})$ *then one can compute* $\deg W'(c_1, \lambda_1, L^{(1)})$, $\deg W'(c_1, \lambda_1)$, $a(c_1, \lambda_1, L^{(1)})$, $b(c_1, \lambda_1)$, *and construct* $(c_4, \lambda_4, L^{(4)}) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}}$ *satisfying (*), condition 5) and such that* $a(c_4, \lambda_4, L^{(4)}) = a(c_1, \lambda_1, L^{(1)})$. *Finally,* $(c_4, \lambda_4, L^{(4)})$ *is obtained from* $(c_1, \lambda_1, L^{(1)})$ *by the substitution in (17)–(19) values* τ'_i *for* τ_i, $1 \le i \le n_1$, *where all* τ'_i *are integers of lengths* $O(n \log d + (n-\sigma) \log d')$.

The working time of the algorithm from (b) is polynomial in d^n, $(d')^n$, d_1, d_2, d'_2, M, M', M_1, m, r, *the size* $\mathrm{L}(x^{(0)})$ *of the point* $x^{(0)}$. *The working time of the algorithm from (a), (respectively (c), (d), (e)) is polynomial in the same values and the size of input, i.e., for (a) it is additionally* $\mathrm{L}(c_1)$, $\mathrm{L}(\lambda_1)$*; for (c) it is* $\mathrm{L}(c_1)$, $\mathrm{L}(\lambda_1)$, $\mathrm{L}(L')$*; for (d) it is* $\mathrm{L}(L')$*; and for (e) it is* $\mathrm{L}(c_1)$, $\mathrm{L}(\lambda_1)$, $\mathrm{L}(L^{(1)})$.

REMARK 9 *One can compute also* $b(c_1, \lambda_1, L^{(1)})$ *within the time for the algorithm from assertion (e). But we do not need it in the present paper.*

REMARK 10 *In what follows in many cases, when one says that* $a_1, \ldots, a_\mu$ *(here may be any other family of variables or coefficients of some polynomials, may be only one variable) are infinitesimals, par abus de langage it is convenient to understand that every* $a_1, \ldots, a_\mu$ *is an infinitesimals or zero. Usually we state explicitly that* $a_1, \ldots, a_\mu$ *can be equal to zero. But even if we forgot to state this explicitly somewhere, the reader will see it easily from the context. This will not lead to an ambiguity. For example, par abus de langage it is more convenient to say that all the coefficients of a linear form* $\widetilde{L}_0$ *are infinitesimals, but still we usually say that all the nonzero coefficients of a linear form* $\widetilde{L}_0$ *are infinitesimals in this case.*

1 Several lemmas

In the next seven lemmas the ground field is $\overline{K}$, and we consider the points $(c_j, \lambda_j, L^{(j)}) \in \mathbb{A}^\nu(\overline{K})$ for integer j (actually for $j = 0, 3, 4, 5$). Below, see Definition 3, we formulate property (D) for $(\lambda_j, L^{(j)})$. For $(c_j, \lambda_j, L^{(j)})$ the family

$c_j = \{c_{j,i}\}_{i\in I}$ similarly to $j = 0$, see the Introduction, the polynomial $e_{j,n+1}$ corresponds to c_j similarly to (20), $\lambda_j = (L_{j,s+1}, \dots, L_{j,\sigma})$, $L^{(j)} = (L^{(j)}_{\sigma+1}, \dots, L^{(j)}_n)$, all $L_{j,i} \in \overline{K}[X_0, \dots, X_n]$, $s+1 \le i \le \sigma$, and $L^{(j)}_w \in \overline{K}[X_0, \dots, X_r]$ are linear forms, $e^{(j)}_w = L^{(j)}_w(e_0, \dots, e_r)$, $w \in \{0, \sigma+1, \dots, n\}$, see the Introduction. The morphisms $p_{\lambda_j, L^{(j)}}$, $p_{\lambda_j, L^{(j)}, *}$ are defined similarly to (7), (10). The varieties $W'(c_j, \lambda_j, L^{(j)})$, $W'(c_j, \lambda_j, L^{(j)}, *)$ are defined similarly to (13), (14). Put $A_{\lambda_j, L^{(j)}}$ to be the set of all the isolated points from $\mathbb{P}^n(\overline{K})$ of the intersection

$$(W \setminus \mathcal{Z}(e_0, \dots, e_r)) \cap \mathcal{Z}(L_{j,s+1}, \dots, L_{j,\sigma}) \cap \mathcal{Z}(e^{(j)}_{\sigma+1}, \dots, e^{(j)}_n), \qquad (22)$$

cf. (15). The polynomial $\Phi_{c_j, \lambda_j, L^{(j)}}$ for $(c_j, \lambda_j, L^{(j)})$ is also defined similarly to the case $j = 0$, see the Introduction. The integers $b(c_j, \lambda_j, L^{(j)})$, $a(c_j, \lambda_j, L^{(j)})$ are defined similarly to the case of the point (c', λ, L') from the Introduction. These definitions are always possible for the introduced objects corresponding to $(c_j, \lambda_j, L^{(j)})$ under the formulated below assumptions, see also the remarks about our assumptions and their variants in the Introduction.

Set $W_{\lambda_j} = W \cap \mathcal{Z}(L_{j,s+1}, \dots, L_{j,\sigma})$ in $\mathbb{P}^n(\overline{K})$, cf. the Introduction. For every $\sigma \le j \le n$ put $\widetilde{W}_{\lambda_j, i}(L^{(j)})$ to be the closure with respect to the Zariski topology of the algebraic variety $(W_{\lambda_j} \setminus \mathcal{Z}(e_0, \dots, e_r)) \cap \mathcal{Z}(e^{(j)}_{\sigma+1}, \dots, e^{(j)}_i)$ in $\mathbb{P}^n(\overline{K})$.

DEFINITION 3 *We shall say that $(\lambda_j, L^{(j)})$ (respectively $(c_j, \lambda_j, L^{(j)})$) satisfies property (D) if and only if the following conditions holds.*

(i) The number of elements $N_{\lambda_j, L^{(j)}} = \#A_{\lambda_j, L^{(j)}} = \deg q_v \deg W'$.

(ii) $A_{\lambda_j, L^{(j)}} \cap (\mathcal{Z}(e^{(j)}_0) \cup \overline{(V \setminus W)}) = \emptyset$.

(iii) For every $\sigma \le i \le n$ every irreducible over $\overline{K}$ component of $\widetilde{W}_{\lambda_j, i}(L^{(j)}) \setminus \mathcal{Z}(e_0, \dots, e_r)$ contains a point from $A_{\lambda_j, L^{(j)}}$. Hence for every $\sigma \le i \le n$ the dimension $\dim \widetilde{W}_{\lambda_j, i}(L^{(j)}) = n - i$.

(iv) For every $\sigma \le i \le n$ the degree $\deg \widetilde{W}_{\lambda_j, i}(L^{(j)}) = D_i$, see the Introduction.

(the notation $(c_j, \lambda_j, L^{(j)})$ is not essential here; one can take in this definition any other point similar to $(c_j, \lambda_j, L^{(j)})$, say, (c_*, λ_*, L^*) and any other field similar to K; this will be seen from the context). Now we would like to make some observations related to property (D) which are useful in what follows.

By Lemma 17 (ii) [13] if $\#A_{\lambda_j, L^{(j)}} = \deg p_v \deg W'$ then for every $x \in A_{\lambda_j, L^{(j)}}$ the intersection of $n-s+1$ algebraic varieties W, $\mathcal{Z}(L_{j,s+1}), \dots, \mathcal{Z}(L_{j,\sigma})$, $\mathcal{Z}(e^{(j)}_{\sigma+1})$, $\dots$, $\mathcal{Z}(e^{(j)}_n)$ at the point x is transversal (in particular x is a smooth point of these $n-s+1$ algebraic varieties). Therefore, if property (D) holds for $(\lambda_j, L^{(j)})$ then intersection (22) coincides with $A_{\lambda_j, L^{(j)}}$ (one should take $i = n$ in (iii) to see it).

If (D) holds then the morphism $p_{\lambda_j, L^{(j)}, i} : \widetilde{W}_{\lambda_j, i}(L^{(j)}) \setminus \mathcal{Z}(e^{(j)}_0, e^{(j)}_{\sigma+1} \dots, e^{(j)}_n) \to \mathbb{P}^{n-i}(\overline{K})$ induced by $p_{\lambda_j, L^{(j)}}$ is dominant (recall that this means by definition that the restriction of $p_{\lambda_j, L^{(j)}, i}$ to every irreducible component of $\widetilde{W}_{\lambda_j, i}(L^{(j)}) \setminus \mathcal{Z}(e^{(j)}_0, e^{(j)}_{\sigma+1} \dots, e^{(j)}_n)$ is dominant) by Lemma 9 [13]. In particular, the morphisms $p_{\lambda_j, L^{(j)}}$ and $p_{\lambda_j, L^{(j)}, *}$ are dominant.

Suppose that $\dim W_{\lambda_j} = n - \sigma$ and the morphism $p_{\lambda_j, L^{(j)}}$ is dominant (now we do not assume that (D) holds). Then by Lemma 17 (ii) [13] the degree $\deg p_{\lambda_j, L^{(j)}} \leq \deg q_v \deg W'$. Further if for every $\sigma \leq i \leq n$ the dimension $\dim \widetilde{W}_{\lambda_j, i}(L^{(j)}) = n - i$ and the morphism $p_{\lambda_j, L^{(j)}, i}$ is dominant then obviously, cf. also the proof of Lemma 11 [12], the degree $\deg p_{\lambda_j, L^{(j)}, i} \leq \deg p_{\lambda_j, L^{(j)}} \leq \deg q_v \deg W'$. Hence in the considered case if (i) holds then by Lemma 2 [12] the degree $\deg p_{\lambda_j, L^{(j)}, i} = \deg q_v \deg W'$, and (iii) holds. Consider the following condition.

(v) For every $\sigma \leq i \leq n$ the dimension $\dim \widetilde{W}_{\lambda_j, i}(L^{(j)}) = n - i$ and the morphism $\deg p_{\lambda_j, L^{(j)}, i}$ is dominant.

Therefore, the conjugation of (i), (ii), (iv) and (v) is equivalent to property (D), i.e., if one replaces in (D) condition (iii) by (v) then the obtained property is equivalent to (D).

Let k' be a purely transcendental extension of k of finite transcendency degree such that all the coefficients of all linear forms $L_{j,i}$, $L_w^{(j)}$ belong to the algebraic closure of k' (hence W is irreducible over k'). Let $\mathcal{U}'_{k',0}$ (respectively $\mathcal{U}''_{k',0}$) be the set $\mathcal{U}'_0$ (respectively $\mathcal{U}''_0$) defined for the ground field k', see Remark 2 [13] (and Remark 3). The sets $\mathcal{U}'_{k',0}$ and $\mathcal{U}''_{k',0}$ depend only on $\overline{k'}$, see Remark 2 [13]. Let $W_i(L^{(j)})$, see the Introduction, be defined for the ground field k'. Obviously if $(c_j, \lambda_j, L^{(j)})$ satisfies conditions (i) and (ii) from (D) then $L^{(j)} \in \mathcal{U}'_{k',0}$ (to prove this one can assume without loss of generality that the coefficients of the linear forms from v are infinitely close to the corresponding coefficients of the linear forms from λ_j, and apply the implicit function theorem). Further, if $(c_j, \lambda_j, L^{(j)})$ satisfies property (D) then $L^{(j)} \in \mathcal{U}''_{k',0}$, and by the Bézout theorem $\widetilde{W}_{\lambda_j, i}(L^{(j)}) = W_i(L^{(j)}) \cap \mathcal{Z}(L_{j,s+1}, \ldots, L_{j,\sigma})$ for every $\sigma \leq i \leq n$. Consider the following condition.

(vi) $L^{(j)} \in \mathcal{U}''_{k',0}$.

(the truth of this condition does not depend on the choice of k'). Then, assuming as above that the coefficients of the linear forms from v are infinitely close to the corresponding coefficients of the linear forms from λ_j and applying Lemma 17 [13] and the implicit function theorem, one proves that the conjugation of (i), (ii) and (vi) is equivalent to property (D).

Now assume that (v) holds for $(\lambda_j, L^{(j)})$ (we do not suppose that (D) is fulfilled). Let k' and $W_i(L^{(j)})$ be as above. Denote by E the union of all the irreducible over $\overline{K}$ components E_1 of $W \cap \mathcal{Z}(e_{\sigma+1}^{(j)}, \ldots, e_i^{(j)})$ such that $E_1 \cap \mathcal{Z}(L_{j,s+1}, \ldots, L_{j,\sigma})$ contains an irreducible component of $\widetilde{W}_{\lambda_j, i}(L^{(j)})$. Then $E \cap \mathcal{Z}(L_{j,s+1}, \ldots, L_{j,\sigma}) \supset \widetilde{W}_{\lambda_j, i}(L^{(j)})$. Further, obviously the algebraic variety $E \subset W_i(L^{(j)})$. Then $\deg W_i(L^{(j)}) \leq D_i$ according to Lemma 12 [13] with the ground field k' in place of k (actually the proof of this inequality does not depend on the ground field; we leave the details to the reader). Hence by the Bézout theorem $\deg \widetilde{W}_{\lambda_j, i}(L^{(j)}) \leq D_i$.

If λ_j satisfies (C) (in place of λ) and $L^{(j)} \in \mathcal{U}''_{\lambda_j,0}$ (so now all the linear forms $L_{j,w}$ have coefficients from k, and all the linear forms $L_w^{(j)}$ have the coefficients from $\overline{k}$) then by Lemma 11 [12] and Lemma 14 [12] the pair $(\lambda_j, L^{(j)})$ satisfies property (D).

Suppose that the all the coefficients of all linear forms $L_{j,\nu}$, $L_w^{(j)}$ belong to a subfield k'' of $\overline{K}$ such that $k'' \supset k$. Let $\lambda_* = (L_{*,s+1}, \ldots, L_{*,\sigma})$, $L^* = (L_0^*, L_{\sigma+1}^*, \ldots, L_n^*)$ where all $L_{*,\nu} \in \overline{K}[X_0, \ldots, X_n]$, $L_w^* \in \overline{K}[X_0, \ldots, X_r]$ are linear forms such that all the nonzero coefficients of linear forms $L_{j,\nu} - L_{*,\nu}$, $s+1 \le \nu \le \sigma$, $L_w^{(j)} - L_w^*$, $w \in \{0, \sigma+1, \ldots, n\}$, are infinitesimals with respect to the field k'. In this case by the implicit function theorem, Lemma 13 [13] and condition (2) (for L^* in place of L') if $(\lambda_j, L^{(j)})$ satisfies (D) then (λ_*, L^*) also satisfies (i), (ii) and (vi). Therefore, (D) holds for (λ_*, L^*).

LEMMA 1 *Let us extend the ground field k till $\overline{K}$. Let $(c_0, \lambda_0, L^{(0)}) \in \mathbb{A}^\nu(\overline{K})$ (so all the elements $c_{0,i}$, the coefficients of all linear forms $L_{0,j}$, $L_w^{(0)}$ are elements from $\overline{K}$). Suppose that $(\lambda_0, L^{(0)})$ satisfies property (D). Then $(\mathrm{lc}_{Z_{n+1}} \Phi_{c_0,\lambda_0,L^{(0)}})(1, 0, \ldots, 0) \ne 0$. Therefore, $(\mathrm{lc}_{Z_{n+1}} \Phi_{c_0,\lambda_0,L^{(0)}})_* \ne 0$, i.e., the degree $\deg_{Z_{n+1}} \Phi_{c_0,\lambda_0,L^{(0)},*} = \deg_{Z_{n+1}} \Phi_{c_0,\lambda_0,L^{(0)}}$. In particular, these assertions hold for any point $(c_0, \lambda_0, L^{(0)}) \in \mathbb{A}^\nu(\overline{k})$ satisfying conditions 1), 2) from (*).*

PROOF By property (D) and our observations the dimensions $\dim W_{\lambda_0} = n - \sigma$, $\dim(W_{\lambda_0} \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e_{\sigma+1}^{(0)}, \ldots, e_{n-1}^{(0)}) = 1$, and the morphisms $p_{\lambda_0,L^{(0)}}$, $p_{\lambda_0,L^{(0)},*}$ are dominant. Consider the regular morphisms

$$r_{c_0,\lambda_0,L^{(0)}} : W'(c_0, \lambda_0, L^{(0)}) \setminus \{(0 : \ldots : 0 : 1)\} \to \mathbb{P}^{n-\sigma}(\overline{k}), \quad (23)$$
$$(X_0 : \ldots : X_{n-\sigma+1}) \mapsto (X_0 : \ldots : X_{n-\sigma});$$
$$q_{c_0,\lambda_0,L^{(0)}} : W_\lambda \setminus \mathcal{Z}(e_0', e_{\sigma+1}', \ldots, e_n') \to W'(c_0, \lambda_0, L^{(0)}). \quad (24)$$
$$(X_0 : \ldots : X_n) \mapsto (e_0' : e_{\sigma+1}' : \ldots : e_{n+1}').$$

Then $p_{\lambda_0,L^{(0)}} = r_{c_0,\lambda_0,L^{(0)}} \circ q_{c_0,\lambda_0,L^{(0)}}$, the morphisms $r_{c_0,\lambda_0,L^{(0)}}$ and $q_{c_0,\lambda_0,L^{(0)}}$ are dominant. Recall that $W'(c_0, \lambda_0, L^{(0)}) = \mathcal{Z}(\Phi_{c_0,\lambda_0,L^{(0)}}(X_0, \ldots, X_{n-\sigma+1}))$. The number of elements $\#A_{\lambda_0,L^{(0)}} = \deg p_v' \deg W'$ is maximal possible (when it is finite), see Lemma 17 (i) [13]. Hence by Lemma 2 [12] there is an affine neighborhood U' of the point $z = (1 : 0 : \ldots : 0) \in \mathbb{P}^{n-\sigma}(\overline{k})$ such that $p_{\lambda_0,L^{(0)}}^{-1}(U') \to U'$ is a finite dominant morphism. The composite of two dominant morphisms is finite if and only if the both of them are finite. Hence the morphism $r_{c_0,\lambda_0,L^{(0)}}^{-1}(U') \to U'$ induced by $r_{c_0,\lambda_0,L^{(0)}}$ is finite dominant. The variety U' is normal. It is known that in this case the leading coefficient $\mathrm{lc}_{Z_{n+1}} \Phi_{c_0,\lambda_0,L^{(0)}}$ does not vanish at any point of $r_{c_0,\lambda_0,L^{(0)}}^{-1}(U')$. Hence $(\mathrm{lc}_{Z_{n+1}} \Phi_{c_0,\lambda_0,L^{(0)}})(1, 0, \ldots, 0) \ne 0$. The lemma is proved.

LEMMA 2 *Under conditions of Lemma 1 suppose additionally that the number of elements $\#(e_{0,n+1}/e_0^{(0)})(A_{\lambda_0,L^{(0)}}) = \deg p_v' \deg W'$. Then the polynomial $\Phi_{c_0,\lambda_0,L^{(0)}}(1, 0, \ldots, 0, Z_{n+1})$ is separable. Hence $\Phi_{c_0,\lambda_0,L^{(0)},*}$ is separable with respect to Z_{n+1}. Therefore, $R_{c_0,\lambda_0,L^{(0)},*}$ and $R_{c_0,\lambda_0,L^{(0)},*,0}$ are nonzero. In particular, these assertions hold for any point $(c_0, \lambda_0, L^{(0)}) \in \mathbb{A}^\nu(\overline{k})$ satisfying conditions 1)–3) from (*).*

PROOF Indeed, the polynomial $\Phi_{c_0,\lambda_0,L^{(0)}}(1, 0, \ldots, 0, Z_{n+1})$ is nonzero by Lemma 1. Hence under our conditions the polynomial $\Phi_{c_0,\lambda_0,L^{(0)}}(1, 0, \ldots, 0, Z_{n+1})$ is separable. The lemma is proved.

DEFINITION 4 *Suppose that the polynomial $\Phi_{c_0,\lambda_0,L^{(0)}} \in \overline{K}[Z_0, Z_{\sigma+1}, \ldots, Z_n]$ is defined for $(c_0, \lambda_0, L^{(0)})$, see the Introduction. Suppose that $\Phi_{c_0,\lambda_0,L^{(0)},*} =$*

$\Phi_{c_0,\lambda_0,L^{(0)}}(Z_0,0,\dots,\ 0,Z_n,Z_{n+1}) \not\equiv 0$. *Let us represent* $\Phi_{c_0,\lambda_0,L^{(0)},*} = g_1g_2$ *where* $g_1 \in \overline{K}[Z_0,Z_n]$, $g_2 \in \overline{K}[Z_0,Z_n,Z_{n+1}]$ *and* $\deg g_1$ *is maximal possible. Let us choose and fix* g_2 *(it is uniquely defined up to a nonzero factor from* $\overline{K}$*). Denote* $\Phi_{c_0,\lambda_0,L^{(0)},+} = g_2$.

LEMMA 3 *Under conditions of Lemma 2 the algebraic variety* $W'(c',\lambda,L',*) = \mathcal{Z}(\Phi_{c,\lambda,L,+}(X_0,X_1,X_2))$*, and the polynomial* $\Phi_{c,\lambda,L,+}$ *is separable.*

PROOF Let

$$q_{c_0,\lambda_0,L^{(0)},*} : W_\lambda \cap \mathcal{Z}(e'_{\sigma+1},\dots,e'_{n-1}) \setminus \mathcal{Z}(e'_0,e'_n)$$
$$\to W'(c_0,\lambda_0,L^{(0)},*), \tag{25}$$
$$r_{c_0,\lambda_0,L^{(0)},*} : W'(c_0,\lambda_0,L^{(0)},*) \setminus \{(0:0:1)\} \to \mathbb{P}^1(\overline{k}), \tag{26}$$

be the morphisms induced by $q_{c_0,\lambda_0,L^{(0)}}$, $r_{c_0,\lambda_0,L^{(0)}}$, see the proof of Lemma 2. By Lemma 2 the morphism $p_{\lambda_0,L^{(0)},*}$ is dominant. Hence $q_{c_0,\lambda_0,L^{(0)},*}$, $r_{c_0,\lambda_0,L^{(0)},*}$ are also dominant. Obviously $W'(c_0,\lambda_0,L^{(0)},*) \subset \mathcal{Z}(\Phi_{c_0,\lambda_0,L^{(0)},*}(X_0,X_1,X_2))$. Since $\#(e_{0,n+1}/e_0^{(0)})(A_{\lambda_0,L^{(0)}}) = \deg p'_v \deg W'$ the variety $W'(c_0,\lambda_0,L^{(0)},*) = \mathcal{Z}(\Phi_{c_0,\lambda_0,L^{(0)},+}(X_0,X_1,X_2))$. The lemma is proved.

LEMMA 4 *Under conditions of Lemma 1 let* $\mathfrak{l} \subset \mathbb{A}^\nu(\overline{K})$ *be a line and the point* $y_0 = (c_0,\lambda_0,L^{(0)}) \in \mathfrak{l}$. *Let* $(c_4,\lambda_4,L^{(4)})$ *be a generic point of the line* $\mathfrak{l}$. *Then for all* $y = (c_3,\lambda_3,L^{(3)}) \in \mathfrak{l}$*, except at most a polynomial in* $d^n(d')^{n-\sigma}$ *number,* y *satisfies property (D), the number of elements* $\#(e_{3,n+1}/e_0^{(3)})(A_{\lambda_3,L^{(3)}}) = \#(e_{4,n+1}/e_0^{(4)})(A_{\lambda_4,L^{(4)}}) \geq \#(e_{1,n+1}/e_0^{(1)})(A_{\lambda_0,L^{(0)}})$*, the degree* $\deg W'(c_3,\lambda_3,L^{(3)}) = \deg W'(c_4,\lambda_4,L^{(4)}) \geq \deg W'(c_0,\ \lambda_0,L^{(0)})$.

PROOF Let ε be the coordinate function on $\mathbb{A}^1(\overline{K})$. Let $(c_5,\lambda_5,L^{(5)}) \in \mathfrak{l}$ and $(c_0,\lambda_0,L^{(0)}) \neq (c_5,\lambda_5,L^{(5)})$. We have $e_{3,n+1} = e_{0,n+1} + \varepsilon^{(3)}e_{5,n+1}$, $L_{3,i} = L_{0,i}+\varepsilon^{(3)}L_{5,i}$, $L_w^{(3)} = L_w^{(0)}+\varepsilon^{(3)}L_w^{(5)}$, where $\varepsilon^{(3)} \in \overline{K}$ for all i,w. We shall assume without loss of generality that $e_{4,n+1} = e_{0,n+1} + \varepsilon e_{5,n+1}$, $L_{4,\nu} = L_{0,\nu} + \varepsilon L_{5,\nu}$, $L_w^{(4)} = L_w^{(0)} + \varepsilon L_w^{(5)}$ for the generic point $(c_4,\lambda_4,L^{(4)})$ of the line $\mathfrak{l}$. Consider the subvariety of $\mathbb{P}^n(\overline{K}) \times \mathbb{A}^1(\overline{K})$

$$\mathcal{V}_i = (V \times \mathbb{A}^1(\overline{K})) \cap \mathcal{Z}(L_{0,s+1} + \varepsilon L_{5,s+1},\dots,L_{0,\sigma} + \varepsilon L_{5,\sigma})$$
$$\cap \mathcal{Z}(e_{\sigma+1}^{(0)} + \varepsilon e_{\sigma+1}^{(5)},\dots,e_i^{(0)} + \varepsilon e_i^{(5)}), \quad i = s,n-1,n \tag{27}$$

(so for $i = s$ one can omit here the variety $\mathcal{Z}(e_{\sigma+1}^{(0)} + \varepsilon e_{\sigma+1}^{(5)},\dots,e_i^{(0)} + \varepsilon e_i^{(5)})$). Denote by $\mathcal{W}_i$ the union of all the irreducible over $\overline{K}$ components E of $\mathcal{V}_i$ such that $E \cap (A_{\lambda_0,L^{(0)}} \times \{0\}) \neq \emptyset$. Hence $\mathcal{W}_i \subset W \times \mathbb{A}^1(\overline{K})$. The dimension of every irreducible component of $\mathcal{W}_i$ is $n-i+1$. Further, $\mathcal{W}_i \cap \mathcal{Z}(\varepsilon) = \widetilde{W}_{\lambda_0,i}(L^{(0)}) \times \{0\}$. For every $\sigma \leq i \leq n$ by the Bézout theorem there is at most a polynomial in $d^n(d')^{n-\sigma}$ number of point $y \in \mathfrak{l}$ such that $\mathcal{W}_i \cap \mathcal{Z}(\varepsilon - \varepsilon^{(3)})$ has an irreducible component E contained in $\mathcal{Z}(e_0,\dots,e_r)$ (since in this case E is necessarily the irreducible component of $\mathcal{W}_i \cap \mathcal{Z}(e_0,\dots,e_r)$). Therefore, for all $y = (c_3,\lambda_3,L^{(3)}) \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, $\mathcal{W}_i \cap \mathcal{Z}(\varepsilon-\varepsilon^{(3)})$ is a union of some irreducible components of $\widetilde{W}_{\lambda_3,i}(L^{(3)}) \times \{\varepsilon^{(3)}\}$. We shall show that $\deg \mathcal{W}_i \cap \mathcal{Z}(\varepsilon-\varepsilon^{(3)}) \geq D_i$ for all $y = (c_3,\lambda_3,L^{(3)}) \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number for every $\sigma \leq i \leq n$. On the other

hand, we know, see the observations above, that always $\deg \widetilde{W}_{\lambda_3,i}(L^{(3)}) \le D_i$. Hence this will imply that $\mathcal{W}_i \cap \mathcal{Z}(\varepsilon - \varepsilon^{(3)}) = \widetilde{W}_{\lambda_3,i}(L^{(3)}) \times \{\varepsilon^{(3)}\}$ for all $y = (c_3, \lambda_3, L^{(3)}) \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number for every $\sigma \le i \le n$.

There are homogeneous polynomials $h_1, \dots, h_s \in k[X_0, \dots, X_n]$ such that W is an irreducible component of $\mathcal{Z}(h_1, \dots, h_s)$ in $\mathbb{P}^n(\overline{k})$, and the intersection of s hypersurfaces $\mathcal{Z}(h_1), \dots, \mathcal{Z}(h_s)$ at each point from $A_{\lambda_0, L^{(0)}}$ is transversal. We choose $h_1, \dots, h_s$ using linear projections $\mathbb{P}^n(\overline{k}) \to \mathbb{P}^{n-s+1}(\overline{k})$ in general position such that $\deg h_i \le \deg W \le d^s$ (we leave the details to the reader). There are linearly independent linear forms $\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_n \in k[X_0, \dots, X_n]$ such that for every $\sigma \le i \le n$ the intersection of n hypersurfaces $\mathcal{Z}(h_1), \dots, \mathcal{Z}(h_s)$, $\mathcal{Z}(L_{0,s+1}), \dots, \mathcal{Z}(L_{0,\sigma})$, $\mathcal{Z}(e^{(0)}_{\sigma+1}), \dots, \mathcal{Z}(e^{(0)}_i)$, $\mathcal{Z}(\widetilde{L}_{i+1}), \dots, \mathcal{Z}(\widetilde{L}_n)$ is transversal at every point $\xi \in \widetilde{W}_{\lambda_0,i}(L^{(0)}) \cap \mathcal{Z}(\widetilde{L}_{i+1}, \dots, \widetilde{L}_n) = B_i$ (hence by (iv) the number of elements $\#B_i = D_i$), for every point $\xi \in B_i$ the differential $d_\xi p_{\lambda_0, L^{(0)}, i}$ is an isomorphism, and $B_i \cap (\mathcal{Z}(e^{(0)}_0) \cup \overline{(V \setminus W)}) = \emptyset$. These are conditions of general position for $\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_n$ due to property (D) for $(\lambda_0, L^{(0)})$ and our first observation related to (D), see above. Denote by Δ_i the Jacobian of n polynomials $h_1, \dots, h_s$, $L_{0,s+1} + \varepsilon L_{5,s+1}, \dots, L_{0,\sigma} + \varepsilon L_{5,n}$, $e^{(0)}_{\sigma+1} + \varepsilon e^{(5)}_{\sigma+1}, \dots, e^{(0)}_i + \varepsilon e^{(5)}_i$, $\widetilde{L}_{i+1}, \dots, \widetilde{L}_n$ with respect to $X_1, \dots, X_n$. Obviously $\deg_{X_0, \dots, X_n, Z} \Delta_i$ is bounded from above by a polynomial in $d^n(d')^{n-\sigma}$. Performing a nondegenerate linear transformation of coordinates in $\mathbb{P}^n(\overline{K})$ we shall assume without loss of generality that $X_0(\xi) \ne 0$ for every $\xi \in B_i$ and $(B_i \times \{0\}) \cap \mathcal{Z}(\Delta_i) = \emptyset$ for all $\sigma \le i \le n$.

Set C_i to be the union of all the irreducible components C of $\mathcal{W}_i \cap \mathcal{Z}(\widetilde{L}_{i+1}, \dots, \widetilde{L}_n)$ such that $C \cap \mathcal{Z}(\varepsilon) \cap (B_i \times \{0\}) \ne \emptyset$. Hence $C_i \cap \mathcal{Z}(\varepsilon) = B_i \times \{0\}$ and C_i is a curve. By the Bézout theorem the degree $\deg C_i \le 2^{n-s} d^s (d')^{n-\sigma}$ ($\deg C_i$ is by definition the degree of $C \setminus \mathcal{Z}(L_0)$ in the affine space $\mathbb{A}^n(\overline{K}) \times \mathbb{A}^1(\overline{K})$ where $L_0 \in k[X_0, \dots, X_n]$ is an arbitrary linear form which does not vanish on any irreducible component of C_i). The linear projection $C_i \to \mathbb{A}^1(\overline{K})$, $((X_0 : \dots : X_n), \varepsilon) \mapsto \varepsilon$, is dominant. Hence for all $\varepsilon^{(3)} \in \overline{K}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, the number of elements $+\infty > \#(C_i \cap \mathcal{Z}(\varepsilon - \varepsilon^{(3)})) \ge \#(C_i \cap \mathcal{Z}(\varepsilon)) = D_i$ for every $\sigma \le i \le n$. Recall that C_i is a union of irreducible components of $\mathcal{W}_i \cap \mathcal{Z}(\widetilde{L}_{i+1}, \dots, \widetilde{L}_n)$. Hence by the Bézout theorem for all $\varepsilon^{(3)} \in \overline{K}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, every isolated point of $C_i \cap \mathcal{Z}(\varepsilon - \varepsilon^{(3)})$ is an isolated point of $\mathcal{W}_i \cap \mathcal{Z}(\widetilde{L}_{i+1}, \dots, \widetilde{L}_n) \cap \mathcal{Z}(\varepsilon - \varepsilon^{(3)})$. Therefore for all $\varepsilon^{(3)} \in \overline{K}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, $\deg \mathcal{W}_i \cap \mathcal{Z}(\varepsilon - \varepsilon^{(3)}) \ge D_i$. For these y the algebraic variety $\mathcal{W}_i \cap \mathcal{Z}(\varepsilon - \varepsilon^{(3)}) = \widetilde{W}_{\lambda_3,i}(L^{(3)}) \times \{\varepsilon^{(3)}\}$ and $\deg \widetilde{W}_{\lambda_3,i}(L^{(3)}) = D_i$, see above.

By the Bézout theorem the number of points from $C_i \cap \mathcal{Z}(\Delta_i)$ is bounded from above by a polynomial in $d^n(d')^{n-\sigma}$. Hence for all $y \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, for every point $(\xi, \varepsilon^{(3)}) \in (\widetilde{W}_{\lambda_3,i}(L^{(3)}) \times \{\varepsilon^{(3)}\}) \cap \mathcal{Z}(\widetilde{L}_{i+1}, \dots, \widetilde{L}_n)$ the differential $d_\xi p_{\lambda_3, L^{(3)}, i}$ is an isomorphism for all $\sigma \le i \le n$. Thus, we have proved that for all $y \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, conditions (i), (iv) and (v), see the beginning of the section, hold with $j = 3$.

Recall that $B_i \cap (\mathcal{Z}(e^{(0)}_0) \cup \overline{(V \setminus W)}) = \emptyset$, see above. Hence the number of elements $\#(C_i \cap (\mathcal{Z}(e^{(0)}_0 + \varepsilon e^{(5)}_0) \cup \overline{(V \setminus W)}))$ is finite and it is bounded from

above by a polynomial in $d^n(d')^{n-\sigma}$ by the Bézout theorem. Taking $i = n$ we get that (ii) with $j = 3$ is fulfilled for all $y \in \mathfrak{l}$ except at most a polynomial in $d^n(d')^{n-\sigma}$ number. For such y conditions (i), (ii), (iv), (v) hold. Hence also (i)–(iv) are satisfied, see observations at the beginning of the section. Thus, property (D) holds for all $y \in \mathfrak{l}$ except at most a polynomial in $d^n(d')^{n-\sigma}$ number. In particular, (D) is fulfilled for the generic point $(c_4, \lambda_4, L^{(4)})$ since there are infinitely many isomorphic over $\overline{K}$ generic points of $\mathfrak{l}$ in $\mathfrak{l}(\overline{K(\varepsilon)})$.

The polynomial $\Phi_{c_4,\lambda_4,L^{(4)}} \in \overline{k}[\varepsilon, Z_0, Z_{\sigma+1}, \ldots, Z_{n+1}]$ is separable and ε does not divide $\Phi_{c_4,\lambda_4,L^{(4)}}$. Hence $\Phi_{c_4,\lambda_4,L^{(4)}}|_{\varepsilon=0} \neq 0$ and $\Phi_{c_0,\lambda_0,L^{(0)}}$ divides $\Phi_{c_4,\lambda_4,L^{(4)}}|_{\varepsilon=0}$. Hence $\deg W'(c_4, \lambda_4, L^{(4)}) \geq \deg W'(c_0, \lambda_0, L^{(0)})$, $\#(e_{4,n+1}/e_0^{(4)})(A_{\lambda_4,L^{(4)}}) = \deg_{Z_{n+1}} \Phi_{c_4,\lambda_4,L^{(4)}} \geq \deg_{Z_{n+1}} \Phi_{c_0,\lambda_0,L^{(0)}} =$ $\#(e_{0,n+1}/e_0^{(0)})(A_{\lambda_0,L^{(0)}})$. Let us represent $\Phi_{c_4,\lambda_4,L^{(4)}} = \sum_{0\leq i\leq\delta} \Phi_i Z_{n+1}^i$, where all $\Phi_i \in \overline{k}[\varepsilon, Z_0, Z_{\sigma+1}, \ldots, Z_n]$, $\delta = \deg_{Z_{n+1}} \Phi_{c_4,\lambda_4,L^{(4)}}$. Then $\mathrm{G\,C\,D}_{0\leq i\leq\delta}\{\Phi_i\} =$ 1 according to the definition of $\Phi_{c_4,\lambda_4,L^{(4)}}$.

On the other hand, for all $y = (c_3, \lambda_3, L^{(3)}) \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, the polynomial $\Phi_{c_4,\lambda_4,L^{(4)}}|_{\varepsilon=\varepsilon^{(3)}}$ is separable, the degree $\deg_{Z_{n+1}} \Phi_{c_3,\lambda_3,L^{(3)}} = \#(e_{3,n+1}/e_0^{(3)})(A_{\lambda_3,L^{(3)}}) = \#(e_{4,n+1}/e_0^{(4)})(A_{\lambda_4,L^{(4)}}) =$ $\deg_{Z_{n+1}} \Phi_{c_4,\lambda_4,L^{(4)}}$, and, cf. the proof of Lemma 8 below, $\mathrm{G\,C\,D}_{0\leq i\leq\delta}\{\Phi_i|_{\varepsilon=\varepsilon^{(3)}}\} = 1$. Hence for these $(c_3, \lambda_3, L^{(3)})$ the polynomials $\Phi_{c_3,\lambda_3,L^{(3)}}$ and $\Phi_{c_4,\lambda_4,L^{(4)}}|_{\varepsilon=\varepsilon^{(3)}}$ coincide up to a nonzero factor from $\overline{K}$. Thus, $\deg W'(c_3, \lambda_3, L^{(3)}) = \deg W'(c_4, \lambda_4, L^{(4)}) \geq \deg W'(c_0, \lambda_0, L^{(0)})$, and $\#(e_{3,n+1}/e_0^{(3)})(A_{\lambda_3,L^{(3)}}) = \#(e_{4,n+1}/e_0^{(4)})(A_{\lambda_4,L^{(4)}}) \geq \#(e_{0,n+1}/e_0^{(0)})(A_{\lambda_0,L^{(0)}})$. The lemma is proved.

LEMMA 5 *Under conditions of Lemma 4 suppose that the number of elements* $\#(e_{0,n+1}/e_0^{(0)})(A_{\lambda_0,L^{(0)}}) = \#(e_{4,n+1}/e_0^{(4)})(A_{\lambda_4,L^{(4)}}) = \deg p'_v \deg W'$ *and* $\deg W'(c_0, \lambda_0, L^{(0)}) = \deg W'(c_4, \lambda_4, L^{(4)})$. *Then for all* $y = (c_3, \lambda_3, L^{(3)}) \in \mathfrak{l}$, *except at most a polynomial in* $d^n(d')^{n-\sigma}$ *number,* $b(c_3, \lambda_3, L^{(3)}) = b(c_4, \lambda_4, L^{(4)}) \geq b(c_0, \lambda_0, L^{(0)})$.

PROOF Let $\varepsilon^{(0)} = 0$. For $j = 0, 3$ for all $y = (c_3, \lambda_3, L^{(3)}) \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, the polynomials $\Phi_{c_j,\lambda_j,L^{(j)}}$ coincide with $\Phi_{c_4,\lambda_4,L^{(4)}}|_{\varepsilon=\varepsilon^{(j)}}$. Hence $(R_{c_4,\lambda_4,L^{(4)},0}|_{\varepsilon=0})_0 = R_{c_0,\lambda_0,L^{(0)},0}$ and $R_{c_4,\lambda_4,L^{(4)},0}|_{\varepsilon=\varepsilon^{(3)}} = R_{c_3,\lambda_3,L^{(3)},0}$ for all $y = (c_3, \lambda_3, L^{(3)}) \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number. This implies the assertion of the lemma. The lemma is proved.

LEMMA 6 *Under conditions of Lemma 4 suppose additionally that the number of elements* $\#(e_{0,n+1}/e_0^{(0)})(A_{\lambda_0,L^{(0)}}) = \deg p'_v \deg W'$. *Then for all* $y = (c_3, \lambda_3, L^{(3)}) \in \mathfrak{l}$, *except at most a polynomial in* $d^n(d')^{n-\sigma}$ *number,* $\deg W'(c_3, \lambda_3, L^{(3)}, *) = \deg W'(c_4, \lambda_4, L^{(4)}, *) \geq \deg W'(c_0, \lambda_0, L^{(0)}, *)$.

PROOF Let us choose in the Definition 4 (with the field $\overline{K}$ in place of $\overline{k}$) for $\Phi_{c_4,\lambda_4,L^{(4)},+}$ the polynomials $g_1 \in \overline{k}[\varepsilon, Z_0, Z_n]$ of maximal possible degree with respect to ε, Z_0, Z_n and $g_2 \in \overline{k}[\varepsilon, Z_0, Z_n, Z_{n+1}]$. Notice that $\Phi_{c_4,\lambda_4,L^{(4)},+}$ is separable, ε does not divide $\Phi_{c_4,\lambda_4,L^{(4)},+}$. Hence $\Phi_{c_4,\lambda_4,L^{(4)},+}|_{\varepsilon=0} \neq 0$, and $\Phi_{c_0,\lambda_0,L^{(0)},+}$ divides $\Phi_{c_4,\lambda_4,L^{(4)},+}|_{\varepsilon=0}$. Therefore, the degree $\deg W'(c_4, \lambda_4, L^{(4)}, *) \geq \deg W'(c_0, \lambda_0, L^{(0)}, *)$ by Lemma 3. Let $\Phi_{c_4,\lambda_4,L^{(4)},+} = \sum_{0\leq i\leq\delta} \varphi_i Z_{n+1}^i$, where all $\varphi_i \in \overline{k}[\varepsilon, Z_0, Z_{\sigma+1}, \ldots, Z_n]$, $\delta = \deg p'_v \deg W'$. Then $\mathrm{G\,C\,D}_{0\leq i\leq\delta}\{\varphi_i\}$

$= 1$ according to the definition of $\Phi_{c_4,\lambda_4,L^{(4)},+}$. On the other hand, for all $y = (c_3, \lambda_3, L^{(3)}) \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, the polynomial $\Phi_{c_4,\lambda_4,L^{(4)},+}|_{\varepsilon=\varepsilon^{(3)}}$ is separable, and, cf. the proof of Lemma 8, the greatest common divisor $\mathrm{G\,C\,D}_{\,0\le i\le\delta}\{\varphi_i|_{\varepsilon=\varepsilon^{(3)}}\} = 1$. Hence for these $(c_3, \lambda_3, L^{(3)})$ the polynomials $\Phi_{c_3,\lambda_3,L^{(3)},+}$ and $\Phi_{c_4,\lambda_4,L^{(4)},+}|_{\varepsilon=\varepsilon^{(3)}}$ coincides up to a nonzero factor from $\overline{K}$, and, therefore, $\deg W'(c_3, \lambda_3, L^{(3)}, *) = \deg W'(c_4, \lambda_4, L^{(4)}, *) \ge \deg W'(c_0, \lambda_0, L^{(0)}, *)$. The lemma is proved.

LEMMA 7 *Under conditions of Lemma 4 suppose the that the number of elements* $\#(e_{0,n+1}/e_0^{(0)})(A_{\lambda_0,L^{(0)}}) = \#(e_{4,n+1}/e_0^{(4)})(A_{\lambda_4,L^{(4)}})$ *and the degrees* $\deg W'(c_0, \lambda_0, L^{(0)}) = \deg W'(c_4, \lambda_4, L^{(4)}) = \deg W'(c_0, \lambda_0, L^{(0)}, *) = \deg W'(c_4, \lambda_4, L^{(4)}, *)$. *Then for all* $y = (c_3, \lambda_3, L^{(3)}) \in \mathfrak{l}$, *except at most a polynomial in* $d^n(d')^{n-\sigma}$ *number* $a(c_3, \lambda_3, L^{(3)}) = a(c_4, \lambda_4, L^{(4)}) \ge a(c_0, \lambda_0, L^{(0)})$.

PROOF Let $\varepsilon^{(0)} = 0$ and $\varepsilon^{(3)}$ be as above. By Lemma 3 and Lemma 6 for all $y = (c_3, \lambda_3, L^{(3)}) \in \mathfrak{l}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number $\Phi_{c_j,\lambda_j,L^{(j)},+} = \Phi_{c_j,\lambda_j,L^{(j)},*} = \Phi_{c_4,\lambda_4,L^{(4)},*}|_{\varepsilon=\varepsilon^{(j)}}$, $j = 0, 3$. Now the required assertion follows from the definitions of $a(c_j, \lambda_j, L^{(j)})$, $j = 0, 3, 4$ immediately. The lemma is proved.

Recall that in the notation from the Introduction we have the natural projection $\mathbb{A}^{\nu}(\overline{k}) \to \mathbb{A}^{\nu_1}(\overline{k})$ such that $(c, \lambda, L') \mapsto (c', \lambda)$.

LEMMA 8 *Let* $\mathbb{A}^1(\overline{k})$ *be defined over* k *and have a fixed coordinate function* ε *defined over* k. *In the notation from the Introduction let* $\iota : \mathbb{A}^1(\overline{k}) \to \mathbb{A}^{\nu}(\overline{k})$ *(respectively* $\mathbb{A}^1(\overline{k}) \to \mathbb{A}^{\nu_1}(\overline{k})$*) be an affine linear embedding inducing the defined over* k *morphism* $\iota : \mathbb{A}^1(\overline{k}) \to \mathbb{A}^{\nu_1}(\overline{k})$ *(respectively defined over* k *itself). Let* $y_\varepsilon = \iota(\varepsilon) = (c_4, \lambda_4, L^{(4)})$ *be the generic point of the line* $\mathfrak{l} = \iota(\mathbb{A}^1(\overline{k}))$. *Suppose that the point* $\iota(0) = (c_0, \lambda_0, L^{(0)})$ *(respectively* $\iota(0) = (c_0, \lambda_0)$*) satisfies property (*) from the Introduction and* $\deg W'(c_0, \lambda_0, L^{(0)}) = \deg W'(c_4, \lambda_4, L^{(4)})$ *(respectively satisfies property (*) and* $\deg W'(c_0, \lambda_0, U) = \deg W'(c_4, \lambda_4, U)$*). Then for all* $y \in k$, *except at most a polynomial in* $d^n(d')^{n-\sigma}$ *number, property (*) holds for* $\iota(y) = (c_3, \lambda_3, L^{(3)})$ *(respectively for* $\iota(y) = (c_3, \lambda_3)$*) and the degree* $\deg W'(c_3, \lambda_3, L^{(3)}) = \deg W'(c_4, \lambda_4, L^{(4)})$ *(respectively* $\deg W'(c_3, \lambda_3, U) = \deg W'(c_4, \lambda_4, U)$*). More than that, the similar statement holds if one replaces (*) by the conjugation of conditions 1) and 2) (or 1), 2) and 3)) from (*).*

PROOF If (c_0, λ_0) satisfies property (*) and $\deg W'(c_0, \lambda_0, U) = \deg W'(c_4, \lambda_4, U)$ then there exists $L^{(0)} \in \mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$ such that $\deg W'(c_0, \lambda_0, L^{(0)}) = \deg W'(c_4, \lambda_4, U)$. For any point (c_3, λ_3) from the statement of the lemma $\deg W'(c_3, \lambda_3, L^{(0)}) \le \deg W'(c_3, \lambda_3, U) \le \deg W'(c_4, \lambda_4, U)$. Hence the case of (c_0, λ_0) is reduced to $(c_0, \lambda_0, L^{(0)})$. The last assertion of the lemma is proved similarly to the first one.

So we shall consider the first assertion for the case of $(c_0, \lambda_0, L^{(0)})$. The intersection (15) with $j = 0$ is finite and is transversal, see [13]. Considering the corresponding Jacobian we get that it remains transversal for all points $(c_3, \lambda_3, L^{(3)}) \in \iota(\mathbb{A}^1(\overline{k}))$ in place of $(c_0, \lambda_0, L^{(0)})$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number. Hence assertion (d) of Theorem 1 [12], assertion (f) of Theorem 2 [13] and the Bézout theorem imply that properties 1), 2) and 3) from (*) hold for all $(c_3, \lambda_3, L^{(3)}) \in \iota(\mathbb{A}^1(\overline{k}))$ in place of, except at most a polynomial in $d^n(d')^{n-\sigma}$ number.

Consider in details condition 4) from (*). Let us choose $\Phi_{c_4,\lambda_4,L^{(4)}} \in \overline{k}[\varepsilon, Z_0, Z_{\sigma+1}, \ldots, Z_{n+1}]$ where ε does not divide the last polynomial. Denote for brevity $\delta = \deg_{Z_{n+1}} \Phi_{c_4,\lambda_4,L^{(4)}}$. Let $\Phi_{c_4,\lambda_4,L^{(4)},*} = \sum_{0\le i\le\delta} \varphi_i Z_{n+1}^i$ where all $\varphi_i \in \overline{k}[\varepsilon, Z_0, Z_n]$. By the Bézout theorem all the degrees $\deg_{\varepsilon,Z_0,Z_n} \varphi_i$ are bounded from above by a polynomial in $d^n(d')^{n-\sigma}$, cf. [12] Lemma 7. The greatest common divisor $\mathrm{G\,C\,D}_{0\le i\le\delta}\{\varphi_i(0,\ Z_0, Z_n)\} = 1$ in $\overline{k}[Z_0, Z_n]$ by condition 4) from (*) for $\iota(0)$ and since $\deg W'(c_0,\lambda_0,L^{(0)}) = \deg W'(c_4,\lambda_4,L^{(4)})$. Hence there are polynomials $A_{j,i} \in \overline{k}[\varepsilon, Z_0, Z_n]$, $j = 0, n$, $0 \le i \le \delta$, and $0 \ne A_j \in \overline{k}[\varepsilon]$, $j = 0, n$, such that the degrees with respect to ε, Z_0, Z_n of all $A_{j,i}, A_j$ are bounded from above by a polynomial in $d^n(d')^{n-\sigma}$, and $\sum_{0\le i\le\delta} A_{j,i}\varphi_i = A_j Z_j^{\mu}$, $j = 0, n$. Now if $\varepsilon^{(3)} \in k$, $\iota(\varepsilon^{(3)}) = (c_3,\lambda_3,L^{(3)})$, properties 1), 2), 3) hold for $(c_3,\lambda_3,L^{(3)})$ and $(A_0A_n)(\varepsilon^{(3)}) \ne 0$ then the polynomial $\Phi_{c_3,\lambda_3,L^{(3)},*} = \Phi_{c_4,\lambda_4,L^{(4)},*}|_{\{\varepsilon=\varepsilon^{(3)}\}}$ does not have irreducible factors from $\overline{k}[Z_0, Z_n]$, i.e., property 4) holds and $\deg W'(c_3,\lambda_3,L^{(3)}) = \deg W'(c_4,\lambda_4,L^{(4)})$. The lemma is proved.

LEMMA 9 *Let $(c_1,\lambda_1,L^{(1)})$ (respectively (c_1,λ_1)) be given by (17)–(19) (respectively (17), (18)), see the Introduction. In the notation from the Introduction suppose that $(c_0,\lambda_0,L^{(0)})$ (respectively (c_0,λ_0)) satisfies property (*) and 5) (respectively satisfies (*) and $\deg W'(c_0,\lambda_0,U) = \deg W'(c_1,\lambda_1,U)$). Then $(c_1,\lambda_1,L^{(1)})$ (respectively (c_1,λ_1)) satisfy (*). More than that, the similar statement holds if one replaces (*) by the conjunction of conditions 1) and 2) (or 1), 2) and 3)) from (*).*

PROOF The generic point $(c_4,\lambda_4,L^{(4)})$ (respectively (c_4,λ_4)) of the line $\iota(\mathbb{A}^1(\overline{k}))$ from Lemma 8 satisfies (*) and 5) (respectively (*) and $\deg W'(c_0,\lambda_0,U) = \deg W'(c_4,\lambda_4,U)$) if one extends ground field. Hence extending subsequently k till $k(\tau_1)$, $k(\tau_1,\tau_2),\ldots,k(\tau_1,\ldots,\tau_{n_1})$ and applying each time Lemma 8 we get the assertion of the lemma. The lemma is proved.

LEMMA 10 *Let (c_1,λ_1) be given by (17), (18) and satisfy 1)–3) from (*), see the Introduction. Then the following assertions hold.*

(a) $R_{c_1,\lambda_1,0,*} = R_{c_1,\lambda_1,0}(\tau_1,\ldots,\tau_{n_1},u,Z_0,0,\ldots,0,Z_n)$ is a nonzero separable element of the ring of polynomials $\overline{k}[\tau_1,\ldots,\tau_{n_1},u,Z_0,Z_n]$ (with respect to all the variables $\tau_1,\ldots,\tau_{n_1}$, u, Z_0, Z_n). In the other words, $a(c_1,\lambda_1) = b(c_1,\lambda_1)$, see the Introduction.

(b) The polynomial $\Phi_{c_1,\lambda_1,*}$ does not have irreducible factors from $k[\tau_1,\ldots,\tau_{n_1}, u, Z_0, Z_n]$.

In particular (see the notation in the Introduction), $\Delta_{0,*}$ is a nonzero separable element of the ring of polynomials $\overline{k}[c,\upsilon,u,Z_0,Z_n]$, and F_* does not have irreducible factors from $\overline{k}[c,\upsilon,u,Z_0,Z_n]$.

PROOF Let $\widetilde{Z}_{\sigma+1},\ldots,\widetilde{Z}_{n-1}$ be linear forms in $Z_0, Z_{\sigma+1},\ldots,Z_n$ with coefficients from k in general position. Let us identify $k[Z_0, Z_{\sigma+1},\ldots,Z_n]/(\widetilde{Z}_{\sigma+1},\ldots,\widetilde{Z}_{n-1}) = k[Z_0, Z_n]$.

Let $\delta = \deg p'_v \deg W'$. By Lemma 2 $\Phi_{c_1,\lambda_1,*} \ne 0$. Let us represent $\Phi_{c_1,\lambda_1} = \sum_{0\le i\le\delta} \Phi_i Z_{n+1}^i$ where all $\Phi_i \in \overline{k}[\tau_1,\ldots,\tau_{n_1},u,Z_0,Z_n]$. By its definition the polynomial Φ_{c_1,λ_1} does not have irreducible factor from $\overline{k}[\tau_1,\ldots,\tau_{n_1},u,Z_0,\ldots,Z_n]$. Hence $\mathrm{G\,C\,D}_{0\le i\le\delta}\{\Phi_i\} = 1$ in the last ring, i.e., $\mathrm{codim}\mathcal{Z}(\Phi_0,\ldots,\Phi_\delta) \ge$

2 in the corresponding affine space. Recall that $\widetilde{Z}_{\sigma+1},\dots,\widetilde{Z}_{n-1}$ are in general position. Hence considering the codimension we get $\mathrm{G\,C\,D}_{0\le i\le\delta}\{\Phi_i \bmod (\widetilde{Z}_{\sigma+1},\dots,\widetilde{Z}_{n-1})\}=1$ in the ring $\overline{k}[\tau_1,\dots,\tau_{n_1},u,Z_0,\dots,Z_n]/(\widetilde{Z}_{\sigma+1},\dots,\widetilde{Z}_{n-1})=\overline{k}[\tau_1,\dots,\tau_{n_1},u,Z_0,Z_n]$.

Consider the affine space $\mathbb{A}^N(\overline{k})$ with the coordinate functions from the families $\tau_1,\dots,\tau_{n_1}$, u, $Z_0,Z_{\sigma+1},\dots,Z_n$. Let $\mathbb{A}^N(\overline{k})\subset\mathbb{P}^N(\overline{k})$ in the natural way. Let $\mathcal{W}$ be an irreducible component of the closure in $\mathbb{P}^N(\overline{k})$ of the algebraic variety $\mathcal{Z}(R_{c_1,\lambda_1,0})$. Let us apply Theorem 1 [13] to every morphism $\mathcal{W}\to\mathbb{P}^{n-\sigma+1}(\overline{k})$ induced by the linear projection to given by $Z_0,Z_{\sigma+1},\dots,Z_n$. Then since $\widetilde{Z}_{\sigma+1},\dots,\widetilde{Z}_{n-1}$ are in general position, we obtain that $0\ne R_{c_1,\lambda_1,0} \bmod (\widetilde{Z}_{\sigma+1},\dots,\widetilde{Z}_{n-1})\in\overline{k}[\tau_1,\dots,\tau_{n_1},u,Z_0,Z_{\sigma+1},\dots,Z_n]/(\widetilde{Z}_{\sigma+1},\dots,\widetilde{Z}_{n-1})$ is a separable element of the last ring. By their definition U_j are generic linear forms. Hence U_0, $\widetilde{Z}_j(U_0,U_{\sigma+1},\dots,U_{n-1})$, $\sigma+1\le j\le n-1$, U_n is also a family of generic linear forms in $X_0,\dots,X_n$ with respect to the field k. Therefore, the polynomials $\Phi_{i,*}$, $R_{c_1,\lambda_1,0,*}$ are obtained from $\Phi_i \bmod (\widetilde{Z}_{\sigma+1},\dots,\widetilde{Z}_{n-1})$, $R_{c_1,\lambda_1,0} \bmod (\widetilde{Z}_{\sigma+1},\dots,\widetilde{Z}_{n-1})$ by an automorphism induced by a linear transformation over k of the variables from the family u. Therefore, the greatest common divisor $\mathrm{G\,C\,D}_{0\le i\le\delta}\{\Phi_{i,*}\}=1$ and $R_{c_1,\lambda_1,0,*}$ is nonzero separable. The lemma is proved.

LEMMA 11 *Let (c_1,λ_1) be from the Introduction and satisfies (*). Then the set of all L' such that $L'\in\mathcal{U}'''_{\lambda_1,0}(\overline{k})$ and conditions (i), (ii) (respectively (i)–(iii)) from Definition 2 hold for (c_1,λ_1,L') is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$. Hence $\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$.*

PROOF By conditions 3) and 2) from (*) the polynomial $e_{1,n+1}$ does not vanish on any irreducible component of W_{λ_1}. Hence X_{r+1} does not vanish on any irreducible component of $W'(c_1,\lambda_1)$. Therefore, $W'(c_1,\lambda_1)\cap\mathcal{Z}(U_0,U_{\sigma+1},\dots,U_n,X_{r+1})=\emptyset$, the degree $\deg W'(c_1,\lambda_1)=$
$\deg_{Z_0,Z_{\sigma+1},\dots,Z_n}\Phi_{c_1,\lambda_1}$, cf. Remark 5 from [12]. Hence, see Lemma 13 [8] and Remark 6 [13] if $\deg_{Z_0,Z_{\sigma+1},\dots,Z_n}\Phi_{c_1,\lambda_1,L'}=\deg W'(c_1,\lambda_1)$ then the polynomials $\Phi_{c_1,\lambda_1,L'}$ and $\Phi_{c_1,\lambda_1}|_{u=l'}$ (the last polynomial is obtained by the substitution of $l'_{i,j}$ for $u_{i,j}$ for all i,j) coincide up to a nonzero factor from $\overline{k}[\tau_1,\dots,\tau_{n_1}]$.

Further, let $L'\in\mathcal{U}'''_{\lambda_1,0}(\overline{k})$ and (c_1,λ_1,L') satisfies 2), 3) from (*) (the set of all such L' is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$). Then condition (ii) of Definition 2 holds for L' if and only if $W'(c_1,\lambda_1)\cap\mathcal{Z}(L'_0,L'_{\sigma+1},\dots,L'_n,X_{r+1})=\emptyset$. Therefore the set $\mathcal{W}$ of all $L'\in\mathcal{U}'''_{\lambda_1,0}(\overline{k})$ such that conditions 2), 3) from (*) and (ii) of Definition 2 are fulfilled for (c_1,λ_1,L') is a nonempty open in the Zariski topology subset of $\mathbb{A}^{(r+1)(n-\sigma+1)}(\overline{k})$.

Let $\delta=\deg p'_v\deg W'$. Let us represent $\Phi_{c_1,\lambda_1,*}=\sum_{0\le i\le\delta}\varphi_i Z^i_{n+1}$ where all $\Phi_i\in\overline{k}[\tau_1,\dots,\tau_{n_1},u,Z_0,Z_n]$. By Lemma 10 there are polynomials $A_{j,i}\in\overline{k}[\tau_1,\dots,\tau_{n_1},u,Z_0,Z_n]$, $j=0,n$, $0\le i\le\delta$, and $0\ne A_j\in\overline{k}[\tau_1,\dots,\tau_{n_1},u]$, $j=0,n$, such that $\sum_{0\le i\le\delta}A_{j,i}\varphi_i=A_jZ^\mu_j$, $j=0,n$. Now let $L'\in\mathcal{W}$. Then $(A_0A_n)|_{\{u=l'\}}\ne0$ implies condition 4) from (*) for (c_1,λ_1,L'). Hence the set of $L'\in\mathcal{W}$ such that condition 4) from (*) holds for (c_1,λ_1,L') is nonempty. On the other hand, obviously if $L'\in\mathcal{W}$ and $\mathrm{G\,C\,D}\{\varphi_i|_{\{u=l'\}}\}=1$ in $\overline{k}(\tau_1,\dots,\tau_{n_1})[Z_0,Z_n]$ then there is a neighborhood $\mathcal{W}'\subset\mathcal{W}$ of L' with respect to Zariski topology such that for every $L''\in\mathcal{W}'$ we have $\mathrm{G\,C\,D}\{\varphi_i|_{\{u=l''\}}\}=1$

(here l'' for L'' is similar to l' defined for L') in $\overline{k}(\tau_1,\dots,\tau_{n_1})[Z_0,Z_n]$. This implies the assertion of the lemma related to conditions (i) and (ii).

Denote by $\mathcal{W}''$ the set of all $L'\in\mathcal{U}'''_{\lambda_1,0}(\overline{k})$ such that condition (i), (ii) of Definition 2 hold. Then $L'\in\mathcal{U}^{(5)}_{c_1,\lambda_1}$ if and only if $L'\in\mathcal{W}''$ and $R_{c_1,\lambda_1,0,*}|_{\{u=l'\}}$ coincides with $R_{c_1,\lambda_1,L',*,0}$ up to a nonzero factor from $\overline{k}[\tau_1,\dots,\tau_{n_1}]$, i.e., if and only if $R_{c_1,\lambda_1,0,*}|_{\{u=l'\}}$ is separable with respect to the variables Z_0,Z_n. The polynomial $R_{c_1,\lambda_1,0,*}$ is nonzero and separable by Lemma 10. Hence, considering the discriminants of $R_{c_1,\lambda_1,0,*}$ with respect to Z_0 and Z_n we deduce that the set of all $L'\in\mathcal{W}'$ such that $R_{c_1,\lambda_1,0,*}|_{\{u=l'\}}$ coincides with $R_{c_1,\lambda_1,L',*,0}$ is a nonempty open in the Zariski topology subset of $\mathcal{W}''$. The lemma is proved.

LEMMA 12 *In the notation from the Introduction let* $(c_2,\lambda_2,L^{(2)})\le(c_1,\lambda_1,L^{(1)})$. *Assume that* $(c_i,\lambda_i,L^{(i)})$ *satisfies conditions 1)–3) from (*) if one extends the field* k *till* $k(\tau_1,\dots,\tau_{n_i})$, $i=1,2$. *Suppose that* $n_1=n_2+1$, *and (17)–(19) for* $(c_2,\lambda_2,L^{(2)})$ *are obtained from (17)–(19) for* $(c_1,\lambda_1,L^{(1)})$ *by the substitution of* $\tau'_{n_1}\in k$ *for* τ_{n_1}, *see the Introduction. Then the polynomial* $\Phi_{c_2,\lambda_2,L^{(2)}}$ *coincides with* $\Phi_{c_1,\lambda_1,L^{(1)}}|_{\{\tau_{n_1}=\tau'_{n_1}\}}$ *up to a nonzero factor from* $\overline{k}[\tau_1,\dots,\tau_{n_2},Z_0,Z_{\sigma+1},\dots,Z_n]$. *In particular, if* $\deg_{Z_0,Z_{\sigma+1},\dots,Z_{n+1}}\Phi_{c_2,\lambda_2,L^{(2)}}=\deg_{Z_0,Z_{\sigma+1},\dots,Z_{n+1}}\Phi_{c_1,\lambda_1,L^{(1)}}$ *then* $\Phi_{c_2,\lambda_2,L^{(2)}}$ *coincides with* $\Phi_{c_1,\lambda_1,L^{(1)}}|_{\{\tau_{n_1}=\tau'_{n_1}\}}$ *up to a nonzero factor from* $\overline{k}[\tau_1,\dots,\tau_{n_2}]$.

PROOF Extending the ground field k till $k(\tau_1,\dots,\tau_{n_2})$ we shall suppose without loss of generality that $n_1=1$ and $n_2=0$. Let $\mathbb{A}^1(\overline{k})$ has the coordinate function τ_1. Let $\mathcal{W}$ be the union of all the irreducible over $\overline{k}$ components E of $(W\times\mathbb{A}^1(\overline{k}))\cap\mathcal{Z}(L_{1,s+1},\dots,L_{1,\sigma})$ such that E is not contained in any hyperplane $\mathcal{Z}(\tau_1-z)$, $z\in\overline{k}$. Then an arbitrary polynomial $h\in\overline{k}[X_0,\dots,X_n,\tau_1]$ homogeneous with respect to $X_0,\dots,X_n$ vanishes on $W_{\lambda_1}(\overline{k(\tau_1)})$ if and only if h vanishes on $\mathcal{W}$.

Besides that, $\dim E=n-\sigma+1$ for every component E of $\mathcal{W}$. Hence $\dim\mathcal{W}\cap\mathcal{Z}(\tau_1-\tau'_1)=n-\sigma$. The dimension of every irreducible component E_1 of $(W\times\mathbb{A}^1(\overline{k}))\cap\mathcal{Z}(L_{1,s+1},\dots,L_{1,\sigma})$ is at least $n-\sigma+1$. In particular this is true for every irreducible component $E_1\subset\mathcal{Z}(\tau_1-\tau'_1)$. Obviously $(W\times\mathbb{A}^1(\overline{k}))\cap\mathcal{Z}(L_{1,s+1},\dots,L_{1,\sigma})\cap\mathcal{Z}(\tau_1-\tau'_1)=W_{\lambda_2}\times\{\tau'_1\}$ and $\dim W_{\lambda_2}=n-\sigma$. Hence $\mathcal{W}\cap\mathcal{Z}(\tau_1-\tau'_1)=W_{\lambda_2}\times\{\tau'_1\}$.

The element $\tau_1-\tau'_1$ does not divide $\Phi_{c_1,\lambda_1,L^{(1)}}$. Hence the polynomial $\Phi_{c_1,\lambda_1,L^{(1)}}|_{\{\tau_1=\tau'_1\}}\ne0$. Let us substitute in $\Phi_{c_1,\lambda_1,L^{(1)}}$ the elements $L^{(1)}_j(e_0,\dots,e_r)$ for Z_j, $j\in\{0,\sigma+1,\dots,n\}$, and $e_{1,n+1}$ for Z_{n+1}. The obtained polynomial vanishes on $W_{\lambda_1}(\overline{k(\tau_1)})$ and, therefore, on $\mathcal{W}$. Hence, if one substitutes in $\Phi_{c_1,\lambda_1,L^{(1)}}|_{\{\tau_1=\tau'_1\}}$ the elements $L^{(2)}_j(e_0,\dots,e_r)$ for Z_j, $j\in\{0,\sigma+1,\dots,n\}$, and $e_{2,n+1}$ for Z_{n+1}, then the obtained polynomial vanishes on W_{λ_2}. Therefore, the degree $\deg_{Z_{n+1}}\Phi_{c_1,\lambda_1,L^{(1)}}|_{\{\tau_1=\tau'_1\}}=\deg p'_v\deg W'$. This implies the required assertions immediately. The lemma is proved.

LEMMA 13 *Under conditions of Lemma 12 let* $\gamma\ge0$ *be the most integer such that* $(\tau_{n_1}-\tau'_{n_1})^\gamma$ *divides* $\Phi_{c_1,\lambda_1,L^{(1)},*}$ *(it exists by Lemma 2). Then the polynomial* $(\Phi_{c_1,\lambda_1,L^{(1)},*}/(\tau_{n_1}-\tau'_{n_1})^\gamma)|_{\{\tau_{n_1}=\tau'_{n_1}\}}$ *coincides with* $\Phi_{c_2,\lambda_2,L^{(2)},*}$ *up to a nonzero factor from* $k(\tau_1,\dots,\tau_{n_2},Z_0,Z_n)$. *Suppose additionally that* $\deg_{Z_0,Z_{\sigma+1},\dots,Z_{n+1}}\Phi_{c_2,\lambda_2,L^{(2)}}=\deg_{Z_0,Z_{\sigma+1},\dots,Z_{n+1}}\Phi_{c_1,\lambda_1,L^{(1)}}$. *Then* $\gamma=0$ *and* $\Phi_{c_1,\lambda_1,L^{(1)},*}|_{\{\tau_{n_1}=\tau'_{n_1}\}}$ *coincides with* $\Phi_{c_2,\lambda_2,L^{(2)},*}$ *up to a nonzero factor from* $k(\tau_1,\dots,\tau_{n_2})$.

PROOF Similarly to the proof of Lemma 12 we shall assume without loss of generality that $n_2 = 0$ and $n_1 = 1$. Let $\mathcal{W}_1$ be the union of all the irreducible over $\overline{k}$ components E of $(W \times \mathbb{A}^1(\overline{k})) \cap \mathcal{Z}(L_{1,s+1}, \ldots, L_{1,\sigma}) \cap \mathcal{Z}(e^{(1)}_{\sigma+1}, \ldots, e^{(1)}_{n-1})$ such that E is not contained in any hyperplane $\mathcal{Z}(\tau_1 - z)$, $z \in \overline{k}$ and $E \not\subset \mathcal{Z}(e_0, \ldots, e_r)$. Then an arbitrary polynomial $h \in \overline{k}[X_0, \ldots, X_n, \tau_1]$ homogeneous with respect to $X_0, \ldots, X_n$ vanishes on $W_{\lambda_1}(\overline{k(\tau_1)}) \cap \mathcal{Z}(e^{(1)}_{\sigma+1}, \ldots, e^{(1)}_{n-1}) \setminus \mathcal{Z}(e_0, \ldots, e_r)$ if and only if h vanishes on $\mathcal{W}_1$.

By property 2) from (*) the dimension $\dim(W_{\lambda_j}(\overline{k(\tau_1, \ldots, \tau_{n_j})} \cap \mathcal{Z}(e^{(1)}_{\sigma+1}, \ldots, e^{(1)}_{n-1}) \setminus \mathcal{Z}(e_0, \ldots, e_r)) = 1$, $j = 1, 2$. Hence $\dim \mathcal{W}_1 = 2$. The dimension of every irreducible component of $(W \times \mathbb{A}^1(\overline{k})) \cap \mathcal{Z}(L_{1,s+1}, \ldots, L_{1,\sigma}) \cap \mathcal{Z}(e^{(1)}_{\sigma+1}, \ldots, e^{(1)}_{n-1})$ is at least 2. Therefore, similarly to the proof of Lemma 12 we get $(\mathcal{W}_1 \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(\tau_1 - \tau'_1) = (W_{\lambda_2} \cap \mathcal{Z}(e^{(1)}_{\sigma+1}, \ldots, e^{(1)}_{n-1}) \setminus \mathcal{Z}(e_0, \ldots, e_r)) \times \{\tau'_1\}$.

By Lemma 2 the polynomial $\Phi_{c_1,\lambda_1,L^{(1)},*}/(\tau_1 - \tau'_1)^\gamma$ is nonzero. Let us substitute $Z_0 = L^{(1)}_0(e_0, \ldots, e_r)$, $Z_n = L^{(1)}_n(e_0, \ldots, e_r)$, $Z_{n+1} = e_{1,n+1}$ in $\Phi_{c_1,\lambda_1,L^{(1)},*}/(\tau_1 - \tau'_1)^\gamma$. By Lemma 3 the obtained polynomial vanishes on $W_{\lambda_1}(\overline{k(\tau_1)}) \cap \mathcal{Z}(e^{(1)}_{\sigma+1}, \ldots, e^{(1)}_{n-1}) \setminus \mathcal{Z}(e_0, \ldots, e_r)$ and, therefore, on $\mathcal{W}_1$. Hence if one substitutes $Z_0 = L^{(2)}_0(e_0, \ldots, e_r)$, $Z_n = L^{(2)}_n(e_0, \ldots, e_r)$ in $(\Phi_{c_1,\lambda_1,L^{(1)},*}/(\tau_1 - \tau'_1)^\gamma)|_{\{\tau_1=\tau'_1\}}$ then the obtained polynomial vanishes on $W_{\lambda_2} \cap \mathcal{Z}(e^{(1)}_{\sigma+1}, \ldots, e^{(1)}_{n-1}) \setminus \mathcal{Z}(e_0, \ldots, e_r)$. Therefore, by Lemma 3 the polynomial $\Phi_{c_2,\lambda_2,L^{(2)},+}$ divides $(\Phi_{c_1,\lambda_1,L^{(1)},*}/(\tau_1 - \tau'_1)^\gamma)|_{\{\tau_1=\tau'_1\}}$. By Lemma 2 the degree $\deg_{Z_{n+1}} \Phi_{c_1,\lambda_1,L^{(1)},*}/(\tau_{n_1} - \tau'_{n_1})^\gamma = \deg_{Z_{n+1}} \Phi_{c_2,\lambda_2,L^{(2)},*} = \deg p'_v \deg W'$. This implies the first assertion of the lemma. The second assertion follows from the second one of Lemma 12 immediately. The lemma is proved.

Suppose that the field k is supplied with a real structure (it induces the real structure on $\overline{k}$), see [6]. Let $\varepsilon_i > 0$ be an infinitesimal with respect to the field $k_i = k(\varepsilon_1, \ldots, \varepsilon_{i-1})$ for every integer $i \geq 1$. The field $\overline{k_i}$ is supplied with the real structure, see [6]. The mapping of the standard part $\mathrm{st}_i : \mathbb{P}^n(\overline{k_i}) \to \mathbb{P}^n(\overline{k})$ is defined (we shall use the same notation for all n, it will not lead to an ambiguity), see [6]. Namely, for every $z \in \mathbb{P}^n(\overline{k_i})$ the element $\mathrm{st}_i(z) \in \mathbb{P}^n(\overline{k})$ is infinitely close to z ($\mathrm{st}_i(z)$ is uniquely defined by this property). For every $z \in \mathbb{A}^n(\overline{k_i})$ such that every the coordinate of z is not infinitely large with respect to the field k the element $\mathrm{st}_i(z) \in \mathbb{A}^n(\overline{k})$. In particular, if $z \in \overline{k_i}$ is not infinitely large with respect to the field k then $\mathrm{st}_i(z) \in \overline{k}$.

LEMMA 14 *In the notation from the Introduction suppose that the point $(c', \lambda, L') \in \mathbb{A}^\nu(\overline{k})$ satisfies properties 1) and 2) from (*). Let $i \geq 1$ be an integer. Suppose that $(c'', \lambda', L'') \in \mathbb{A}^\nu(\overline{k_i})$, and $\mathrm{st}_i((c'', \lambda', L'')) = (c', \lambda, L')$. Let us replace in (8)–(14), (7), (10) the elements c', λ, L', and the field k by c'', λ', L'', and the field $\overline{k_i}$ respectively (we do not assume now that (C) is fulfilled for (c'', λ', L'') over any field). Then $\dim W'(c'', \lambda') = \dim W'(c'', \lambda', L'') = n - \sigma$, $\dim W'(c'', \lambda', *) = \dim W'(c'', \lambda', L'', *) = 1$, the morphisms $p_{\lambda', L''}$, $p_{\lambda', L'', *}$ are dominant, and $\mathrm{st}_i(W'(c'', \lambda')) \supset W'(c', \lambda)$, $\mathrm{st}_i(W'(c'', \lambda', L'')) \supset W'(c', \lambda, L')$, $\mathrm{st}_i(W'(c'', \lambda', *)) \supset W'(c', \lambda, *)$, $\mathrm{st}_i(W'(c'', \lambda', L'', *)) \supset W'(c', \lambda, L', *)$.*

PROOF Let $\lambda' = (L'_{s+1}, \ldots, L'_\sigma)$, $L'' = (L''_0, L''_{\sigma+1}, \ldots, L''_n)$, and $e''_j = L''_j(e_0, \ldots, e_r)$ for all j. Properties 1) and 2) for (c', λ, L') imply that (c', λ, L') satisfies (D), see the beginning of the section. Therefore, see the last observation before Lemma 1, the point (c'', λ', L'') also satisfies (D). Therefore, the morphisms

$p_{\lambda',L''}$ and $p_{\lambda',L'',*}$ are dominant.

Therefore, $\dim W'(c'',\lambda') = \dim W'(c'',\lambda',L'') = n-\sigma$, $\dim W'(c'',\lambda',*) = \dim W'(c'',\lambda',L'',*) = 1$. For every $y \in W_{\lambda'}$ which is not infinitely close to $\mathcal{Z}(e'_0, e'_{\sigma+1}, \ldots, e'_n)$ with respect to the field k the equality $p_{c',\lambda,L'}(\mathrm{st}_i(y)) = \mathrm{st}_i(p_{c',\lambda,L'}(y))$ holds. Hence using the transfer principle or directly one obtains that $\mathrm{st}_i(W'(c'',\lambda',L''))$ contains a dense with respect to the Zariski topology subset of $W'(c',\lambda,L')$. By Lemma 5 [13] every irreducible component of $\mathrm{st}_i(W'(c'',\lambda',L''))$ has dimension $n-\sigma$. Therefore, $\mathrm{st}_i(W'(c'',\lambda',L'')) \supset W'(c',\lambda,L')$. The inclusions $\mathrm{st}_i(W'(c'',\lambda')) \supset W'(c',\lambda)$, $\mathrm{st}_i(W'(c'',\lambda',*)) \supset W'(c',\lambda,*)$ and $\mathrm{st}_i(W'(c'',\lambda',L'',*)) \supset W'(c',\lambda,L',*)$ are proved in the similar way. The lemma is proved.

2 Some facts from linear algebra

By definition in any characteristic of the ground field there exists only one (0×0)–matrix, and its determinant is 1. Further, for every integer $m_1 \geq 0$ there is only one $(m_1 \times 0)$–matrix (respectively $(0\times m_1)$–matrix). Let $m_2 \geq 0$ be an integer. If A_1 is an $(m_1\times m_2)$–matrix and A_2 is the $(m_2\times 0)$–matrix then A_1A_2 is the $(m_1\times 0)$–matrix. If A_1 is the $(0\times m_1)$–matrix and A_2 is an $(m_1\times m_2)$–matrix then A_1A_2 is the $(0\times m_2)$–matrix.

According to [14] for every integers $m_1 \geq m_2 \geq 0$ one can construct within the time polynomial in m_1 a family of $((m_1-m_2)\times m_1)$–matrices A_i, $1 \leq i \leq m_2(m_1-m_2)+1$ (they depend on m_1 and m_2) satisfying the following conditions.

- All the matrices A_i have integer coefficients of length $O(m_1)$ (here and below the constant of $O(\ldots)$ is absolute) and the rank $\mathrm{rank}(A_i) = m_1 - m_2$.
- For every $(m_2\times m_1)$–matrix A with coefficients from a field of zero–characteristic (actually in [13] the case of arbitrary characteristic is considered) with $\mathrm{rank}(A) = m_2$ there exists $1 \leq i \leq m_2(m_1-m_2)+1$ such that the determinant
$$\det \begin{pmatrix} A \\ A_i \end{pmatrix} \neq 0.$$

Hence one can construct within the time polynomial in m_1 an $(m_1\times m_1)$–matrix B_i with integer coefficients of length $O(m_1)$ such that $A_iB_i = (0,\Delta_i)$ where Δ_i is a diagonal $(m_1-m_2)\times(m_1-m_2)$–matrix, all the nonzero coefficients of Δ_i coincide, and the determinant $\det\Delta_i \neq 0$. Let us represent $B_i = (C_i, C'_i)$ where C_i (respectively C'_i) is the matrix with m_2 (respectively m_1-m_2) columns. We shall denote also A_i, B_i, C_i C'_i by $A_{m_1,m_2,i}$, $B_{m_1,m_2,i}$, $C_{m_1,m_2,i}$ $C'_{m_1,m_2,i}$ respectively if the dependence on m_1, m_2 is important.

LEMMA 15 *Let m_1, m_2, m_3 be arbitrary integers such that $m_1 \geq m_3$, $m_2 \geq m_3$, $m_3 \geq 0$. Let $A = (a_{i,j})$ be a $(m_2\times m_1)$-matrix with coefficients from $\overline{k_\alpha}$ where the field $k_\alpha = k(\varepsilon_1,\ldots,\varepsilon_\alpha)$, see Section 1, $\alpha \geq 0$. Suppose that* $\mathrm{rank}(A) = m_3$. *Let Δ be an $(m_3\times m_3)$–minor of A (here it is up to a sign the determinant of the matrix (a_{i_v,j_w}) for some integers $1 \leq i_1 < \ldots < i_{m_3} \leq m_2$, $1 \leq j_1 < \ldots < j_{m_3} \leq m_3$) such that $\Delta \neq 0$ and for any other $(m_3\times m_3)$–minor Δ_1 of A the element Δ_1/Δ is not infinitely large with respect to the field k. Then the following assertions hold*

(i) There are integers i, j such that $1 \leq j \leq m_3(m_1 - m_3) + 1$, $1 \leq i \leq m_3(m_2 - m_3) + 1$, the determinant $\Delta_{i,j} = \det(C^T_{m_2,m_3,i} A C_{m_1,m_3,j}) \neq 0$ (here T denotes transposition of matrices), and $\Delta/\Delta_{i,j}$ is not infinitely large with respect to the field k (hence the standard part $\mathrm{st}_\alpha(\Delta/\Delta_{i,j})$ is a nonzero element of $\overline{k}$).

(ii) Suppose that $m_2 = m_3$. Then there is $1 \leq j \leq m_3(m_1 - m_3) + 1$ such that the determinant $\Delta_j = \det(AC_{m_1,m_3,j}) \neq 0$ and Δ/Δ_j is not infinitely large with respect to the field k. Hence the standard part $\mathrm{st}_\alpha(\Delta/\Delta_j)$ is a nonzero element of $\overline{k}$.

(iii) Suppose that $m_2 = m_3$. Then there exists an $(m_1 \times m_1)$–matrix $G = (g_{i,j})$ such that every the coefficient $g_{i,j} \in \overline{k_\alpha}$ is not infinitely large with respect to the field k; $\det(G) \neq 0$ and $\det(G)$ is not an infinitesimal with respect to k; the matrix $AG = (A', 0)$, where A' has m_3 columns. The determinant $\det(A') \neq 0$, and $\Delta/\det(A')$ is not infinitely large with respect to the field k.

PROOF We shall assume without loss of generality that $m_3 \geq 1$. Let us show that (ii) implies (i). Indeed denote by A''' the submatrix of A constituted from the rows with induces $i_1, \ldots, i_{m_3}$. Let us apply (ii) to A'''. Then $A'''C_{m_1,m_3,j}$ is a submatrix of $AC_{m_1,m_3,j}$, the determinant $\det(A'''C_{m_1,m_3,j}) \neq 0$, and $\Delta/\det(A'''C_{m_1,m_3,j})$ is not infinitely large with respect to the field k. So to prove (i) it is sufficient to apply (ii) for the second time to the matrix $(AC_{m_1,m_3,j})^T$. The required assertion is proved.

Let us prove (iii). Notice that, see [6], for all $z_1, z_2 \in \overline{k_\alpha}$ such that $z_1 \neq 0$ and z_2/z_1 is not infinitely large with respect to the field k there is $c \in \overline{k}$ such that $(z_2 - cz_1)/z_1$ is an infinitesimal with respect to the field k. Set $Q = A$ as the base of the recursion. We shall describe a recursion by an $(m_3 \times m_1)$–matrix $Q = (q_{i,j})$ with coefficients from $\overline{k_\alpha}$. At each step of this recursion we use the elementary transformations of columns of Q over the field $\overline{k_\alpha}$ and permutations of rows (the permutations of rows are used only for convenience of notation) such that all the coefficients of the matrices corresponding to these transformations of columns are not infinitely large with respect to the field k. At the end of the recursion we shall obtain a matrix Q which has the form $(A', 0)$. This is sufficient for the proof of assertion (iii).

Let $0 \leq i < m_3$ be an integer. The recursive assumption is that at the beginning of $(i+1)$–th step

$$Q = Q_i = \begin{pmatrix} Q'_i, & 0 \\ Q''_i & Q'''_i \end{pmatrix}, \tag{28}$$

where $Q'_i = (q'_{i,j})$ is a $(i \times i)$–matrix, coefficients $q'_{1,1}, \ldots, q'_{i,i}$ are nonzero, $q'_{i,j} = 0$ if $j > i$, and for all j, j_1, i_1 satisfying the inequalities $0 \leq j \leq i$, $j \leq j_1 \leq m_1$, $1 \leq i_1 \leq m_2$ the quotient $q_{i_1,j_1}/q_{j,j}$ is not infinitely large with respect to the field k. The matrix $Q'''_i \neq 0$.

Let us describe the general step of the recursion. Let us choose a nonzero coefficient z of the matrix Q'''_i such that for any other coefficient z' of Q'''_i the element z'/z is not infinitely large with respect to the field k. Using the permutations of rows and columns we shall suppose without loss of generality that $z = q_{i+1,i+1}$. Now by an appropriate elementary transformation of columns we

transfer Q_i''' to $\widetilde{Q}_i$ where the first row of $\widetilde{Q}_i$ is $(q_{i+1,i+1}, 0, \dots, 0)$. By definition the image the matrix Q_i after all these transformations is Q_{i+1} (to define Q'_{i+1}, Q''_{i+1}, Q'''_{i+1} one need to replace i by $i+1$ in (28)). Since rank rank$(A) = m_3$ the matrix $Q'''_{i+1} \neq 0$ or $i+1 = m_3$. Thus, if $i+1 < m_3$ we proceed to the next step of the recursion. If $i+1 = m_3$ we set $A' = Q'_{m_3}$. The existence of G and A' is proved. We have $A = (A', 0)G^{-1}$. Hence the minor $\Delta = \det(A')\det(G')$ for an $(m_3 \times m_3)$–submatrix G' of G^{-1}. This implies the last assertion. Thus, (iii) is proved.

Let us prove (ii). We shall use the notation from the proof of (iii). Let $G_0 = \mathrm{st}_\alpha(G) = (\mathrm{st}_\alpha(g_{i,j}))$ be the standard part of the matrix G (it is matrix with coefficients from $\overline{k}$). Hence by (iii) the determinant $\det(\mathrm{st}_\alpha(G)) \neq 0$. By (iii) and the formula for the adjoint matrix $(E+H)G = G_0$ where E is identity matrix, and all the nonzero coefficients of H are elements from $\overline{k_\alpha}$ which are infinitely small with respect to the field k. There is $1 \leq i \leq m_3(m_1 - m_3) + 1$ such that

$$\det \begin{pmatrix} (A', 0)G_0^{-1} \\ A_{m_1,m_3,i} \end{pmatrix} = c\det(A'),$$

where $0 \neq c \in \overline{k}$. Then

$$\det \begin{pmatrix} A \\ A_{m_1,m_3,i} \end{pmatrix} = \det \begin{pmatrix} (A', 0)G_0^{-1}(E+H) \\ A_{m_1,m_3,i} \end{pmatrix} = c_1\det(A'),$$

where $0 \neq c_1 \in \overline{k_\alpha}$, and c_1, c_1^{-1} are not infinitely large with respect to the field k. Therefore, $\det(AC_{m_1,m_3,i}) = c_2\det(A')$, where $0 \neq c_2 \in \overline{k_\alpha}$, and c_2, c_2^{-1} are not infinitely large with respect to the field k. By (iii) $\Delta/\det(A')$ is not infinitely large with respect to the field k. This implies (ii). The lemma is proved.

Let $(c_1, \lambda_1, L^{(1)})$ be given by (17)–(19), see the Introduction. Now we shall suppose without loss of generality that n_1 is minimal possible and hence $n_1 \leq \nu$, see the Introduction, is bounded from above by a polynomial in $rd(d'+1)^n$, see the Introduction. Consider systems of polynomial equations and inequalities with squares of absolute values containing generic homogeneous polynomials. Recall that in [13] a method for solving such systems is suggested (only generic linear forms occur in [13]). Now we are going to strengthen this method for the case of a system satisfying the following conditions.

1) The coefficients of all the equalities and inequalities of this system are elements from a finite extension of some $k_\alpha = k(\varepsilon_1, \dots, \varepsilon_\alpha)$, see Section 1, where $\alpha = O(1)$.

2) The indeterminates of this system are $\tau_1, \dots, \tau_{n_1}$; $X_{i,j}$, $0 \leq j \leq n$, $1 \leq i \leq \nu_2$, where $\nu_2 = O(1)$ (is considered as a constant, $\nu_2 < n$); and some other $O(n)$ variables. Denote them, say, by Y_γ, $\gamma \in \Gamma$.

3) This system contains the inequality $\sum_{1 \leq i \leq n_1} |\tau_i|^2 \leq \varepsilon_1$. All other equations and inequalities of this system have coefficients from a finite extension k' of $k(\varepsilon_2, \dots, \varepsilon_\alpha)$.

4) Every equation or inequality of this system, apart of $\sum_{1 \leq i \leq n_1} |\tau_i|^2 \leq \varepsilon_1$, is equivalent to $A \geq 0$ or $A = 0$ where A belongs to the least class of functions satisfying the following properties. This class is closed under the

operations of the addition of two functions, multiplication of two functions, multiplication of a function to a scalar from $\overline{k(\varepsilon_2,\dots,\varepsilon_\alpha)}$. The function $B \mapsto |B|^2$ belongs to this class. Finally, this class contains all the scalars from $\overline{k(\varepsilon_2,\dots,\varepsilon_\alpha)}$; the functions $X_{i,j}$, $0 \le j \le n$, $1 \le i \le \nu_2$; $e_{1,n+1}(X_{i,0},\dots,X_{i,n})$, see (20); $L_{1,j}(X_{i,0},\dots,X_{i,n})$, $s+1 \le j \le \sigma$; $e_w^{(1)}(X_{i,0},\dots,X_{i,n})$, $w \in \{0,\sigma+1,\dots,n\}$, where $1 \le i \le \nu_2$; and Y_γ, $\gamma \in \Gamma$.

Denote the considered system by $(\star)$ for brevity.

Now suppose that for every $1 \le i \le \nu_2$ the algebraic varieties $V^{(i)}$, $W^{(i)}$ are similar (and given in the analogous way) to V, W from the Introduction. We are interested whether system $(\star)$ has a solution $\tau_w = \tau_w^*$, $X_{i,j} = x_{i,j}^*$, $Y_\gamma = y_\gamma^*$ such that all τ_w^*, $x_{i,j}^*$, y_γ^* are elements from $\overline{k_\alpha}$, and

$$(x_{i,0}^* : \dots : x_{i,n}^*) \in W^{(i)}(\overline{k_\alpha}), \quad 1 \le i \le \nu_2. \tag{29}$$

The total number of variables of system $(\star)$ may be $\ge \mathcal{P}(rd(d'+1)^n)$ for a polynomial $\mathcal{P}$. Hence applying the general algorithms for constructing the described solution of such systems, see [6] and Theorem 2 [12], we get an estimate polynomial in $(d')^{\mathcal{P}(rd(d'+1)^n)}$ for the working time of the decision algorithm and the algorithm for constructing a solution of system $(\star)$. But the suggested method allows to reduce the working time.

Namely, let $1 \le n_3 \le \min\{n_1, \nu_2(n-s+2)\}$ be an integer. Let $B_{n_1,n_3,j}$, $C_{n_1,n_3,j}$, $1 \le j \le n_3(n_1-n_3)+1$, be the matrices defined at the beginning of the section. Put

$$\begin{pmatrix} T_{1,n_3,j} \\ \vdots \\ T_{n_1,n_3,j} \end{pmatrix} = B_{n_1,n_3,j}^{-1} \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_{n_1} \end{pmatrix}.$$

For an arbitrary linear form $H = \sum_{1 \le i \le n_1} h_i \tau_i$ with respect to $\tau_1,\dots,\tau_{n_1}$ represent $H = \sum_{1 \le i \le n_1} h_{i,n_3,j} \times T_{i,n_3,j}$ (hence all $h_{i,n_3,j}$ are linear combinations of h_i with integer coefficients). By definition set

$$H_{n_3,j} = \sum_{1 \le i \le n_3} h_{i,n_3,j} T_{i,n_3,j}.$$

Thus, $e_{1,n+1,n_3,j}$; $L_{1,w_1,n_3,j}$, $s+1 \le w_1 \le \sigma$; $L_{w_2,n_3,j}^{(1)}$, $w_2 \in \{0,\sigma+1,\dots,n\}$, are given.

Let us replace in system $(\star)$ in all the equations and inequalities $e_{1,n+1}$; L_{1,w_1}; $e_{w_2}^{(1)}$ by $e_{1,n+1,n_3,j}$; $L_{1,w_1,n_3,j}$, $s+1 \le w_1 \le \sigma$; $e_{w_2,n_3,j}^{(1)}$, $w_2 \in \{0,\sigma+1,\dots,n\}$ respectively. Let us replace the inequality $\sum_{1 \le i \le n_1} |\tau_i|^2 \le \varepsilon_1$ by $\sum_{1 \le i \le n_3} |T_{i,n_3,j}|^2 \le \varepsilon_1$. Let us replace the indeterminates $\tau_1,\dots,\tau_{n_1}$ of system $(\star)$ by $T_{1,n_3,j},\dots,T_{n_3,n_3,j}$. Denote the obtained system by $(\star,n_3,j)$. Hence system $(\star,n_3,j)$ has $O(n)$ indeterminates $T_{1,n_3,j},\dots,T_{n_3,n_3,j}$; $X_{i,j}$, $0 \le j \le n$, $1 \le i \le \nu_2$; Y_γ, $\gamma \in \Gamma$.

Similarly to system $(\star)$ we shall consider a solution $T_{i,n_3,j} = T'_{i,n_3,j}$, $X_{i,j} = x_{i,j}^*$, $Y_\gamma = y_\gamma^*$ such that all $T'_{i,n_3,j}$, $x_{i,j}^*$, y_γ^* are elements from $\overline{k_\alpha}$ and (29) hold.

LEMMA 16 *System $(\star)$ has a solution over the field $\overline{k_\alpha}$ satisfying (29) if and only if there exist $0 \le n_3 \le \min\{n_1, \nu_2(n-s+2)\}$ and $1 \le j \le n_3(n_1 -$*

$n_3)+1$ *such that system* $(\star, n_3, j)$ *has a solution over the field* $\overline{k_\alpha}$ *satisfying (29). Besides that, using the proof of the lemma for a given solution of system* $(\star, n_3, j)$ *over the field* $\overline{k_\alpha}$ *satisfying (29) one can construct explicitly the solution of system* $(\star)$ *over the field* $\overline{k_\alpha}$ *satisfying (29).*

PROOF Suppose that system $(\star, n_3, j)$ has a solution over the field $\overline{k_\alpha}$ satisfying (29) for some $0 \le n_3 \le \min\{n_1, \nu_2(n-s+2)\}$ and $1 \le j \le n_3(n_1-n_3)+1$. Set $T'_{i,n_3,j}=0$ for all $n_3+1 \le i \le n_1$. Then the linear system $T_{i,n_3,j}=T'_{i,n_3,j}$, $1 \le i \le n_1$, with respect to $\tau_1, \dots, \tau_{n_1}$ has a solution $\tau_i=\tau_i^* \in \overline{k(\varepsilon_2, \dots, \varepsilon_\alpha)}$, $1 \le i \le n_1$. Hence $\sum_{1 \le i \le n_1} |\tau_i|^2 \le c\varepsilon_1$ for an integer $c \ne 0$ with length $O(n_1)$. Now to get a solution of system $(\star)$ satisfying (29) it is sufficient to perform the replacement $\varepsilon_1 \mapsto c^{-1}\varepsilon_1$. Thus, system $(\star)$ has a solution over the field $\overline{k_\alpha}$ satisfying (29).

Conversely, suppose that system $(\star)$ has a solution satisfying (29). By 3) and the transfer principle , see [3], one can suppose without loss of generality that all τ_w^*, $x_{i,j}^*$, y_γ^* are elements from $\overline{k(\varepsilon_2, \dots, \varepsilon_\alpha)}$. Set $e_{1,n+1}(x_{i,0}^*, \dots, x_{i,n}^*) = e_{1,n+1,i}^*$, $L_{1,j}(x_{i,0}^*, \dots, x_{i,n}^*) = L_{1,j,i}^*$, $e_w^{(1)}(x_{i,0}^*, \dots, x_{i,n}^*) = e_{w,i}^{(1,*)}$ for all j, w, i. Hence $e_{1,n+1,i}^*$, $L_{1,j,i}^*$, $e_{w,i}^{(1,*)}$ are linear forms in $\tau_1, \dots, \tau_{n_1}$ with coefficients from $\overline{k(\varepsilon_2, \dots, \varepsilon_n)}$. Let A be the $((n-s+2)\nu_2 \times n_1)$–matrix constituted of the coefficients from $\overline{k(\varepsilon_2, \dots, \varepsilon_n)}$ of the linear forms $e_{1,n+1,i}^*$, $1 \le i \le \nu_2$; $L_{1,j,i}^*$, $s+1 \le j \le \sigma$, $1 \le i \le \nu_2$; $e_{w,i}^{(1,*)}$, $w \in \{0, \sigma+1, \dots, n\}$, $1 \le i \le \nu_2$. Set $n_3 = \mathrm{rank}(A)$. Hence $0 \le n_3 \le \min\{n_1, \nu_2(n-s+2)\}$.

Let us apply Lemma 15 (i) to the matrix A with $m_1=n_1$, $m_2=\nu_2(n-s+2)$, $m_3=n_3$. Thus the matrices $B_{n_1,n_3,j}$ and $C_{n_1,n_3,j}$ are defined by this lemma. In what follows j is fixed in the proof.

Set $T_{i,n_3,j}^* = T_{i,n_3,j}(\tau_1^*, \dots, \tau_{n_1}^*)$ for all $1 \le i \le n_1$. Let us define linear forms $\widetilde{e}_{1,n+1,i}$, $\widetilde{L}_{1,w_1,i}$, $\widetilde{e}_{w_2,i}^{(1)}$ by the equalities $\widetilde{e}_{1,n+1,i}(T_{1,n_3,j}, \dots, T_{n_1,n_3,j}) = e_{1,n+1,i}^*$, $\widetilde{L}_{1,w_1,i}(T_{1,n_3,j}, \dots, T_{n_1,n_3,j}) = L_{1,w_1,i}^*$, $\widetilde{e}_{w_2,i}^{(1)}(T_{1,n_3,j}, \dots, T_{n_1,n_3,j}) = e_{w_2,i}^{(1,*)}$. Put $e_{1,n+1,i}^*(\tau_1^*, \dots, \tau_{n_1}^*) = l_{1,n+1,i}^*$, $L_{1,w_1,i}^*(\tau_1^*, \dots, \tau_{n_1}^*) = l_{1,w_1,i}^*$, $e_{w_2,i}^{(1,*)}(\tau_1^*, \dots, \tau_{n_1}^*) = l_{w_2,i}^{(1,*)}$ for all w_1, w_2, i.

Consider the linear system relatively to the variables $T_{1,n_3,j}, \dots, T_{n_1,n_3,j}$

$$\begin{cases} \widetilde{e}_{1,n+1,i}(T_{1,n_3,j}, \dots, T_{n_1,n_3,j}) = l_{1,n+1,i}^*, & \\ \widetilde{L}_{1,w_1,i}(T_{1,n_3,j}, \dots, T_{n_1,n_3,j}) = l_{1,w_1,i}^*, & s+1 \le w_1 \le \sigma,\ 1 \le i \le \nu_2, \\ \widetilde{e}_{w_2,i}^{(1)}(T_{1,n_3,j}, \dots, T_{n_1,n_3,j}) = l_{w_2,i}^{(1,*)}, & w_2 \in \{0, \sigma+1, \dots, n\},\ 1 \le i \le \nu_2. \end{cases} \tag{30}$$

System (30) has a solution $T_{i,n_3,j}=T_{i,n_3,j}^*$, $1 \le i \le n_1$. Let us represent the matrix $AB_{n_1,n_3,j}=(A', A'')$ where A' has n_3 columns. By Lemma 15 we have $A''=A'Z$ where $Z=(z_{i,w})$, and every the coefficient $z_{i,w} \in \overline{k(\varepsilon_2, \dots, \varepsilon_\alpha)}$ is not infinitely large with respect to the field k. Therefore, there is also a solution of linear system (30) $T_{i,n_3,j}=T'_{i,n_3,j}$, $1 \le i \le n_3$, and $T_{i,n_3,j}=0$, $n_3+1 \le i \le n_1$, where all $T'_{i,n_3,j} = T_{i,n_3,j}^* + \sum_{n_3+1 \le w \le n_1} z_{i,w} T_{w,n_3,j}^*$. Hence system $(\star, n_3, j)$ has a solution over the field $\overline{k_\alpha}$ satisfying (29). The lemma is proved.

REMARK 11 *In what follows applying Lemma 16 for the considered* j *and* n_3 *we shall decide whether system* $(\star, n_3, j)$ *has a solution and construct this solution if it exists using Theorem 2 [12] and Theorem 2 [11].*

3 A decision algorithm for property (*)

In this section a point $(c_1, \lambda_1, L^{(1)})$ satisfying (**) is given, see the Introduction. The following observations will be used in the algorithm for assertion (e) of Theorem 2 described in this section and Section 6. Suppose that $k_i \subset K$ for some i

- Suppose that $(c_4, \lambda_4, L^{(4)}) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}} \otimes \overline{k_i}$ satisfies conditions of Lemma 1 (in place of $(c_0, \lambda_0, L^{(0)})$). Then using the algorithms from [5], [11] one can compute the number of elements $\#(e_{4,n+1}/e_0^{(4)})(A_{\lambda_4,L^{(4)}})$. In particular, this can be done if conditions 1), 2) from (*) hold for $(c_4, \lambda_4, L^{(4)})$.

- Suppose that $(c_4, \lambda_4, L^{(4)}) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}} \otimes \overline{k_i}$ satisfies conditions of Lemma 2 (in place of $(c_0, \lambda_0, L^{(0)})$). Then using the algorithm from [12] one can compute $\deg W'(c_4, \lambda_4, L^{(4)})$. In particular, this can be done if conditions 1)–3) from (*) hold for $(c_4, \lambda_4, L^{(4)})$.

- Suppose that $(c_4, \lambda_4, L^{(4)}) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}} \otimes \overline{k_i}$ satisfies conditions of Lemma 2 (in particular, this is true if conditions 1)–3) from (*) hold for $(c_4, \lambda_4, L^{(4)})$), and $\deg W'(c_4, \lambda_4, L^{(4)}) = \deg W'(c_1, \lambda_1, L^{(1)})$. Then, see Definition 4 and Lemma 3, by the algorithms from [5], [11], [12] one can compute $W'(c_4, \lambda_4, L^{(4)}, *)$ and after that $\deg W'(c_4, \lambda_4, L^{(4)}, *)$.

- Suppose that $(c_4, \lambda_4, L^{(4)}) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}} \otimes \overline{k_i}$ satisfies conditions of Lemma 2 and $\deg W'(c_4, \lambda_4, L^{(4)}) = \deg W'(c_1, \lambda_1, L^{(1)}) = \deg W'(c_4, \lambda_4, L^{(4)}, *) = \deg W'(c_1, \lambda_1, L^{(1)}, *)$ (in particular, it is true if property (*) hold for $(c_4, \lambda_4, L^{(4)})$ and $\deg W'(c_4, \lambda_4, L^{(4)}) = \deg W'(c_1, \lambda_1, L^{(1)})$, $\deg W'(c_4, \lambda_4, L^{(4)}, *) = \deg W'(c_1, \lambda_1, L^{(1)}, *)$). Then $\Phi_{c_4,\lambda_4,L^{(4)},*} = \Phi_{c_4,\lambda_4,L^{(4)},+}$ and, therefore, by the algorithms from [5], [11], [12] one can compute the polynomials $\Phi_{c_4,\lambda_4,L^{(4)},*}$, $R_{c_4,\lambda_4,L^{(4)},*}$, and $R_{c_4,\lambda_4,L^{(4)},*,0}$. Hence also one can find the integer $a(c_4, \lambda_4, L^{(4)})$.

We shall use this observations in the algorithm for assertion (e) of Theorem 2 described in this and the next sections.

Assume that property (**) is satisfied, see the Introduction. At first our aim is to decide whether 1)–3) from (*) hold for $(c_1, \lambda_1, L^{(1)})$. If 1)–3) are fulfilled for $(c_1, \lambda_1, L^{(1)})$, we shall construct $(c_5, \lambda_5, L^{(5)})$ similar to $(c_3, \lambda_3, L^{(3)})$ and such that conditions 1)–3) from (*) hold for $(c_5, \lambda_5, L^{(5)})$. In what follows in this and next sections we shall assume that $c_5, \lambda_5, L^{(5)}$ are obtained by substituting in (17)–(19) values $\tau_i = \tau_{5,i}$, $1 \le i \le n_1$.

We describe a recursive algorithm by $(c_5, \lambda_5, L^{(5)})$. As the base of the recursion set $(c_5, \lambda_5, L^{(5)}) = (c_3, \lambda_3, L^{(3)})$. Let us describe the general step of this recursion. Let us compute $\deg p'_v \deg W'$ points from the set $S = A_{\lambda_5,L^{(5)}} = (W_{\lambda_5} \setminus \mathcal{Z}(e_0, \dots, e_r)) \cap \mathcal{Z}(e_{\sigma+1}^{(5)}, \dots, e_n^{(5)})$. Properties 1) and 2) from (*) for $(c_5, \lambda_5, L^{(5)})$ imply the transversality of intersection (22) with $j = 5$, see Lemma 17 (ii) [13]. Let us compute the number of elements $\#(e_{5,n+1}/e_0^{(5)})(S)$. If $\#(e_{5,n+1}/e_0^{(5)})(S) < \deg p'_v \deg W'$ then we construct a pair of distinct points $x_1, x_2 \in S$ such that $(e_{5,n+1}/e_0^{(5)})(x_1) \ne (e_{5,n+1}/e_0^{(5)})(x_2)$. Let us construct a linear form $L_{1,0} \in k[X_0, \dots, X_n]$ with integer coefficients of length $O(1)$ such that $L_{1,0}(x_j) \ne 0$ for $j = 1, 2$. Denote for brevity $X^{(j)} = (X_{j,0}, \dots, X_{j,n})$, $j =$

$1, 2$, and for an arbitrary polynomial p in $X_0, \ldots, X_n$ set $p(X^{(j)}) = p(X_{j,0}, \ldots, X_{j,n})$. Consider the system of polynomial equations and inequalities with respect to the indeterminates $X_{j,0}, \ldots, X_{j,n}$, $j = 1, 2$; $\tau_1, \ldots, \tau_{n_1}$ with coefficients from the field $\overline{k_3}$

$$\begin{cases} \sum_{1\le i\le n_1} |\tau_i|^2 \le \varepsilon_1, \\ \sum_{0\le i\le n} |X_{j,i} - (X_i/L_{1,0})(x_j)L_{1,0}|^2 \le \varepsilon_2, & j = 1, 2, \\ \sum_{0\le i\le n} |(e_{5,n+1} - e_{0,n+1} + e_{1,n+1})(X^{(1)}) - \\ (e_{5,n+1} - e_{0,n+1} + e_{1,n+1})(X^{(2)})|^2 \ge \varepsilon_3, \\ (L_{5,i} - L_{0,i} + L_{1,i})(X^{(j)}) = 0, & s+1 \le i \le \sigma,\ j = 1, 2, \\ (e_w^{(5)} - e_w^{(0)} + e_w^{(1)})(X^{(j)}) = 0, & \sigma+1 \le w \le n,\ j = 1, 2, \\ (e_0^{(5)} - e_0^{(0)} + e_0^{(1)})(X^{(j)}) = 1, & j = 1, 2. \end{cases} \tag{31}$$

LEMMA 17 *If $(c_1, \lambda_1, L^{(1)})$ satisfies conditions 1)–3) from (*) then system (31) has a solution $X_{j,w} = x^*_{j,w}$, $\tau_i = \tau_i^*$ from $\overline{k_3}$ such that*

$$(x^*_{j,0} : \ldots : x^*_{j,n}) \in W(\overline{k_3}), \quad j = 1, 2. \tag{32}$$

PROOF There is an element $\varepsilon' > 0$ such that ε' is an infinitesimal with respect to the field k_2 and ε_3 is an infinitesimal with respect to the field $k_2(\varepsilon')$. By the transfer principle, see [3], it is sufficient to construct a solution of system (31) with $\overline{k(\varepsilon')}$ in place of $\overline{k_3}$ in (32). Set in Lemma 4 the element $(c_0, \lambda_0, L^{(0)})$ to be the generic point $(c_1, \lambda_1, L^{(1)})$ of $\mathcal{L}_{c_1,\lambda_1,L^{(1)}}$ (one need to extend here the ground field k till $k(\tau_1, \ldots, \tau_{n_1})$). Subsequently applying n_1 times Lemma 4 we get a point $(c_3, \lambda_3, L^{(3)}) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}}$ satisfying conditions 1)–3) from (*) and such that it is obtained from $(c_1, \lambda_1, L^{(1)})$ by the substitution integers τ_i' for τ_i in (17)–(19). Consider the line $\mathfrak{l}$ containing $(c_5, \lambda_5, L^{(5)})$ and $(c_3, \lambda_3, L^{(3)})$. Let us choose a point $(c_*, \lambda_*, L^*) \in \mathfrak{l}(\overline{k(\varepsilon')})$ which is infinitely close to $(c_5, \lambda_5, L^{(5)})$. Let $\tau_i = \tau_i^*$, $1 \le i \le n_1$, corresponds to (c_*, λ_*, L^*). Then obviously for the considered τ_i^*, $1 \le i \le n_1$, there is a solution (32) of system (31) with $\overline{k(\varepsilon')}$ in place of $\overline{k_3}$. The lemma is proved.

Notice that system (31) satisfies conditions 1)–4) from Section 2. Hence using Lemma 16 and Remark 11 we decide whether system (31) has a solution satisfying (32), and if it exists we construct such a solution. Suppose that such a solution does not exist. Then by Lemma 17 the point $(c_1, \lambda_1, L^{(1)})$ does not satisfy 3), and hence (*) does not hold.

Suppose that we have constructed a solution of system (31) satisfying (32). Substituting in (17)–(19) and (20) $\tau_{5,i} + \tau_i^*$ for τ_i, $1 \le i \le n_1$, we get a point (c_*, λ_*, L^*) and the polynomial e^*_{n+1} with coefficients from a subfield k' of $\overline{k_3}$, where the extension $k' \supset k$ is finitely generated. We can suppose without loss of generality that $k' \subset \overline{K}$. We have $L^{(5)} \in \mathcal{U}_0''$, $A_{\lambda_5,L^{(5)}} \cap \mathcal{Z}(e_0^{(5)}) = \emptyset$ and $A_{\lambda_5,L^{(5)}} \cap \overline{V \setminus W} = \emptyset$ by 1) and 2). Hence $L^* \in \mathcal{U}_0''$ (where $\mathcal{U}_0''$ is defined for the ground field $\overline{k'}$, see Remark 2 [13]), $A_{\lambda_*,L^*} \cap \mathcal{Z}(e_0^*) = \emptyset$ and $A_{\lambda_*,L^*} \cap \overline{V \setminus W} = \emptyset$. Since intersection (22) is transversal for $j = 5$, the implicit function theorem implies that intersection (22) is transversal with λ_*, L^* in place of λ_j, $L^{(j)}$, and $\#A_{\lambda_*,L^*} = \deg p'_v \deg W'$. We have $\#(e^*_{n+1}/e_0^*)(A_{\lambda_*,L^*}) > \#(e_{5,n+1}/e_0^{(5)})(A_{\lambda_5,L^{(5)}})$. Now conditions of Lemma 4 hold for (c_*, λ_*, L^*) in place of $(c_0, \lambda_0, L^{(0)})$.

By Lemma 4 and the observations at the beginning of the section one can apply the algorithm of reduction to integer coefficients, see [12], Proposition 2, and replace subsequently all values τ_i^* by integers τ_i' of length $O(n\log d+(n-\sigma)\log(d'))$. Denote by (c',λ,L') the obtained point. Then $L'\in\mathcal{U}_0''$, the number of elements $\#A_{\lambda,L'}=\deg p_v'\deg W'$, $A_{\lambda,L'}\cap\mathcal{Z}(e_0')=\emptyset$, and $A_{\lambda,L'}\cap\overline{V\setminus W}=\emptyset$, the dimensions $\dim W_\lambda=n-\sigma$, $\dim(W_\lambda\setminus\mathcal{Z}(e_0,\dots,e_r))\cap\mathcal{Z}(e'_{\sigma+1},\dots,e'_{n-1})=1$, the number of elements $\#(e'_{n+1}/e'_0)(A_{\lambda,L'})\geq\#(e^*_{n+1}/e^*_0)(A_{\lambda_*,L^*})>$ $\#(e_{5,n+1}/e_0^{(5)})(A_{\lambda_5,L^{(5)}})$, the morphisms $p_{\lambda,L'}$, $p_{\lambda,L',*}$ are dominant. Let us apply Lemma 8 to the case when $\iota(\mathbb{A}^1(\overline{k}))$ is the line containing (c',λ,L') and $(c_5,\lambda_5,L^{(5)})$, and $\iota(0)=(c_5,\lambda_5,L^{(5)})$. Then using Theorem 2 (f) [13], Lemma 8 and Lemma 4 we construct a point (c'',λ',L'') similar to $(c_5,\lambda_5,L^{(5)})$ and such that the number of elements $\#(e''_{n+1}/e''_0)(A_{\lambda',L''})\geq\#(e'_{n+1}/e'_0)(A_{\lambda,L'})>$ $\#(e_{5,n+1}/e_0^{(5)})(A_{\lambda_5,L^{(5)}})$. After that we replace $(c_5,\lambda_5,L^{(5)})$ by (c'',λ',L'') and proceed to the next step of the recursion. Thus, finally we shall come to the case when the number of elements $\#(e_{5,n+1}/e_0^{(5)})(S)=\deg p_v'\deg W'$. Then $(c_1,\lambda_1,L^{(1)})$ and $(c_5,\lambda_5,L^{(5)})$ satisfies conditions 1)–3). This step of the described recursion is final.

In what follows we shall assume without loss of generality that $(c_1,\lambda_1,L^{(1)})$ satisfies 3). Now we shall describe a new recursion. As the base of the recursion we take the constructed point $(c_5,\lambda_5,L^{(5)})$. At the end of this recursion we shall obtain $(c_5,\lambda_5,L^{(5)})$ such that conditions 1)–3) from (*) hold for $(c_5,\lambda_5,L^{(5)})$ and $\deg W'(c_5,\lambda_5,L^{(5)})=\deg W'(c_1,\lambda_1,L^{(1)})$.

Let us describe the general step of this recursion. At the beginning of the step a point $(c_5,\lambda_5,L^{(5)})$ satisfying conditions 1)–3) from (*) is given. Let us define $r_{c_5,\lambda_5,L^{(5)}}$, $p_{c_5,\lambda_5,L^{(5)}}$, $q_{c_5,\lambda_5,L^{(5)}}$, $p_{\lambda_5,L^{(5)}}$ similarly to $r_{c_0,\lambda_0,L^{(0)}}$, $p_{c_0,\lambda_0,L^{(0)}}$, $q_{c_0,\lambda_0,L^{(0)}}$, $p_{\lambda_0,L^{(0)}}$, see the proof of Lemma 1 and the Introduction. At first our aim is to construct linearly independent linear forms $\widetilde{L}_j\in k[X_0,\dots,X_{n-\sigma}]$, $j\in\{0,\sigma+1,\dots,n\}$ with integer coefficients of length $O(n^2\log(dd')$ (additionally applying the algorithm of reduction to integer coefficients, see Proposition 2 [12] one can obtain the estimate $O(n\log d+(n-\sigma)\log(d'))$, but it is not principal now) and an integer t of the same length satisfying the following conditions.

(i) The $(n-\sigma+1)$–tuples of linear forms $(\widetilde{L}_0,\widetilde{L}_{\sigma+1},\dots,\widetilde{L}_{n-1},\widetilde{L}_n+tX_{n-\sigma+1})$ satisfies the assertion of Theorem 1 [12] for the morphism $q_{c_5,\lambda_5,L^{(5)}}$ (in place of the $(n-\sigma+1)$–tuples L' for p'). Hence $\#W'(c_5,\lambda_5,L^{(5)})\cap\mathcal{Z}(\widetilde{L}_{\sigma+1},\dots,\widetilde{L}_{n-1},\widetilde{L}_n+tX_{n-\sigma+1})=\deg W'(c_5,\lambda_5,L^{(5)})$, $W'(c_5,\lambda_5,L^{(5)})\cap\mathcal{Z}(\widetilde{L}_{\sigma+1},\dots,\widetilde{L}_{n-1},\widetilde{L}_n+tX_{n-\sigma+1})\cap\mathcal{Z}(\widetilde{L}_0)=\emptyset$ (here we assume that $X_0,\dots,X_{n-\sigma+1}$ are the coordinate functions on $\mathbb{P}^{n-\sigma+1}(\overline{k})$), for every $z\in W'(c_5,\lambda_5,L^{(5)})\cap\mathcal{Z}(\widetilde{L}_{\sigma+1},\dots,\widetilde{L}_{n-1},\widetilde{L}_n+tX_{n-\sigma+1})$ the inverse image $q^{-1}_{c_5,\lambda_5,L^{(5)}}(z)$ consists of one point z_1, and z_1 is a smooth point of W_{λ_5}, see [12].

(ii) Put $L_j=\widetilde{L}_j(L_0^{(5)},L_{\sigma+1}^{(5)},\dots,L_n^{(5)})$, and $L=(L_0,L_{\sigma+1},\dots,L_n)$, $j\in\{0,\sigma+1,\dots,n\}$. Then the element (c_5,λ_5,L) satisfies conditions 1)–3) from (*).

(iii) For every $z\in W'(c_5,\lambda_5,L^{(5)})\cap\mathcal{Z}(\widetilde{L}_{\sigma+1},\dots,\widetilde{L}_{n-1},\widetilde{L}_n+tX_{n-\sigma+1})$ put $L_z=(L_0,L_{\sigma+1},\dots,L_{n-1},L_n-(\widetilde{L}_n/\widetilde{L}_0)(z)L_0)$. Then (c_5,λ_5,L_z) (in place of $(c_0,\lambda_0,L^{(0)})$) satisfies conditions of Lemma 2.

(iv) Denote $\eta_j^{(5)} = \widetilde{L}_j(e_0^{(5)}, e_{\sigma+1}^{(5)}, \ldots, e_n^{(5)}) = L_j(e_0, \ldots, e_r)$ for all $j \in \{0, \sigma+1, \ldots, n\}$. For every $z \in W_{\lambda_5} \cap \mathcal{Z}(\eta_{\sigma+1}^{(5)}, \ldots, \eta_{n-1}^{(5)}, \eta_n^{(5)} + te_{5,n+1}) \setminus \mathcal{Z}(e_0, \ldots, e_r)$ the number of elements $\#p_{\lambda_5,L}^{-1} p_{\lambda_5,L}(z) = \deg p'_v \deg W'$, and the intersection $p_{\lambda_5,L}^{-1} p_{\lambda_5,L}(z) \cap (\overline{V \setminus W}) = \emptyset$. Hence by Lemma 2 [12] the differential $d_z p_{\lambda_5,L}$ is an isomorphism. Therefore, again by this lemma and (i) the differentials $d_z q_{c_5,\lambda_5,L}$ are isomorphisms for all considered z. This implies, see below, that the differentials $d_z q_{c_5,\lambda_5,L^{(5)}}$ are isomorphisms for all considered z.

Applying Theorem 1 [12] and Remark 2 to the morphism $q_{c_5,\lambda_5,L^{(5)}}$ we construct linear forms $\widetilde{L}_j \in k[X_0, \ldots, X_{n-\sigma}]$, $j \in \{0, \sigma+1, \ldots, n\}$ and an integer t' satisfying (i) with t' in place of t. By assertion (d) Theorem 1 [12] we shall assume without loss of generality that $\widetilde{L}_j$, $j \in \{0, \sigma+1, \ldots, n\}$ are linearly independent. Consider a replacement $\widetilde{L}_0$ by $\widetilde{L}_0 + t''X_0$ and $\widetilde{L}_j$ by $\widetilde{L}_j + t''X_{j-\sigma}$, $j \in \{\sigma+1, \ldots, n\}$, with an integer t'' of length $O(n \log d + (n-\sigma)\log d')$. By Theorem 2 (f) [13] and Lemma 8 using such a replacement with an appropriate t'' (it is constructed by the enumeration) we shall assume without loss of generality that additionally the element (c_5, λ_5, L) satisfies (ii). Notice that the morphism $p_{\lambda_5,L}$ is distinct from $p_{\lambda_5,L^{(5)}}$ only by an automorphism of $\mathbb{P}^{n-\sigma}(\overline{k})$. Similarly $p_{c_5,\lambda_5,L}$ is distinct from $p_{c_5,\lambda_5,L^{(5)}}$ only by an automorphism of $\mathbb{P}^{n-\sigma+1}(\overline{k})$. Hence the algebraic variety $W'(c_5, \lambda_5, L)$ is distinct from $W'(c_5, \lambda_5, L^{(5)})$ only by an automorphism of $\mathbb{P}^{n-\sigma+1}(\overline{k})$. Therefore, $\deg W'(c_5, \lambda_5, L^{(5)}) = \deg W'(c_5, \lambda_5, L)$. Note also that $W'(c_5, \lambda_5, L^{(5)}) = \mathcal{Z}(\Phi_{c_5,\lambda_5,L^{(5)}}(X_0, \ldots, X_{n-\sigma+1}))$, $W'(c_5, \lambda_5, L) = \mathcal{Z}(\Phi_{c_5,\lambda_5,L}(X_0, \ldots, X_{n-\sigma+1})))$ and

$$\Phi_{c_5,\lambda_5,L^{(5)}}(X_0, \ldots, X_{n-\sigma+1}) = \mu\Phi_{c_5,\lambda_5,L}(\widetilde{L}_0, \widetilde{L}_{\sigma+1}, \ldots, \widetilde{L}_n, X_{n-\sigma+1}), \quad (33)$$

where $0 \neq \mu \in \overline{k}$.

Now by (i) every irreducible component of the variety $C_{\lambda_5,L} = W_{\lambda_5} \cap \mathcal{Z}(\eta_{\sigma+1}^{(5)}, \ldots, \eta_{n-1}^{(5)}) \setminus \mathcal{Z}(e_0, \ldots, e_r)$ is a curve. By conditions 1)–3) from (*) for (c_5, λ_5, L) the morphism $p_{\lambda_5,L,*}$ is dominant, and for all points $z \in C_{\lambda_5,L}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, the differential $d_z p_{\lambda_5,L}$ is an isomorphism, $\#p_{\lambda_5,L}^{-1} p_{\lambda_5,L}(z) = \deg p'_v \deg W'$, the intersection $p_{\lambda_5,L}^{-1} p_{\lambda_5,L}(z) \cap (\overline{V \setminus W}) = \emptyset$. Let $z \in C_{\lambda_5,L}$ and $z' = q_{c_5,\lambda_5,L^{(5)}}(z)$. Then by Lemma 2 for all points $z \in C_{\lambda_5,L}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, the point $(c_5, \lambda_5, L_{z'})$ (in place of $(c_0, \lambda_0, L^{(0)})$) satisfies conditions of Lemma 2. Notice that by [5] for every integer t one can compute the algebraic variety $C_{\lambda_5,L} \cap \mathcal{Z}(\widetilde{e}_n + te_{5,n+1})$. Hence one can construct an integer t of length $O(n^2 \log(dd'))$ such that (i) and (ii) hold, the variety $C_{\lambda_5,L} \cap \mathcal{Z}(\widetilde{e}_n + te_{5,n+1})$ is zero–dimensional, and conditions (iii) and (iv) are fulfilled. Thus, the required linear forms $\widetilde{L}_j$ and the integer t are constructed.

Denote for brevity $\eta_j = \widetilde{L}_j(e_0^{(5)} - e_0^{(0)} + e_0^{(1)}, e_{\sigma+1}^{(5)} - e_{\sigma+1}^{(0)} + e_{\sigma+1}^{(1)}, \ldots, e_n^{(5)} - e_j^{(0)} + e_j^{(1)})$, $j \in \{0, \sigma+1, \ldots, n\}$. Let $\tau_i^* \in \overline{K}$ be arbitrary (in what follows we assume that $K \supset k_3$). Let us define $(c_*, \lambda_*, L^*) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}} \otimes \overline{K}$ substituting in (17)–(19) the values $\tau_i = \tau_{5,i} + \tau_i^*$, $1 \leq i \leq n_1$. Let $\lambda_* = (L_{*,s+1}, \ldots, L_{*,\sigma})$, $L^* = (L_0^*, L_{\sigma+1}^*, \ldots, L_n^*)$, $e_j^* = L_j^*(e_0, \ldots, e_r)$, $j \in \{0, \sigma+1, \ldots, n\}$. Set $\eta_j^* = \widetilde{L}_j(e_0^*, e_{\sigma+1}^*, \ldots, e_n^*)$, $j \in \{0, \sigma+1, \ldots, n\}$. Thus, $\eta_j^* = \eta_j|_{\tau_i = \tau_i^*,\, 1 \leq i \leq n_1}$ and

$\eta_j^{(5)} = \eta_j|_{\tau_i=0,\,1\le i\le n_1}$. Put $\overline{L}_j = \widetilde{L}_j(L_0^*, L_{\sigma+1}^*, \dots, L_n^*)$, $j \in \{0, \sigma+1, \dots, n\}$, and $\overline{L} = (\overline{L}_0, \widetilde{L}_{\sigma+1}, \dots, \overline{L}_n)$. Hence $\eta_j^* = \overline{L}_j(e_0, \dots, e_r)$ for all $j \in \{0, \sigma+1, \dots, n\}$.

Consider the system of equations and inequalities with respect to the indeterminates $X_0, \dots, X_n$, $Y_{\sigma+1}, \dots, Y_n$, $\tau_1, \dots, \tau_{n_1}$ with coefficients from the field k_2

$$\begin{cases} \sum_{1\le i\le n_1} |\tau_i|^2 \le \varepsilon_1, \\ \eta_j - Y_j\eta_0 = 0, & \sigma+1 \le j \le n-1, \\ \eta_n - Y_n\eta_0 + \\ +t(e_{5,n+1} - e_{0,n+1} + e_{1,n+1}) = 0, \\ \eta_0 = 1, \\ L_{5,j} - L_{0,j} + L_{1,j} = 0, & s+1 \le j \le \sigma, \\ \sum_{\sigma+1\le i\le n} |Y_i|^2 \le \varepsilon_2, \\ \sum_{0\le i\le n} |e_i|^2 \ge \varepsilon_2^{-1}. \end{cases} \tag{34}$$

LEMMA 18 *Suppose that $(c_1, \lambda_1, L^{(1)})$ satisfies conditions 1)–3) from (*), conditions (i)–(iv) hold,* $\deg W'(c_1, \lambda_1, L^{(1)}) > \deg W'(c_5, \lambda_5, L^{(5)})$. *Then system (34) has a solution $X_w = x_w^*$, $0 \le w \le n$; $\tau_i = \tau_i^*$, $1 \le i \le n_1$; $Y_j = y_j^*$, $\sigma+1 \le j \le n$, from $\overline{k_2}$ such that*

$$(x_0^* : \dots : x_n^*) \in W(\overline{k_2}). \tag{35}$$

For arbitrary solution (35) the point (c_*, λ_*, L^*) satisfies conditions of Lemma 2 (in place of $(c_0, \lambda_0, L^{(0)})$) and $\deg W'(c_*, \lambda_*, L^*) > \deg W'(c_5, \lambda_5, L^{(5)})$.

PROOF Obviously it is sufficient to prove this lemma with k_3 in place of $\underline{k_2}$. Put $S' = W'(c_5, \lambda_5, L^{(5)}) \cap \mathcal{Z}(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_{n-1}, \widetilde{L}_n + tX_{n+1})$. Let $\tau_i^*, y_j^* \in \overline{k_3}$, $1 \le i \le n_1$, $\sigma+1 \le j \le n$ be arbitrary (not necessarily related to a solution of system (34)) such that $\sum_{1\le i\le n_1} |\tau_i^*|^2 \le \varepsilon_1$ and $\sum_{\sigma+1\le j\le n} |y_j^*|^2 \le \varepsilon_2$. Notice that according to the observations from the beginning of Section 1 the point (c_*, λ_*, L^*) satisfies (D) and conditions of Lemma 2. Therefore, the objects q_{c_*,λ_*,L^*}, $W'(c_*, \lambda_*, L^*)$ and so on are defined. Let $z \in S'$ and $q_{c_5,\lambda_5,L^{(5)}}(z_1) = z$, see (i).

Let us show that for every $z \in S'$ there is the unique point $z^* \in S^* = W'(c_*, \lambda_*, L^*) \cap \mathcal{Z}(\widetilde{L}_{\sigma+1} - y_{\sigma+1}^*\widetilde{L}_0, \dots, \widetilde{L}_{n-1} - y_{n-1}^*\widetilde{L}_0, \widetilde{L}_n - y_n^*\widetilde{L}_0 + tX_{n+1})$ such that $q^{-1}_{c_*,\lambda_*,L^*}(z^*)$ contains a point z_1^* which is infinitely close to z_1 (hence also z^* is infinitely close to z). Note that by (i) $\widetilde{L}_{\sigma+1}/\widetilde{L}_0, \dots, \widetilde{L}_{n-1}/\widetilde{L}_0, (\widetilde{L}_n + tX_{n+1})/\widetilde{L}_0$ generate the linear space $\mathfrak{m}_z/\mathfrak{m}_z^2$ where $\mathfrak{m}_z$ is the maximal ideal of the local ring $\mathcal{O}_{z,W'(c_5,\lambda_5,L^{(5)})}$ of the algebraic variety $W'(c_5, \lambda_5, L^{(5)})$ at z. By (iv) the differential $d_{z_1}q_{c_5,\lambda_5,L^{(5)}}$ is an isomorphism. Therefore, the dual mapping to $d_{z_1}q_{c_5,\lambda_5,L^{(5)}}$ is also an isomorphism. Hence $\eta_{\sigma+1}^{(5)}/\eta_0^{(5)}, \dots, \eta_{n-1}^{(5)}/\eta_0^{(5)}, (\eta_n^{(5)} + te_{5,n+1})/\eta_0^{(5)}$ generate the linear space $\mathfrak{m}_{z_1}/\mathfrak{m}_{z_1}^2$ where $\mathfrak{m}_{z_1}$ is the maximal ideal of the local ring $\mathcal{O}_{z_1,W_{\lambda_5}}$ of the algebraic variety W_{λ_5} at z_1. Hence z_1 is a smooth point of hypersurfaces $\mathcal{Z}(\eta_j^{(5)})$, $\sigma+1 \le j \le n-1$; $\mathcal{Z}(\eta_n^{(5)} + te_{5,n+1})$, and the intersection of the tangent spaces of the algebraic varieties W_{λ_5}, $\mathcal{Z}(\eta_{\sigma+1}^{(5)}), \dots, \mathcal{Z}(\eta_{n-1}^{(5)})$, $\mathcal{Z}(\eta_n^{(5)} + te_{5,n+1})$ at the point z_1 is transversal.

On the other hand, by Lemma 17 (ii) [13] and (iii) the intersection $W \cap \mathcal{Z}(L_{5,s+1}, \dots, L_{5,\sigma}) \cap \mathcal{Z}(\eta_{\sigma+1}^{(5)}, \dots, \eta_{n-1}^{(5)}, \eta_n^{(5)} - (\widetilde{L}_n/\widetilde{L}_0)(z)\eta_0^{(5)})$ is transversal at z_1. Hence the intersection $W \cap \mathcal{Z}(L_{5,s+1}, \dots, L_{5,\sigma})$ is also transversal at z_1.

Therefore, the intersection $W \cap \mathcal{Z}(L_{5,s+1}, \dots, L_{5,\sigma}) \cap \mathcal{Z}(\eta^{(5)}_{\sigma+1}) \cap \dots \cap \mathcal{Z}(\eta^{(5)}_{n-1}) \cap \mathcal{Z}(\eta^{(5)}_n + te_{5,n+1})$ is transversal at z_1. Finally, by the implicit function theorem there is one and only one point $z_1^* \in q^{-1}_{c_*,\lambda_*,L^*}(S^*)$ such that z_1^* is infinitely close to z_1. The required assertion is proved.

Now suppose that $y \in S^*$, $q_{c_*,\lambda_*,L^*}(y_1) = y$, and y_1 is not infinitely close to $\mathcal{Z}(e_0, \dots, e_r)$. Then obviously y_1 is infinitely close to $q^{-1}_{c_5,\lambda_5,L^{(5)}}(S')$ and, hence, y is infinitely close to S'. Conversely suppose that y_1 is infinitely close to $\mathcal{Z}(e_0, \dots, e_r)$. Let us show that y is not infinitely close to S'. Indeed, suppose contrary, that y is infinitely close to $z \in S'$. Therefore, $\widetilde{L}_0(y) \neq 0$. Set $\overline{L}_y = (\overline{L}_0, \overline{L}_{\sigma+1}, \dots, \overline{L}_{n-1}, \overline{L}_n - (\widetilde{L}_n/\widetilde{L}_0)(y)\overline{L}_0)$. Notice that the coefficients of the linear forms from $\overline{L}_y$ are infinitely close to the coefficients of the corresponding linear forms from L_z, see condition (iii). By property (D) for (c_5, λ_5, L_z), see (iii), the intersection $(W \setminus \mathcal{Z}(e_0, \dots, e_r)) \cap \mathcal{Z}(L_{5,s+1}, \dots, L_{5,\sigma}) \cap \mathcal{Z}(\eta^{(5)}_{\sigma+1}) \cap \dots \cap \mathcal{Z}(\eta^{(5)}_{n-1}) \cap \mathcal{Z}(\eta^{(5)}_n - (\widetilde{L}_n/\widetilde{L}_0)(z)\eta^{(5)}_0) = A_{\lambda_5,L_z}$ is transversal and consists of $\deg p'_v \deg W'$ points. By Lemma 17 [13] and the implicit function theorem the intersection $(W \setminus \mathcal{Z}(e_0, \dots, e_r)) \cap \mathcal{Z}(L_{*,s+1}, \dots, L_{*,\sigma}) \cap \mathcal{Z}(\eta^*_{\sigma+1}) \cap \dots \cap \mathcal{Z}(\eta^*_{n-1}) \cap \mathcal{Z}(\eta^*_n - (\widetilde{L}_n/\widetilde{L}_0)(y)\eta^*_0) = A_{\lambda_*,\overline{L}_y}$ is transversal and consists of $\deg p'_v \deg W'$ points. By the implicit function theorem the points from $A_{\lambda_*,\overline{L}_y}$ are infinitely close to the corresponding points from A_{λ_5,L_z}. But by (i) $q^{-1}_{c_5,\lambda_5,L^{(5)}}(z)$ consist of one point, and $q^{-1}_{c_5,\lambda_5,L^{(5)}}(z) \subset A_{\lambda_5,L_z}$, $q^{-1}_{c_*,\lambda_*,L^*}(y) \subset A_{\lambda_*,\overline{L}_y}$. Hence $q^{-1}_{c_*,\lambda_*,L^*}(y)$ also consists of one point which is infinitely close to the one from $q^{-1}_{c_5,\lambda_5,L^{(5)}}(z)$. Thus, y_1 is not infinitely close to $\mathcal{Z}(e_0, \dots, e_r)$. This contradiction proves the required assertion.

Let us show that $\widetilde{L}_0$ does not vanish on any irreducible component of $W'(c_*, \lambda_*, L^*)$. Indeed by (ii) every irreducible component of W_{λ_5} contains a point from $A_{\lambda_5,L}$, and $\eta^{(5)}_0$ does not vanish at any points from $A_{\lambda_5,L}$. Notice that the coefficients of the linear forms $\overline{L}_j$ are infinitely close to the corresponding coefficients of the linear forms L_j, $j \in \{0, \sigma+1, \dots, n\}$. Hence by Lemma 17 [13] and the implicit function theorem, see the last observation related to property (D) at the beginning of Section 1, every irreducible component of W_{λ_*} contains a point from $A_{\lambda_5,\overline{L}}$, and $\eta^*_0 = \widetilde{L}_0(e^*_0, e^*_{\sigma+1}, \dots, e^*_n)$ does not vanish at any points from $A_{\lambda_5,\overline{L}}$. Therefore, every irreducible component of $W'(c_*, \lambda_*, L^*)$ contains a point from $q_{c_*,\lambda_*,L^*}(A_{\lambda_5,\overline{L}})$, and $\widetilde{L}_0$ does not vanish at any points from $q_{c_*,\lambda_*,L^*}(A_{\lambda_5,\overline{L}})$. This implies the required assertion.

Let us prove the existence of the required solution of system (34). By Lemma 4 and the algorithm of reduction to integer coefficients, cf. the proof of the existence of (c_*, λ_*, L^*) in Lemma 17, there are $\tau^*_i \in k(\varepsilon_3)$ which are infinitesimal with respect to the field k such that (c_*, λ_*, L^*) satisfies conditions of Lemma 4 (for any line $\mathfrak{l}$ containing (c_*, λ_*, L^*)) in place of $(c_0, \lambda_0, L^{(0)})$, and $\deg W'(c_*, \lambda_*, L^*) = \deg W'(c_1, \lambda_1, L^{(1)})$. We have $\deg W'(c_*, \lambda_*, L^*) > \deg W'(c_5, \lambda_5, L^{(5)})$. Hence as we have proved the Bézout theorem implies the existence of a point $z \in W'(c_*, \lambda_*, L^*) \cap \mathcal{Z}(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_{n-1}, \widetilde{L}_n + tX_{n-\sigma+1})$, which is not infinitely close to S'. Further, there exists a point $y \in W'(c_*, \lambda_*, L^*)$ with the coordinates from $\overline{k(\varepsilon_3)}$ such that y is infinitely close to z with respect to the field k, the inverse image $q^{-1}_{c_*,\lambda_*,L^*}(y) \neq \emptyset$ and $\widetilde{L}_0(y) \neq 0$. Put $y^*_j = (\widetilde{L}_j/\widetilde{L}_0)(y)$ for $\sigma + 1 \leq j \leq n-1$ and $y^*_n = ((\widetilde{L}_n + tX_{n-\sigma+1})/\widetilde{L}_0)(y)$.

Then every nonzero $y_j^* \in \overline{k(\varepsilon_3)}$, $j \in \{\sigma+1, \ldots, n\}$, is infinitesimal with respect to the field k. Let $y_1 \in q^{-1}_{c_*,\lambda_*,L^*}(y)$. By previous considerations the point y_1 is infinitely close to $\mathcal{Z}(e_0, \ldots, e_r)$. Thus, system (34) has a solution satisfying (35) (with k_3 in place of k_2).

Conversely, if (34) has a solution satisfying (35) then by the previous considerations and the Bézout theorem $\deg W'(c_*, \lambda_*, L^*) > \deg W'(c_5, \lambda_5, L^{(5)})$. The lemma is proved.

Let us return to the description of the general step of the considered recursion. Notice that system (34) satisfies conditions 1)–4) from Section 2. Hence using Lemma 16 and Remark 11 we decide whether system (34) has a solution satisfying (35), and if such a solution exists we construct it. Suppose that such a solution does not exist. Then by Lemma 18 the degree $\deg W'(c_1, \lambda_1, L^{(1)}) = \deg W'(c_5, \lambda_5, L^{(5)})$ and the considered step is final.

Suppose that we have constructed a solution of system (34) satisfying (35), see the statement of Lemma 18. Then $\deg W'(c_*, \lambda_*, L^*) > \deg W'(c_5, \lambda_5, L^{(5)})$ by this lemma. Similarly to the previous recursion conditions of Lemma 4 hold for (c_*, λ_*, L^*) in place of $(c_0, \lambda_0, L^{(0)})$. By Lemma 4 and the observations from the beginning of the section one can apply the algorithm of reduction to integer coefficients, see [12], Proposition 2, and replace subsequently all values τ_i^* by integers τ_i' of length $O(n \log d + (n-\sigma)\log(d'))$. Denote by (c', λ, L') the obtained point. Then $L' \in \mathcal{U}_0''$, the number of elements $\#A_{\lambda,L'} = \deg p_v' \deg W'$, $A_{\lambda,L'} \cap \mathcal{Z}(e_0') = \emptyset$, and $A_{\lambda,L'} \cap \overline{V \setminus W} = \emptyset$, the dimensions $\dim W_\lambda = n - \sigma$, $\dim(W_\lambda \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_{n-1}) = 1$, the number of elements $\#(e'_{n+1}/e'_0)(A_{\lambda,L'}) = \#(e_{5,n+1}/e_0^{(5)})(A_{\lambda_5,L^{(5)}})$, the morphisms $p_{\lambda,L'}$, $p_{\lambda,L',*}$ are dominant, and $\deg W'(c', \lambda, L') \geq \deg W'(c_*, \lambda_*, L^*)$. Let us apply Lemma 8 to the case when $\iota(\mathbb{A}^1(\overline{k}))$ is the line containing (c', λ, L') and $(c_5, \lambda_5, L^{(5)})$, and $\iota(0) = (c_5, \lambda_5, L^{(5)})$. Then using Theorem 2 (f) [13], Lemma 8 and Lemma 4 we construct a point (c'', λ', L'') which is similar to $(c_5, \lambda_5, L^{(5)})$, and $\deg W'(c'', \lambda', L'') \geq \deg W'(c', \lambda, L') > \deg W'(c_5, \lambda_5, L^{(5)})$. After that we replace $(c_5, \lambda_5, L^{(5)})$ by (c'', λ', L'') and proceed to the next step of the recursion. The general step and all the recursion are described.

Now we shall describe one more recursion. As the base of the recursion we take the constructed point $(c_5, \lambda_5, L^{(5)})$. At the end of this recursion we shall obtain $(c_5, \lambda_5, L^{(5)})$ such that conditions 1)–3) from (*) hold for $(c_5, \lambda_5, L^{(5)})$ and $\deg W'(c_5, \lambda_5, L^{(5)}) = \deg W'(c_1, \lambda_1, L^{(1)})$, $\deg W'(c_5, \lambda_5, L^{(5)}, *) = \deg W'(c_1, \lambda_1, L^{(1)}, *)$.

Let us describe the general step of this recursion. At the beginning of the step a point $(c_5, \lambda_5, L^{(5)})$ satisfying conditions 1)–3) from (*) is given, and $\deg W'(c_5, \lambda_5, L^{(5)}) = \deg W'(c_1, \lambda_1, L^{(1)})$. Applying the algorithm from [5] let us compute, see the observations at the beginning of the section, the variety $W'(c_5, \lambda_5, L^{(5)}, *)$ and the polynomial $\Phi_{c_5,\lambda_5,L^{(5)},+}$, see Lemma 3. Hence $W'(c_5, \lambda_5, L^{(5)}, *) = \mathcal{Z}(\Phi_{c_5,\lambda_5,L^{(5)},+}(X_0, X_1, X_2))$ by Lemma 3. By properties 1) and 2) from (*) for $(c_5, \lambda_5, L^{(5)})$ the intersection $(W_{\lambda_5} \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e_{\sigma+1}^{(5)}) \cap \ldots \cap \mathcal{Z}(e_n^{(5)}) = A_{\lambda_5,L^{(5)}}$ is transversal at each its point, consists of $\deg p_v' \deg W'$ isolated points, and $A_{\lambda_5,L^{(5)}} \cap \mathcal{Z}(e_0^{(5)}) = \emptyset$. By property 3) from (*) and Lemma 2 [12] the intersection $W'(c_5, \lambda_5, L^{(5)}) \cap \mathcal{Z}(X_1, \ldots, X_{n-\sigma})$ is also transversal at every point from $q_{c_5,\lambda_5,L^{(5)}}(A_{\lambda_5,L^{(5)}})$, the number of elements $\#q_{c_5,\lambda_5,L^{(5)}}(A_{\lambda_5,L^{(5)}}) = \deg p_v' \deg W'$, and for every $z \in A_{\lambda_5,L^{(5)}}$ the differential

$d_z q_{c_5,\lambda_5,L^{(5)}}$ is an isomorphism. Let us define $r_{c_5,\lambda_5,L^{(5)},*}$, $p_{\lambda_5,L^{(5)},*}$, $q_{c_5,\lambda_5,L^{(5)},*}$ similarly to $r_{c_0,\lambda_0,L^{(0)},*}$, $p_{\lambda_0,L^{(0)},*}$, $q_{c_0,\lambda_0,L^{(0)},*}$, see the proof of Lemma 3 and the Introduction. By Lemma 11 [12] every irreducible component of $(W_{\lambda_5} \setminus \mathcal{Z}(e_0,\dots,e_r)) \cap \mathcal{Z}(e^{(5)}_{\sigma+1},\ \dots, e^{(5)}_{n-1}) = C_{\lambda_5,L^{(5)}}$ contains a point from $A_{\lambda_5,L^{(5)}}$. Hence the intersection

$$(W_{\lambda_5} \setminus \mathcal{Z}(e_0,\dots,e_r)) \cap \mathcal{Z}(e^{(5)}_{\sigma+1}) \cap \dots \cap \mathcal{Z}(e^{(5)}_{n-1}) \tag{36}$$

is transversal, each irreducible component of $C_{\lambda_5,L^{(5)}}$ is a curve, and the variety $W'(c_5,\lambda_5,L^{(5)},*)$ is the union of all the irreducible components E of $W'(c_5,\lambda_5,L^{(5)}) \cap \mathcal{Z}(X_1,\dots,X_{n-\sigma-1})$ such that $E \cap q_{c_5,\lambda_5,L^{(5)}}(A_{\lambda_5,L^{(5)}}) \neq \emptyset$. Therefore, the intersection $W'(c_5,\lambda_5,L^{(5)}) \cap \mathcal{Z}(X_1,\dots,X_{n-\sigma-1})$ is transversal at every irreducible component E of $W'(c_5,\lambda_5,L^{(5)},*)$. We have $\deg p_{\lambda_5,L^{(5)},*} = \deg r_{c_5,\lambda_5,L^{(5)},*} = \deg p_{\lambda_5,L^{(5)}} = \deg r_{c_5,\lambda_5,L^{(5)}} = \deg p'_v \deg W'$. Hence the morphism $q_{c_5,\lambda_5,L^{(5)},*}$ induces the isomorphism of algebras of rational functions on $C_{\lambda_5,L^{(5)}}$ and $W'(c_5,\lambda_5,L^{(5)},*)$ respectively (by definition these algebras are isomorphic to the direct products of fields of rational functions of all irreducible components of $C_{\lambda_5,L^{(5)}}$ and $W'(c_5,\lambda_5,L^{(5)},*)$ respectively). Similarly $q_{c_5,\lambda_5,L^{(5)}}$ induces the birational isomorphism. Further, $q_{c_5,\lambda_5,L^{(5)},*}$ (respectively $q_{c_5,\lambda_5,L^{(5)}}$) is an isomorphism in a neighborhood with respect to the Zariski topology of the finite set $A_{\lambda_5,L^{(5)}}$. Let $z \in W'(c_5,\lambda_5,L^{(5)},*)$ and $X_0(z) \neq 0$. Put $L^{(5)}_z = (L^{(5)}_0, L^{(5)}_{\sigma+1},\dots,L^{(5)}_{n-1}, L^{(5)}_n - (X_{n-\sigma}/X_0)(z)L^{(5)}_0)$.

Let $z \in C_{\lambda_5,L^{(5)}}$ and $z' = q_{c_5,\lambda_5,L^{(5)}}(z)$. Then by the Bézout theorem (we leave the details to the reader here) for all $z \in C_{\lambda_5,L^{(5)}}$, except at most a polynomial in $d^n(d')^{n-\sigma}$ number, the morphisms $q_{c_5,\lambda_5,L^{(5)},*}$ and $q_{c_5,\lambda_5,L^{(5)}}$ are isomorphisms in some neighborhoods of z with respect to the Zariski topology, $z \notin \overline{V \setminus W}$, intersection (36) is transversal at z (in particular z is a smooth point of $C_{\lambda_5,L^{(5)}}$), the number of elements $p^{-1}_{\lambda_5,L^{(5)}} p_{\lambda_5,L^{(5)}}(z) = \deg p'_v \deg W'$, $e^{(5)}_0(z) \neq 0$, the element $(c_5,\lambda_5,L^{(5)}_{z'})$ satisfies conditions of Lemma 2 (in place of $(c_0,\lambda_0,L^{(0)})$).

Let t', t'' be integers. Set $B_{\lambda_5,L^{(5)}} = C_{\lambda_5,L^{(5)}} \cap \mathcal{Z}(e^{(5)}_0 + t'e^{(5)}_n + t''e_{5,n+1})$ in $\mathbb{P}^n(\overline{k})$. Recall that we know explicitly the polynomial $\Phi_{c_5,\lambda_5,L^{(5)},+}$. Hence one can construct integers t', t'' with length bounded from above $O(n\log d + (n-\sigma)\log(d'))$ such that:

(v) for every $z \in B_{\lambda_5,L^{(5)}}$ the morphism $q_{c_5,\lambda_5,L^{(5)},*}$ (respectively $q_{c_5,\lambda_5,L^{(5)}}$) is an isomorphism in a neighborhood with respect to the Zariski topology of the point z,

(vi) for every $z \in B_{\lambda_5,L^{(5)}}$ the number of elements $p^{-1}_{\lambda_5,L^{(5)}} p_{\lambda_5,L^{(5)}}(z) = \deg p'_v \deg W'$.

(vii) for every $z \in B_{\lambda_5,L^{(5)}}$ intersection (36) is transversal at z,

(viii) $q_{c_5,\lambda_5,L^{(5)}}(B_{\lambda_5,L^{(5)}}) = W'(c_5,\lambda_5,L^{(5)},*) \cap \mathcal{Z}(X_0 + t'X_1 + t''X_2)$, $\#q_{c_5,\lambda_5,L^{(5)}}(B_{\lambda_5,L^{(5)}}) = \deg W'(c_5,\lambda_5,L^{(5)},*)$, and $q_{c_5,\lambda_5,L^{(5)}}(B_{\lambda_5,L^{(5)}}) \cap \mathcal{Z}(X_0) = \emptyset$,

(ix) for every $z \in q_{c_5,\lambda_5,L^{(5)}}(B_{\lambda_5,L^{(5)}})$ the element $(c_5,\lambda_5,L^{(5)}_z)$ satisfies conditions of Lemma 2 (in place of $(c_0,\lambda_0,L^{(0)})$).

Consider the system of equations and inequalities with respect to the indeterminates $X_0, \ldots, X_n$, Y_0, $\tau_1, \ldots, \tau_{n_1}$ with coefficients from the field k_2

$$\begin{cases} \sum_{1\le i\le n_1} |\tau_i|^2 \le \varepsilon_1, \\ e_j^{(5)} - e_j^{(0)} + e_j^{(1)} = 0, & \sigma+1 \le j \le n-1, \\ e_0^{(5)} - e_0^{(0)} + e_0^{(1)} + \\ +t'(e_n^{(5)} - e_n^{(0)} + e_n^{(1)}) + \\ +t''(e_{5,n+1} - e_{0,n+1} + e_{1,n+1}) = Y_0, \\ e_0^{(5)} - e_0^{(0)} + e_0^{(1)} = 1, \\ L_{5,j} - L_{0,j} + L_{1,j} = 0, & s+1 \le j \le \sigma, \\ |Y_0|^2 \le \varepsilon_2, \\ \sum_{0\le i\le n} |e_i|^2 \ge \varepsilon_2^{-1}. \end{cases} \tag{37}$$

LEMMA 19 *Suppose that* $(c_1, \lambda_1, L^{(1)})$ *satisfies conditions 1)–4) from (*), conditions (v)–(ix) hold,* $\deg W'(c_1, \lambda_1, L^{(1)}) = \deg W'(c_5, \lambda_5, L^{(5)})$, $\deg W'(c_1, \lambda_1, L^{(1)}, *) > \deg W'(c_5, \lambda_5, L^{(5)}, *)$. *Then system (34) has a solution* $X_w = x_w^*$, $0 \le w \le n$; $\tau_i = \tau_i^*$, $1 \le i \le n_1$; $Y_0 = y_0^*$, *from* $\overline{k_2}$ *such that*

$$(x_0^* : \ldots : x_n^*) \in W(\overline{k_2}). \tag{38}$$

For arbitrary solution (38) the point (c_*, λ_*, L^*) *satisfies conditions of Lemma 2 (in place of* $(c_0, \lambda_0, L^{(0)})$*),* $\deg W'(c_1, \lambda_1, L^{(1)}) = \deg W'(c_5, \lambda_5, L^{(5)})$ *and* $\deg W'(c_*, \lambda_*, L^*) > \deg W'(c_5, \lambda_5, L^{(5)})$.

PROOF Obviously it is sufficient to prove this lemma with k_3 in place of k_2. Denote for brevity $S' = q_{c_5,\lambda_5,L^{(5)}}(B_{\lambda_5,L^{(5)}})$ (in the proof of this lemma the notations S' and S^* are distinct from the ones introduced in the proof of Lemma 18, but they have the analogous sense). Let $\tau_i^*, y_0^* \in \overline{k_3}$, $1 \le i \le n_1$, be arbitrary (not necessarily related to a solution of system (34)) such that $\sum_{1\le i\le n_1} |\tau_i^*|^2 \le \varepsilon_1$ and $|y_0^*|^2 \le \varepsilon_2$. Notice that according to the observations from the beginning of Section 1 the point (c_*, λ_*, L^*) satisfies (D) and conditions of Lemma 2. Therefore, the objects q_{c_*,λ_*,L^*}, $W'(c_*, \lambda_*, L^*)$ and so on are defined. Let $z \in S'$ and $q_{c_5,\lambda_5,L^{(5)}}(z_1) = z$.

Let us show that for every $z \in S'$ there is the unique point $z^* \in S^* = W'(c_*, \lambda_*, L^*, *) \cap \mathcal{Z}(X_0 + t'X_1 + t''X_2 - y_0^* X_0)$ such that $q^{-1}_{c_*,\lambda_*,L^*}(z^*)$ contains a point z_1^* which is infinitely close to z_1 (hence also z^* is infinitely close to z). Note that by (v) and (viii) the intersection $C_{\lambda_5,L^{(5)}} \cap \mathcal{Z}(e_0^{(5)} + t'e_n^{(5)} + t''e_{5,n+1})$ is transversal at z_1. Hence by (vii) the intersection $(W_{\lambda_5} \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e_{\sigma+1}^{(5)}) \cap \ldots \cap \mathcal{Z}(e_{n-1}^{(5)}) \cap \mathcal{Z}(e_0^{(5)} + t'e_n^{(5)} + t''e_{5,n+1})$ is transversal at z_1 (in particular z_1 is a smooth point of all the considered varieties).

On the other hand, by Lemma 17 (ii) [13] and (ix) the intersection $W \cap \mathcal{Z}(L_{5,s+1}, \ldots, L_{5,\sigma}) \cap \mathcal{Z}(e_{\sigma+1}^{(5)}, \ldots, e_{n-1}^{(5)}, e_n^{(5)} - (X_{n-\sigma}/X_0)(z)e_0^{(5)})$ is transversal at z_1. Hence the intersection $W \cap \mathcal{Z}(L_{5,s+1}, \ldots, L_{5,\sigma})$ is also transversal at z_1. Therefore, the intersection $W \cap \mathcal{Z}(L_{5,s+1}, \ldots, L_{5,\sigma}) \cap \mathcal{Z}(e_{\sigma+1}^{(5)}) \cap \ldots \cap \mathcal{Z}(e_{n-1}^{(5)}) \cap \mathcal{Z}(e_0^{(5)} + t'e_n^{(5)} + t''e_{5,n+1})$ is transversal at z_1. Finally, by the implicit function theorem there is one and only one point $z_1^* \in q^{-1}_{c_*,\lambda_*,L^*,*}(S^*)$ such that z_1^* is infinitely close to z_1. The required assertion is proved.

Now suppose that $y \in S^*$, $q_{c_*,\lambda_*,L^*,*}(y_1) = y$, and y_1 is not infinitely close to $\mathcal{Z}(e_0, \ldots, e_r)$. Then obviously y_1 is infinitely close to $q^{-1}_{c_5,\lambda_5,L^{(5)},*}(S')$ and,

hence, y is infinitely close to S'. Conversely, suppose that y_1 is infinitely close to $\mathcal{Z}(e_0,\dots,e_r)$. Let us show that y is not infinitely close to S'. Indeed, suppose contrary, that y is infinitely close to $z\in S'$. Therefore, $X_0(y)\neq 0$. Put $L^*_y=(L^*_0,\dots,L^*_{n-1},L^*_n-(X_{n-\sigma}/X_0)(y)L^*_0)$. Then the nonzero coefficients of the linear forms from L^*_y are infinitely close to the coefficients of the corresponding linear forms from $L^{(5)}_z$.

By property (D) for $(c_5,\lambda_5,L^{(5)}_z)$, see (ix), the intersection $(W\setminus\mathcal{Z}(e_0,\dots,e_r))\cap\mathcal{Z}(L_{5,s+1},\dots,L_{5,\sigma})\cap\mathcal{Z}(e^{(5)}_{\sigma+1})\cap\ldots\cap\mathcal{Z}(e^{(5)}_{n-1})\cap\mathcal{Z}(e^{(5)}_n-(X_{n-\sigma}/X_0)(z)e^{(5)}_0)=A_{\lambda_5,L^{(5)}_z}$ is transversal and consists of $\deg p'_v\deg W'$ points. By Lemma 17 [13] and the implicit function theorem the intersection $(W\setminus\mathcal{Z}(e_0,\dots,e_r))\cap\mathcal{Z}(L_{*,s+1},\dots,L_{*,\sigma})\cap\mathcal{Z}(e^*_{\sigma+1})\cap\ldots\cap\mathcal{Z}(e^*_{n-1})\cap\mathcal{Z}(e^*_n-(X_{n-\sigma}/X_0)(y)e^*_0)=A_{\lambda_*,L^*_y}$ is transversal and consists of $\deg p'_v\deg W'$ points. By the implicit function theorem the points from A_{λ_*,L^*_y} are infinitely close to the corresponding points from A_{λ_5,L_z}. But by (v) $q^{-1}_{c_5,\lambda_5,L^{(5)}}(z)$ consist of one point. Further, $q^{-1}_{c_5,\lambda_5,L^{(5)}}(z)\subset A_{\lambda_5,L^{(5)}_z}$, $q^{-1}_{c_*,\lambda_*,L^*}(y)\subset A_{\lambda_*,L^*_y}$. Hence $q^{-1}_{c_*,\lambda_*,L^*}(y)$ also consists of one point which is infinitely close to the one from $q^{-1}_{c_5,\lambda_5,L^{(5)}}(z)$. Thus, y_1 is not infinitely close to $\mathcal{Z}(e_0,\dots,e_r)$. This contradiction proves the required assertion.

Let us show that X_0 does not vanish on any irreducible component of $W'(c_*,\lambda_*,L^*,*)$. Indeed as we have seen every irreducible component of $C_{\lambda_5,L^{(5)}}$ contains a point from $A_{\lambda_5,L^{(5)}}$, and $e^{(5)}_0$ does not vanish at any points from $A_{\lambda_5,L^{(5)}}$. Notice that the coefficients of the linear forms L^*_j are infinitely close to the corresponding coefficients of the linear forms $L^{(5)}_j$, $j\in\{0,\sigma+1,\dots,n\}$. Hence by Lemma 17 [13] and the implicit function theorem, see the last observation related to property (D) at the beginning of Section 1, every irreducible component of $(W_{\lambda_*}\setminus\mathcal{Z}(e_0,\dots,e_r))\cap\mathcal{Z}(e^*_{s+1},\dots,e^*_\sigma)$ contains a point from A_{λ_*,L^*}, and $e^*_0=\widetilde{L}_0(e^*_0,e^*_{\sigma+1},\dots,e^*_n)$ does not vanish at any points from A_{λ_*,L^*}. Therefore, every irreducible component of $W'(c_*,\lambda_*,L^*,*)$ contains a point from $q_{c_*,\lambda_*,L^*,*}(A_{\lambda_*,L^*})$, and X_0 does not vanish at any points from $q_{c_*,\lambda_*,L^*,*}(A_{\lambda_*,L^*})$. This implies the required assertion.

Let us prove the existence of the required solution of system (37). By Lemma 4, Lemma 6 and the algorithm of reduction to integer coefficients, cf. the proof of the existence of (c_*,λ_*,L^*) in Lemma 17, there are $\tau^*_i\in k(\varepsilon_3)$, which are infinitesimals with respect to the field k or zeroes, such that (c_*,λ_*,L^*) satisfies conditions of Lemma 4, Lemma 6 (for any line $\mathfrak{l}$ containing (c_*,λ_*,L^*)) in place of $(c_0,\lambda_0,L^{(0)})$, and $\deg W'(c_*,\lambda_*,L^*)=\deg W'(c_1,\lambda_1,L^{(1)})$, $\deg W'(c_*,\lambda_*,L^*,*)=\deg W'(c_1,\lambda_1,L^{(1)},*)$. We have $\deg W'(c_*,\lambda_*,L^*,*)>\deg W'(c_5,\lambda_5,L^{(5)},*)$. Hence as we have proved the Bézout theorem implies the existence of a point $z\in W'(c_*,\lambda_*,L^*,*)\cap\mathcal{Z}(X_0+t'X_1+t''X_2)$, which is not infinitely close to S'. Further, there exists a point $y\in W'(c_*,\lambda_*,L^*,*)$ with the coordinates from $\overline{k(\varepsilon_3)}$ such that y is infinitely close to z with respect to the field k, the inverse image $q^{-1}_{c_*,\lambda_*,L^*,*}(y)\neq\emptyset$ and $X_0(y)\neq 0$. Put $y^*_0=((X_0+t'X_1+t''X_2)/X_0)(y)$. Then $y^*_0\in\overline{k(\varepsilon_3)}$ is infinitesimal with respect to the field k. Let $y_1\in q^{-1}_{c_*,\lambda_*,L^*,*}(y)$. By previous considerations the point y_1 is infinitely close to $\mathcal{Z}(e_0,\dots,e_r)$. Thus, system (37) has a solution satisfying (38) (with k_3 in place of k_2).

Conversely, if (37) has a solution satisfying (38) then by the previous considerations and the Bézout theorem $\deg W'(c_*, \lambda_*, L^*, *) > \deg W'(c_5, \lambda_5, L^{(5)}, *)$. The lemma is proved.

Let us return to the description of the general step of the considered recursion. Notice that system (37) satisfies conditions 1)–4) from Section 2. Hence using Lemma 16 and Remark 11 we decide whether system (37) has a solution satisfying (38), and if such a solution exists we construct it. Suppose that such a solution does not exist. Then by Lemma 19 the degree $\deg W'(c_1, \lambda_1, L^{(1)}, *) = \deg W'(c_5, \lambda_5, L^{(5)}, *)$ and the considered step is final. In this case we can decide whether $\deg W'(c_1, \lambda_1, L^{(1)}) = \deg W'(c_1, \lambda_1, L^{(1)}, *)$. If $\deg W'(c_1, \lambda_1, L^{(1)}) \neq \deg W'(c_1, \lambda_1, L^{(1)}, *)$ then condition 4) from (*) does not hold for $(c_1, \lambda_1, L^{(1)})$, and hence (*) also is not satisfied for $(c_1, \lambda_1, L^{(1)})$. If $\deg W'(c_1, \lambda_1, L^{(1)}) = \deg W'(c_1, \lambda_1, L^{(1)}, *)$ then condition 4) from (*) holds for $(c_1, \lambda_1, L^{(1)})$, and hence (*) also is fulfilled for $(c_1, \lambda_1, L^{(1)})$.

Suppose that we have constructed a solution of system (37) satisfying (38), see the statement of Lemma 19. Then $\deg W'(c_*, \lambda_*, L^*, *) > \deg W'(c_5, \lambda_5, L^{(5)}, *)$ by this lemma. Similarly to the previous recursions the conditions of Lemma 4, Lemma 6 hold for (c_*, λ_*, L^*) in place of $(c_0, \lambda_0, L^{(0)})$. By Lemma 4, Lemma 6 and the observations from the beginning of the section one can apply the algorithm of reduction to integer coefficients, see [12], Proposition 2, and replace subsequently all τ_i^* by integers τ_i' of length $O(n \log d + (n - \sigma) \log(d'))$. Denote by (c', λ, L') the obtained point. Then $L' \in \mathcal{U}_0''$, the number of elements $\#A_{\lambda, L'} = \deg p_v' \deg W'$, $A_{\lambda, L'} \cap \mathcal{Z}(e_0') = \emptyset$, and $A_{\lambda, L'} \cap \overline{V \setminus W} = \emptyset$, the dimensions $\dim W_\lambda = n - \sigma$, $\dim(W_\lambda \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e_{\sigma+1}', \ldots, e_{n-1}') = 1$, the number of elements $\#(e_{n+1}'/e_0')(A_{\lambda, L'}) = \#(e_{5,n+1}/e_0^{(5)})(A_{\lambda_5, L^{(5)}})$, the morphisms $p_{\lambda, L'}$, $p_{\lambda, L', *}$ are dominant, $\deg W'(c', \lambda, L') = \deg W'(c_5, \lambda_5, L^{(5)})$, and $\deg W'(c', \lambda, L', *) \geq \deg W'(c_*, \lambda_*, L^*, *)$. Let us apply Lemma 8 to the case when $\iota(\mathbb{A}^1(\overline{k}))$ is the line containing (c', λ, L') and $(c_5, \lambda_5, L^{(5)})$, and $\iota(0) = (c_5, \lambda_5, L^{(5)})$. Then using Theorem 2 (f) [13], Lemma 8 and Lemma 4, Lemma 6 we construct a point (c'', λ', L'') which is similar to $(c_5, \lambda_5, L^{(5)})$, and such that $\deg W'(c'', \lambda', L'', *) \geq \deg W'(c', \lambda, L', *) > \deg W'(c_5, \lambda_5, L^{(5)}, *)$. After that we replace $(c_5, \lambda_5, L^{(5)})$ by (c'', λ', L'') and proceed to the next step of the recursion. The general step and all the recursion are described. The decision algorithm for property (*) is described completely. The required estimation for the working time of this algorithm (it is the same as for assertion (e) of Theorem 2, see the Introduction) follows immediately from the estimates for working times of the applied algorithms.

4 Monodromy and irreducibility

The most part of this section and its notations are independent from the ones of the Introduction and other sections. Let $n \geq 1$ and $1 \leq s \leq n$ be integers. Let k be an arbitrary field of zero–characteristic, V_s be a defined over k projective algebraic variety in $\mathbb{P}^n(\overline{k})$, $1 \leq s \leq n - 1$, and all the irreducible components of V_s have the same dimension $n - s$. Let $T_0, T_{s+1}, T_{s+2}, \ldots, T_{n+1}$ be linear forms in $X_0, \ldots, X_n$ with coefficients from k. Consider the defined over k morphisms

$$p : V_s \setminus \mathcal{Z}(T_0, T_{s+1}, T_{s+2}, \ldots, T_n) \to \mathbb{P}^{n-s}(\overline{k}),$$

$$(X_0 : \ldots : X_n) \mapsto (T_0 : T_{s+1} : T_{s+2} : \ldots : T_n), \tag{39}$$
$$p' : V_s \setminus \mathcal{Z}(T_0, T_{s+1}, T_{s+2}, \ldots, T_{n+1}) \to \mathbb{P}^{n-s+1}(\overline{k}),$$
$$(X_0 : \ldots : X_n) \mapsto (T_0 : T_{s+1} : T_{s+2} : \ldots : T_{n+1}) \tag{40}$$

(one should not confuse these morphisms p and p' with the ones from the Introduction). In this section we shall suppose that $T_0, T_{s+1}, T_{s+2}, \ldots, T_n$, (respectively $T_0, T_{s+1}, T_{s+2}, \ldots, T_{n+1}$) are the coordinate functions on $\mathbb{P}^{n-s}(\overline{k})$, (respectively $\mathbb{P}^{n-s+1}(\overline{k})$). We shall assume that the morphism p is dominant, i.e., by definition for every irreducible component W of V_s the restriction $p|_W$ is a dominant morphism. Then the closure in the Zariski topology

$$\overline{p'(V_s \setminus \mathcal{Z}(T_0, T_{s+1}, T_{s+2}, \ldots, T_{n+1}))} = \mathcal{Z}(\Phi),$$

where $\Phi \in k[T_0, T_{s+1}, T_{s+2}, \ldots, T_{n+1}]$ is a homogeneous nonzero separable polynomial which is uniquely defined up to a nonzero factor from k. We fix Φ. Denote by $q : V_s \setminus \mathcal{Z}(T_0, T_{s+1}, T_{s+2}, \ldots, T_n) \to \mathcal{Z}(\Phi)$ the morphism induced by (40). Denote by $k(V_s)$ and $k(\mathcal{Z}(\Phi))$ the algebras of defined over k rational functions on V_s and $\mathcal{Z}(\Phi)$ respectively (by definition these algebras are isomorphic to the direct products of fields of rational functions of all defined over k components of V_s and $\mathcal{Z}(\Phi)$ respectively). We shall suppose that the morphism q is a birational isomorphism in the sense that it induces the isomorphism of separable algebras $k(\mathcal{Z}(\Phi)) \simeq k(V_s)$. Let $r : \mathcal{Z}(\Phi) \setminus \mathcal{Z}(T_0, T_{s+1}, T_{s+2}, \ldots, T_n) \to \mathbb{P}^{n-s}$ be the morphism of the linear projection to the coordinates $T_0, T_{s+1}, T_{s+2}, \ldots, T_n$. Hence $p = r \circ q$. The morphism r is defined over k.

By definition the degree $\deg p$ of the morphism p is the degree of the extension of algebras of rational functions $[k(V_s) : k(\mathbb{P}^{n-s}(\overline{k}))] = \dim_{k(\mathbb{P}^{n-s}(\overline{k}))} k(V_s)$. Similarly one defines the degree of an arbitrary dominant morphism. In particular, $\deg r$ is defined. We have $\deg p = \deg r = \deg_{T_{n+1}} \Phi$ since q is a birational isomorphism. Conversely, if $\deg p = \deg_{T_{n+1}} \Phi$ then q is a birational isomorphism. Denote $\deg p = D$. Hence the integer $D \geq 1$.

Denote by $R = \mathrm{Res}_{T_{n+1}}(\Phi, \partial\Phi/\partial T_{n+1}) \in k[T_0, T_{s+1}, T_{s+2}, \ldots, T_n]$ the discriminant with respect to T_{n+1} of the polynomial Φ, see the remark above about the definition of discriminant at the beginning of the Introduction. Hence $0 \neq R$ is a homogeneous with respect to $T_0, T_{s+1}, T_{s+2}, \ldots, T_n$ polynomial. Let R_0 be square free part of R, i.e., R_0 is separable and $\mathcal{Z}(R) = \mathcal{Z}(R_0)$. The polynomial R_0 is uniquely defined up to a nonzero factor from k. We choose and fix R_0. Let $\Phi_D \in k[T_0, T_{s+1}, \ldots, T_n]$ be the coefficient in T_{n+1}^D of the polynomial Φ. Hence Φ_D divides R.

Denote by $A^{(s)}$ the set of isolated points of $V_s \cap \mathcal{Z}(T_{s+1}, \ldots, T_n)$ in $\mathbb{P}^n(\overline{k})$. Consider the condition

$$A^{(s)} \cap \mathcal{Z}(T_0) = \emptyset, \ \& \ \#(T_{n+1}/T_0)(A^{(s)}) = \deg p. \tag{41}$$

Let $T_0, T_{s+1}, T_{s+2}, \ldots, T_{n+1}$ be arbitrary linear forms with coefficients from k such that p is dominant and condition (41) holds (we do not suppose now in advance that q is birational). Then by Lemma 2 [12] the morphism q is a birational isomorphism. We shall suppose till the end of the proof of Theorem 3 that p is dominant and condition (41) holds.

LEMMA 20 *Let V_s, p, q, D and Φ, Φ_D be as above. Suppose that for a point $z \in \mathbb{P}^{n-s}(\overline{k}) \setminus \mathcal{Z}(T_0, T_{s+1}, \ldots, T_n)$, the number of elements $\#p^{-1}(z) = D$. Then $\Phi_D(z) \neq 0$.*

PROOF The proof of this lemma is similar to the one of Lemma 2 with $d' = 1$. It is sufficient to replace in Lemma 2 W $p_{\lambda,L'}$, $r_{c',\lambda,L'}$, $q_{c',\lambda,L'}$ by V_s, p, q, r. The only difference here is that V_s is not necessarily irreducible over k. The lemma is proved.

Suppose that the morphism p is regular dominant and finite. Then $p'(V_s) = q(V_s) = \mathcal{Z}(\Phi)$, $\deg \Phi = \deg_{T_{n+1}} \Phi = D$, $\deg p = \deg V_s = \deg q(V_s) = D$, and the morphisms q and r are also finite. In this case we choose leading coefficient $\mathrm{lc}_{T_{n+1}} \Phi = 1$. Now $A^{(s)} = V_s \cap \mathcal{Z}(T_{s+1}, \ldots, T_n)$. Consider the condition

$$A^{(s)} = V_s \cap \mathcal{Z}(T_{s+1}, \ldots, T_n),\ \&\ A^{(s)} \cap \mathcal{Z}(T_0) = \emptyset,\ \&\ \#(T_{n+1}/T_0)(A^{(s)}) = \deg V_s. \tag{42}$$

If $T_0, T_{s+1}, T_{s+2}, \ldots, T_{n+1}$ are arbitrary linear forms with coefficients from k satisfying (42) (we do not suppose now that p is dominant and q is birational) then the morphism p is regular dominant finite and q is birational finite, see e.g., [7], Section 2. Besides that, (42) implies that $\#A^{(s)} = D$ and the intersection $V_s \cap \mathcal{Z}(T_{s+1}, \ldots, T_n)$ is transversal at each point by the Bézout theorem. Condition (41) is a consequence of (42).

For an arbitrary polynomial $g \in \overline{k}[T_0, T_{s+1}, \ldots, T_{n+1}]$ set $g_* = g(T_0, 0, \ldots, 0, T_n, T_{n+1})$. We shall suppose in what follows that $R_{0,*}$ is nonzero. We shall assume without loss of generality that T_0 does not divide $R_{0,*}$ replacing if it is necessary T_0 by $T_0 + \lambda T_n$ with an appropriate integer λ. Since $R_{0,*} \neq 0$ the polynomial $\Phi_{D,*} \neq 0$. Hence $\deg_{T_{n+1}} \Phi_* = \deg_{T_{n+1}} \Phi$. Let $\varphi \in k[T_0, T_n]$ be the greatest common divisor of the coefficients in T_{n+1}^j, $0 \leq j \leq D$, of the polynomial Φ_*. The polynomial φ is uniquely defined up to a nonzero factor from k. Let us choose and fix φ. Set $\Phi_+ = \Phi_*/\varphi$, $R_+ = R_*/\varphi^{2D-1}$. Hence $\deg_{T_{n+1}} \Phi_+ = \deg_{T_{n+1}} \Phi$.

REMARK 12 *Actually by Lemma 11 [12] the equality* $\deg_{T_{n+1}} \Phi = \deg_{T_{n+1}} \Phi_+$ *follows directly from (41)*

Let us identify $\mathcal{Z}(T_{s+1}, \ldots, T_{n-1}) \subset \mathbb{P}^n(\overline{k})$ (respectively $\mathbb{P}^{n-s+1}(\overline{k})$, $\mathbb{P}^{n-s}(\overline{k})$) with $\mathbb{P}^{s+1}(\overline{k})$ (respectively $\mathbb{P}^2(\overline{k})$, $\mathbb{P}^1(\overline{k})$). Denote by $V_{s,+}$ the union of all irreducible components W of $V_s \cap \mathcal{Z}(T_{s+1}, \ldots, T_{n-1}) \subset \mathbb{P}^{s+1}(\overline{k})$ such that $W \not\subset \mathcal{Z}(g)$ for any $0 \neq g \in k[T_0, T_n]$. Let us repeat the described construction to $V_{s,+} \subset \mathbb{P}^{s+1}(\overline{k})$ in place of $V_s \subset \mathbb{P}^n(\overline{k})$. Now n and s are replaced by $s+1$ and s respectively. By Lemma 2 [12], by condition (41), and since $\deg_{T_{n+1}} \Phi_+ = \deg_{T_{n+1}} \Phi$, one can replace the linear forms $T_0, T_{s+1}, T_{s+2}, \ldots, T_{n+1}$ by T_0, T_n, T_{n+1}. The new set $A_+^{(s)} = A^{(s)}$ is the same as it was initially. We get the objects p_+, q_+, r_+, Φ_+, R_+, $R_{0,+}$, $A_+^{(s)}$ which are similar to p, q, r, Φ, R, R_0, $A^{(s)}$ and satisfy the analogous properties by the similar reasons. In particular, $\deg p = \deg p_+$. Note that if p is finite then $\varphi \in k$ and $V_{s,+} = V_s \cap \mathcal{Z}(T_{s+1}, \ldots, T_{n-1})$.

Now we can formulate our irreducibility criterion.

THEOREM 3 *Let $n \geq 3$ and $1 \leq s \leq n-2$ be integers. Let V_s be an equidimensional projective algebraic variety in $\mathbb{P}^n(\overline{k})$ with $\dim V_s = n-s$. Let T_0, T_{s+1}, $T_{s+2}, \ldots, T_{n+1}$ be linear forms from $k[X_0, \ldots, X_n]$ such that the morphism p is dominant and condition (41) holds. Let the polynomial Φ and its discriminant R be as above. Suppose that the polynomial $R_{0,*} = R_0(T_0, 0, \ldots, 0, T_n)$ is nonzero and separable. Then $\deg p = \deg p_+$ and the irreducible over $\overline{k}$ (respectively k)*

components of algebraic varieties V_s and $V_{s,+}$ are in the one–to–one correspondence by the rule $W \mapsto W_1$ where W_1 is the unique irreducible component of $V_{s,+}$ such that $W_1 \subset W \cap \mathcal{Z}(T_{s+1}, \ldots, T_{n-1})$. In particular, if p is finite then $V_{s,+} = V_s \cap \mathcal{Z}(T_{s+1}, \ldots, T_{n-1})$ and $W_1 = W \cap \mathcal{Z}(T_{s+1}, \ldots, T_{n-1})$.

PROOF Since V_s and $V_{s,+}$ are defined over k it is sufficient to prove only the assertion related to the irreducible over $\overline{k}$ components. Denote by N (respectively N_+) the number of all irreducible over $\overline{k}$ components of V_s (respectively $V_{s,+}$). The equality $A^{(s)} = A^{(s)}_+$ entails that for every irreducible component W of V_s there is an irreducible component W_1 of $V_{s,+}$ such that W_1 is an irreducible component of $W \cap \mathcal{Z}(T_{s+1}, \ldots, T_{n-1})$. Condition (41), Lemma 2 [12] and the equality $A^{(s)} = A^{(s)}_+$ imply that the intersection $V_s \cap \mathcal{Z}(T_{s+1}, \ldots, T_{n-1})$ is transversal at each irreducible component of $V_{s,+}$. Hence the inequality $N_+ \geq N$ holds, and if $N_+ = N$ then the required one–to–one correspondence takes place. So it is sufficient to prove that $N = N_+$. Note that N (respectively N_+) is the number of irreducible over $\overline{k}$ components of $\mathcal{Z}(\Phi)$ (respectively $\mathcal{Z}(\Phi_+)$) since q (respectively q_+) is a birational isomorphism.

We shall assume without loss of generality that k is a subfield of the field of complex numbers $\mathbb{C}$. Recall that $R_{0,*} \in k[T_0, T_n]$ is a nonzero homogeneous polynomial and T_0 does not divide $R_{0,*}$. Let $y_1, \ldots, y_u$ be all the pairwise different roots of the polynomial $R_{0,*}(1, T_n)$. Hence $u = \deg R_0$ by the condition of the lemma. Let us choose a point $y \in \mathbb{C}$ such that $R_{0,*}(y) \neq 0$ and for all $1 \leq i_1 \neq i_2 \leq u$ the points y, y_{i_1}, y_{i_2} do not belong to the same line in $\mathbb{C}$ (considered as $\mathbb{R}^2$). Let $0 < m_1 \in \mathbb{R}$. Denote by b_i the closed disc with the radius m_1 around y_i, $1 \leq i \leq u$. Let us choose m_1 so small that any line in $\mathbb{C}$ trough y intersects at most one disc b_i, $1 \leq i \leq u$, and $|y_i - y_j| > 3m_1$, $|y - y_i| > m_1$ for all $1 \leq i \neq j \leq u$. Denote by c_i, $1 \leq i \leq u$, the circumference with the radius m_1 around y_i. Denote by z_i the intersection of the interval $[y, y_i]$ with c_i. Denote by e_i the union of the interval $[y, z_i]$ and c_i. Denote by γ_i, $1 \leq i \leq u$ the path from y to z_i along $[y, z_i]$, then by c_i around y_i counterclockwise, and back from z_i to y along the interval $[z_i, y]$. Set $e = \cup_{1 \leq i \leq u} e_i$ and $E_z = (T_{n+1}/T_0)(r_+^{-1}(1 : z))$ for every $z \in \mathbb{C}$. Note that for every $z \in \mathbb{C} \setminus \mathcal{Z}(R)$ the number of elements $\#E_z = D$.

Set $E = E_y$. Hence $\#E = D$. Let $a \in E$. Denote by ξ_a the analytical function in Z defined in a neighborhood of y such that $\Phi_+(1, Z, \xi_a) = 0$ and $\xi_a(y) = a$. The result of the analytic continuation of ξ_a along γ_i is $\xi_{\sigma_i(a)}$ where σ_i is the uniquely defined permutation of E. Denote by S_D the group of permutations of E and by G the subgroup of S_D generated by all σ_i, $1 \leq i \leq u$. Then it is known, see, e.g., [4], [16], that the number N_+ of irreducible over $\overline{k}$ components of $V_{s,+}$ coincides with the number of classes of transitivity of G on E.

Let $\mu_i, \nu_i \in k$, $s+1 \leq i \leq n-1$ and $0 < \delta \in \mathbb{R}$. Denote by $\mathcal{A}_\delta$ the condition $\sum_{s+1 \leq i \leq n-1}(|\mu_i|^2 + |\nu_i|^2) < \delta$. Set $\widetilde{T}_i = T_i$, $i = 0, n, n+1$ and $\widetilde{T}_i = T_i - \mu_i T_0 - \nu_i T_n$ for all $s+1 \leq i \leq n-1$. By Lemma 2 [12] there is $0 < \delta_0 \in \mathbb{R}$ such that under the condition $\mathcal{A}_{\delta_0}$ the linear forms $\widetilde{T}_i$ satisfy the same properties (considered above) as T_i, $i \in \{0, s+1, s+2, \ldots, n+1\}$ do. In what follows we shall suppose that $\mathcal{A}_{\delta_0}$ holds. Then for $\widetilde{T}_i$ the objects $\widetilde{A}_s$, $\widetilde{\Phi}$, $\widetilde{R}$ and so on are defined. They correspond to $A^{(s)}$, Φ, R and so on. Note that

$\Phi = \widetilde{\Phi}$, $R = \widetilde{R}$ (as polynomials in $X_0, \ldots, X_n$), $N = \widetilde{N}$, $u = \widetilde{u} = \deg R_0$ and

$$\widetilde{\Phi}_* = \Phi(T_0, \mu_{s+1}T_0 + \nu_{s+1}T_n, \ldots, \mu_{n-1}T_0 + \nu_{n-1}T_n, T_n, T_{n+1}), \quad (43)$$

$$\widetilde{R}_* = R(T_0, \mu_{s+1}T_0 + \nu_{s+1}T_n, \ldots, \mu_{n-1}T_0 + \nu_{n-1}T_n, T_n), \quad (44)$$

$$\widetilde{\Phi}_* = \widetilde{\varphi}\widetilde{\Phi}_+, \quad (45)$$

where $\widetilde{\varphi} \in k[T_0, T_n]$ divides $\widetilde{\Phi}_{D,*}$ and $\widetilde{\Phi}_{D,*}$ divides $\widetilde{R}_{0,*}$. Hence by Lemma 1 [11] applied to the polynomial $R_*(1, Z)/\mathrm{lc}_Z(R_*(1, Z))$ there is $0 < \delta_1 < \delta_0$ such that $\mathcal{A}_{\delta_1}$ entails $|\widetilde{y}_i - y_i| < m_1/2$ for every $1 \leq i \leq u$. We shall suppose in what follows that $\mathcal{A}_{\delta_1}$ holds. Now we choose $\widetilde{y} = y$ and $\widetilde{m}_1 = m_1/2$.

Let $m_2 = \inf\{|a_1 - a_2| : a_1, a_2 \in E_z \,\&\, a_1 \neq a_2 \,\&\, z \in e\}$. Then $m_2 > 0$ due to the compactness of e. By Lemma 20 for every $z \in e$ the polynomial $\Phi_*(1, z, Z)/\mathrm{lc}_Z\Phi_*(1, z, Z) = \Phi_+(1, z, Z)/\mathrm{lc}_Z\Phi_+(1, z, Z)$. Hence by Lemma 1 [11] applied to the polynomials $\Phi_*(1, z, Z)/\mathrm{lc}_Z\Phi_*(1, z, Z)$, $z \in e$, by (43), (45) and due to the compactness of e there is $0 < \delta_2 \leq \delta_1$ such that $\mathcal{A}_{\delta_2}$ entails that for every $z \in e$ for every $a \in E_z$ there is the unique $\widetilde{a} \in \widetilde{E}_z$ such that $|a - \widetilde{a}| < m_2/3$. We shall suppose in what follows that $\mathcal{A}_{\delta_2}$ holds.

Now there is the bijection $\tau : E \to \widetilde{E}$, $a \mapsto \widetilde{a}$, $|a - \widetilde{a}| < m_2/3$. Hence τ induces the isomorphism $S_D \to \widetilde{S}_D$. By our construction the path γ_i is homotopic to $\widetilde{\gamma}_i$ in $\mathbb{C} \setminus \mathcal{Z}(\widetilde{R}_*)$, for every $1 \leq i \leq u$. Therefore, $\widetilde{\sigma}_i = \tau \circ \sigma_i \circ \tau^{-1}$, for every $1 \leq i \leq u$. Hence the groups $G \simeq \widetilde{G}$ and $N_+ = \widetilde{N}_+$.

Denote by $\mathcal{L}_{n-s+1}$, (respectively $\mathcal{L}_{n-s-1}$) the vector space of linear forms with coefficients from $\overline{k}$ with the basis $T_0, T_{s+1}, \ldots, T_n$ (respectively $T_{s+1}, , \ldots, T_{n-1}$). Let $\mathcal{L}$ be a linear subspace of the space $\mathcal{L}_{n-s+1}$. Denote by $I(\mathcal{L}) \subset \overline{k}[T_0, T_{s+1}, , \ldots, T_{n+1}]$ the ideal generated by $\mathcal{L}$. For every polynomial $g \in \overline{k}[T_0, T_{s+1}, \ldots, T_{n+1}]$ denote

$$g \bmod \mathcal{L} = g \bmod I(\mathcal{L}) \in \big(\overline{k}[T_0, T_{s+1}, , \ldots, T_n]/I(\mathcal{L})\big)[T_{n+1}] = A_{\mathcal{L}}.$$

The ring $A_{\mathcal{L}}$ is isomorphic to the ring of polynomials in $n-s+2-\dim\mathcal{L}$ variables over $\overline{k}$. For every $\psi \in A_{\mathcal{L}}$ the degree $\deg_{T_{n+1}}\psi$ is defined in the natural way. According to the first Bertini theorem applied subsequently $n - s - 1$ times (one should delete each time fixed components) in any neighborhood in the classic topology of the vector subspace $\mathcal{L}_{n-s-1} \subset \mathcal{L}_{n-s+1}$ (the neighborhood of $\mathcal{L}_{n-s-1}$ is considered here in the corresponding Grassman variety) there is an $(n - s - 1)$–dimensional vector subspace $\mathcal{L} \subset \mathcal{L}_{n-s+1}$ satisfying the following properties.

- The polynomial $R \bmod \mathcal{L} \neq 0$. Hence $\Phi_D \bmod \mathcal{L} \neq 0$, $\deg_{T_{n+1}}\Phi = D$, and the polynomial $\Phi \bmod \mathcal{L}$ does not have multiple factors ψ with $\deg_{T_{n+1}}\psi \geq 1$.

- For every irreducible over $\overline{k}$ factor Ψ of Φ with $\deg_{T_{n+1}}\Psi \geq 1$ the polynomial $\Psi \bmod \mathcal{L}$ has only one irreducible over $\overline{k}$ factor ψ with $\deg_{T_{n+1}}\psi \geq 1$, and hence $\deg_{T_{n+1}}\psi = \deg_{T_{n+1}}\Psi$.

Therefore, one can choose $\mathcal{L}$ to be a linear subspace generated by $\widetilde{T}_{s+1}, \ldots, \widetilde{T}_{n-1}$ for some μ_i, ν_i, $s + 1 \leq i \leq n - 1$, satisfying $\mathcal{A}_{\delta_2}$. In this case $\widetilde{N}_+ = \widetilde{N}$. Thus, $N = \widetilde{N} = \widetilde{N}_+ = N_+$. The theorem is proved.

Let $n \geq 1$ and $1 \leq s \leq n-1$ be integers. Till the end of the section we shall not suppose that condition (41) holds. We shall assume only that p is a dominant morphism and q is a birational isomorphism. Now let $L \in k[T_0, T_{s+1}, T_{s+2}, \ldots, T_n]$ be a nonzero linear form. Denote $A_L = k[T_0, T_{s+1}, T_{s+2}, \ldots, T_n]/(L)$. So A_L is isomorphic to the ring of polynomials in $n-s$ variables. The following conditions (i), (ii) and (iii) are pairwise equivalent.

(i) $R_0 \bmod L \in A_L$ is separable,

(ii) The intersection $\mathcal{Z}(R) \cap \mathcal{Z}(L)$ is transversal in $\mathbb{P}^{n-s}(\overline{k})$.

(iii) The degree $\deg \mathcal{Z}(R) = \deg(\mathcal{Z}(R) \cap \mathcal{Z}(L))$ in $\mathbb{P}^{n-s}(\overline{k})$.

More than that there is a nonempty open in the Zariski topology subset $\mathcal{U}_{s,1}$ of the space of linear forms in $T_0, T_{s+1}, \ldots, T_n$ such that for every $L \in U_{s,1}$ conditions (i)–(iii) holds.

Denote by $V_{s,L}$ the union of all the irreducible components W of $V_s \cap \mathcal{Z}(L)$ such that $W \not\subset \mathcal{Z}(g)$ for any nonzero element $g \in k[T_0, T_{s+1}, \ldots, T_n]/(L)$.

The following conditions (iv) and (v) are equivalent by Lemma 2 [12].

(iv) The dimension $\dim V_{s,L} = n-s-1$, further p induces the rational dominant morphism $p_L : V_{s,L} \to \mathbb{P}^{n-s-1}(\overline{k})$ (i.e., $p|_W$ is a rational dominant morphism for every irreducible component W of $V_{s,L}$), and $\deg p_L = \deg p$.

(v) The dimension $\dim V_{s,L} = n-s-1$ and there is a point $z \in \mathcal{Z}(L) \subset \mathbb{P}^{n-s}(\overline{k})$ such that the number of elements $\#p^{-1}(z) = \deg p$.

More than that there is a nonempty open in the Zariski topology subset $\mathcal{U}_{s,2}$ of the space of linear forms in $T_0, T_{s+1}, \ldots, T_n$ such that for every $L \in U_{s,2}$ conditions (iv), (v) holds.

Denote by $V_{s,1}$ the union of all irreducible over $\overline{k}$ closed subvarieties W of V_s such that $\dim W = n-s-1$ and the dimension of the closure

$$\dim \overline{p(W \setminus \mathcal{Z}(T_0, T_{s+1}, \ldots, T_n))} < n-s-1.$$

Note that there is only a finite number of such subvarieties W, and the variety $V_{s,1}$ is defined over k (it may be empty). The conditions (vi) and (vii) are equivalent.

(vi) $V_{s,L} = V_s \cap \mathcal{Z}(L)$.

(vii) $\dim V_{s,1} \cap \mathcal{Z}(L) \leq n-s-2$, i.e., for every irreducible component W of $V_{s,1}$ the algebraic variety $W \cap \mathcal{Z}(L) \neq W$.

Besides that, there is a nonzero linear form L satisfying (vi) and (vii) if and only if $\dim(V_s \cap \mathcal{Z}(T_0, T_{s+1}, \ldots, T_n)) \leq \dim(V_s) - 2$. More than that if the last condition holds then there is a nonempty open in the Zariski topology subset $\mathcal{U}_{s,3}$ of the space of linear forms in $T_0, T_{s+1}, \ldots, T_n$ such that for every $L \in U_{s,3}$ conditions (vi) and (vii) are satisfied.

COROLLARY 1 *Let $n \geq 1$ and $1 \leq s \leq n-1$ be integers. Let V_s be an equidimensional projective algebraic variety in $\mathbb{P}^n(\overline{k})$ with $\dim V_s = n-s$. Let $T_0, T_{s+1}, T_{s+2}, \ldots, T_{n+1}$ be linear forms from $k[X_0, \ldots, X_n]$ such that the morphism p is dominant and q is birational, see above. Let $L \in k[T_0, T_{s+1}, T_{s+2}, \ldots, T_n]$ be a*

nonzero linear form. Suppose that the equivalent conditions (i)–(iii) hold and the equivalent conditions (iv) and (v) are satisfied, see above. Then the irreducible over $\overline{k}$ (respectively irreducible over k) components of V and $V_{s,L}$ are in the one–to–one correspondence by the rule $W \mapsto W_1$ where W_1 is the uniquely defined component $V_{s,L}$ such that $W_1 \subset W \cap \mathcal{Z}(L)$. Besides that, if the equivalent conditions (vi) and (vii) hold then $\dim(V_s \cap \mathcal{Z}(T_0, T_{s+1}, \ldots, T_n)) \leq \dim(V_s) - 2$. *The set of nonzero linear forms L satisfying (i)–(iii) (respectively (iv)–(vi)) is a nonempty open in the Zariski topology subset $\mathcal{U}_{s,1}$ (respectively $\mathcal{U}_{s,2}$) of the space of all linear forms in $T_0, T_{s+1}, \ldots, T_n$. If* $\dim(V_s \cap \mathcal{Z}(T_0, T_{s+1}, \ldots, T_n)) \leq \dim(V_s) - 2$ *then the set of nonzero linear forms L satisfying (vi) and (vii) is a nonempty open in the Zariski topology subset $\mathcal{U}_{s,3}$ of the space of all linear forms in $T_0, T_{s+1}, \ldots, T_n$.*

PROOF Let $\mathbb{P}^{n+1}(\overline{k})$ has homogeneous coordinates $X_0, \ldots, X_{n+1}$. Let us identify $\mathbb{P}^n(\overline{k}) = \mathcal{Z}(X_{n+1}) \subset \mathbb{P}^{n+1}(\overline{k})$. Let $I(V_s)$ be homogeneous ideal of V_s. Denote by $\widetilde{V}_s \subset \mathbb{P}^{n+1}(\overline{k})$ the projective algebraic variety which is the set of all common zeroes of $I(V_s)$. Hence $V_s = \widetilde{V}_s \cap \mathcal{Z}(X_{n+1})$ in $\mathbb{P}^{n+1}(\overline{k})$. Set $\widetilde{T}_{s+1} = L$ (one should not confuse the tilde here and below with the one from the proof of the theorem). By (v) and Lemma 20 there are linear forms $\widetilde{T}_0, \widetilde{T}_{s+2}, \ldots, \widetilde{T}_n$ in $T_0, T_{s+1}, T_{s+2}, \ldots, T_n$ with coefficients from k such that the linear forms $\widetilde{T}_0, \widetilde{T}_{s+1}, \ldots, \widetilde{T}_n$ are linearly independent, the number of elements $\#r^{-1}(\mathcal{Z}(\widetilde{T}_{s+1}, \ldots, \widetilde{T}_n)) = \deg \Phi$ and $r^{-1}(\mathcal{Z}(\widetilde{T}_{s+1}, \ldots, \widetilde{T}_n)) \cap \mathcal{Z}(\widetilde{T}_0) = \emptyset$. Let us replace in (41) n, V_s, T_0, $T_{s+1}, \ldots, T_n$ and T_{n+1} by $n+1$, $\widetilde{V}_s$, $\widetilde{T}_0$, $\widetilde{T}_{s+1}, \ldots, \widetilde{T}_n$, X_{n+1} and T_{n+1} respectively. Then the obtained condition holds. We get the polynomials $\widetilde{\Phi}$, $\widetilde{R}$ corresponding $\widetilde{V}_s$ and the new linear forms according to the construction described before the Theorem 3. Obviously $\widetilde{\Phi} = \Phi$ and $\widetilde{R} = R$ as polynomials in $X_0, \ldots, X_{n+1}$. Now applying Theorem 3 in the considered situation we get all the assertions of the corollary. The corollary is proved.

5 Proof of Theorem 1

By Lemma 11 to prove (a) it is sufficient to show that $\mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$ is a subset of $\mathcal{U}_0$. Replacing the field k by $k(\tau_1, \ldots, \tau_{n_1})$ one can suppose without loss of generality that $n_1 = 0$, and $(c_1, \lambda_1) = (c', \lambda)$, where (c', λ) is from the Introduction and satisfies (*).

Let $L' \in \mathcal{U}^{(\mathrm{v})}_{c',\lambda,0}$. Using the Veronese isomorphism of degree d', see [15], we can suppose without loss of generality that $d' = 1$, i.e., that $e'_0, e'_{\sigma+1}, \ldots, e'_{n+1}$ are linear forms in $X_0, \ldots, X_n$ with coefficients from k. Let us replace in Theorem 3 V_s, s and all T_j by W_λ, σ and e'_j respectively and apply this theorem.

Then the irreducible over $\overline{k}$ components of W_λ and $W_{\lambda,n-1}(L')$ are in the one–to–one correspondence by the rule

$$E \mapsto \overline{E \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_{n-1}) \setminus \mathcal{Z}(e_0, \ldots, e_r)}. \tag{46}$$

The irreducible over $\overline{k}$ components of W and $W \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$ are in the one–to–one correspondence by the rule $E \mapsto E \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma)$, see [13]. The intersections $(W \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_{n-1})$ and $(W \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(L_{s+1}, \ldots, L_\sigma) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_{n-1})$ are transversal, see [13]. Therefore, the irreducible over $\overline{k}$ components of W and $W_{\lambda,n-1}(L')$ are in the one–to–one

correspondence by the rule

$$E \mapsto \overline{E \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma) \cap \mathcal{Z}(e'_{\sigma+1}, \dots, e'_{n-1}) \setminus \mathcal{Z}(e_0, \dots, e_r)}.$$

Thus, the irreducible over $\overline{k}$ components of W and $W_{n-1}(L')$ are in the one–to–one correspondence by (46). Hence $L' \in \mathcal{U}_0$. Assertion (a) is proved.

Let us prove (e). We can suppose without loss of generality that $n_1 = n_2 + 1$, and conditions of Lemma 12 hold. Denote for brevity $\Phi_i = \Phi_{c_i,\lambda_i,L^{(i)}}$, $R_i = R_{c_i,\lambda_i,L^{(i)}}$, $i = 1, 2$; $\overline{\Phi_1} = \Phi_1|_{\{\tau_{n_1} = \tau'_{n_1}\}}$, $\overline{R_1} = R_1|_{\{\tau_{n_1} = \tau'_{n_1}\}}$, $\overline{R_{1,0}} = R_{1,0}|_{\{\tau_{n_1} = \tau'_{n_1}\}}$, $\overline{R_{1,*}} = R_{1,*}|_{\{\tau_{n_1} = \tau'_{n_1}\}}$ (here the index 0 denotes taking square free part, the index $*$ is also defined in the Introduction). Set $\delta = \deg p'_v \deg W'$. Then by Lemma 12 we have $\overline{\Phi_1} = \varphi\Phi_2$, where $0 \neq \varphi \in \overline{k}[\tau_1, \dots, \tau_{n_2}, Z_0, Z_{\sigma+1}, \dots, Z_n]$. Hence $\deg_{Z_0,Z_{\sigma+1},\dots,Z_{n+1}} \Phi_{c_2,\lambda_2,L^{(2)}} \leq \deg_{Z_0,Z_{\sigma+1},\dots,Z_{n+1}} \Phi_{c_1,\lambda_1,L^{(1)}}$.

Further, $\overline{R_1} = \varphi^{2\delta-1} R_2$. Therefore, $\overline{R_{1,0}} \neq 0$ and $R_{2,0}$ divides $\overline{R_{1,0}}$. Hence $b(c_2, \lambda_2, L^{(2)}) \leq b(c_1, \lambda_1, L^{(2)})$.

We have $R_{1,*} \neq 0$ by Lemma 2. By Lemma 13 the polynomial $(\Phi_{1,*}/(\tau_{n_1} - \tau'_{n_1})^\gamma)|_{\{\tau_{n_1} = \tau'_{n_1}\}} = \varphi_1\Phi_{2,*}$, $(R_{1,*}/(\tau_{n_1} - \tau'_{n_1})^{\gamma(2\delta-1)})|_{\{\tau_{n_1} = \tau'_{n_1}\}} = \varphi_1^{2\delta-1} R_{2,*}$ where $\varphi_1 \in \overline{k}(\tau_1, \dots, \tau_{n_2}, Z_0, Z_n)$. By condition 4) from (*) for Φ_2 the element $\varphi_1 \in \overline{k}[\tau_1, \dots, \tau_{n_2}, Z_0, Z_n]$. Hence $\deg R_{1,*,0} \geq \deg R_{2,*,0}$. Thus, $a(c_2, \lambda_2, L^{(2)}) \leq a(c_1, \lambda_1, L^{(2)})$. Assertion (e) is proved.

Assertion (d) follows from Lemma 8 and the Bézout theorem immediately.

Let us prove (b) and (c) by the induction on n_1. Let $n_2 + 1 = n_1$, and the point (c_2, λ_2) is obtained from (c_1, λ_1) by the substitution an integer τ'_{n_1} for τ_{n_1} in (17), (18). By (d) there is τ'_{n_1} with length $O(n \log d + (n - \sigma) \log d')$ such that (c_2, λ_2, L') satisfies the same properties as (c_1, λ_1, L') does. Thus, one can replace (c_1, λ_1, L') by (c_2, λ_2, L'), and assertions (b) and (c) are proved by induction.

Let us prove (g). Consider the point (c_1, λ_1, U). If one extends the ground field then $U \in \mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$. Hence by the Bézout theorem and Lemma 8 using the algorithm of reduction to integer coefficients, see Proposition 2 [12], one can replace subsequently all the coefficients of the linear forms U_j, $j \in \{0, \sigma + 1, \dots, n\}$ by integers of lengths $O(n \log d + (n - \sigma) \log d')$ and obtain the required L'. Assertion (g) is proved.

Let us prove (f). The inequalities from (f) are particular cases of the ones from (e), see the definitions from the Introduction. Suppose that $\deg W'(c_2, \lambda_2) = \deg W'(c_1, \lambda_1)$ and $b(c_2, \lambda_2) = b(c_1, \lambda_1)$. Let $L' \in \mathcal{U}^{(\mathrm{v})}_{c_2,\lambda_2,0}$ By (c) we can suppose without loss of generality that $(c_2, \lambda_2) = (c', \lambda)$. Then by Lemma 9 property (*) holds for (c_1, λ_1, L'). Further, (ii) and (iii) from Definition 2 for (c', λ, L') imply (ii) and (iii) for (c_1, λ_1, L'). Thus, $L' \in \mathcal{U}^{(\mathrm{v})}_{c_1,\lambda_1,0}$. Assertion (f) is proved. The theorem is proved.

6 Proof of Theorem 2

Obviously, assertion (a) of this theorem follows from (e) and Definition 2. Assertion (b) follows from (a). Assertion (c) follows from (e) and Definition 2. Assertion (d) is a partial case of (c). So it is sufficient to prove only (e).

Notice that $\deg W'(c_1, \lambda_1, U) = \deg W'(c_1, \lambda_1)$ and $b(c_1, \lambda_1) = b(c_1, \lambda_1, U)$ by definition, and $b(c_1, \lambda_1, U) = a(c_1, \lambda_1, U)$ by Lemma 10. So it is sufficient

to prove all the statements from (e) apart of the ones related to computing $\deg W'(c_1,\lambda_1)$ and $b(c_1,\lambda_1)$. After that, to compute $\deg W'(c_1,\lambda_1)$ and $b(c_1,\lambda_1)$, one can take $L^{(1)}=U$.

Under conditions from Section 3 let $\deg W'(c_1,\lambda_1,L^{(1)}) = \deg W'(c_1,\lambda_1, L^{(1)},*)$ and $(c_5,\lambda_5,L^{(5)}) \in \mathcal{L}_{c_1,\lambda_1,L^{(1)}}$ be the element constructed by the decision algorithm from Section 3. Hence $\deg W'(c_1,\lambda_1,L^{(1)}) = \deg W'(c_5,\lambda_5,L^{(5)}) = \deg W'(c_5,\lambda_5,L^{(5)},*)$ and the both $(c_1,\lambda_1,L^{(1)})$, $(c_5,\lambda_5,L^{(5)})$ satisfy property $(*)$. Further, condition 5), see the Introduction, holds for $(c_5,\lambda_5,L^{(5)})$.

We shall describe a recursive algorithm by $(c_5,\lambda_5,L^{(5)})$. At the end of the recursion we shall obtain an element $(c_5,\lambda_5,L^{(5)})$ such that $a(c_5,\lambda_5,L^{(5)}) = a(c_1,\lambda_1,L^{(1)})$. As the base of the recursion we take the element $(c_5,\lambda_5,L^{(5)})$ constructed in Section 3.

Let us describe the general step of this recursion. Let us compute $a(c_5,\lambda_5, L^{(5)})$ and $R_{c_5,\lambda_5,L^{(5)},*,0}$, see the observations at the beginning of Section 3. Replacing if it is necessary $e_0^{(5)}$ by $e_0^{(5)}+\mu e_n^{(5)}$ for an appropriate integer μ with length $O(n\log d + (n-\sigma)\log(d'))$, we shall assume in what follows without loss of generality that Z_0 does not divide $R_{c_5,\lambda_5,L^{(5)},*,0}$. Denote for brevity $\xi_j = e_j^{(5)} - e_j^{(0)} + e_j^{(1)}$, $j\in\{0,\sigma+1,\ldots,n\}$, and $e_{n+1} = e_{5,n+1} - e_{0,n+1} + e_{1,n+1}$ (the homogeneous polynomials $e_j^{(5)}, e_j^{(0)}, e_j^{(1)}$, $e_{5,n+1}, e_{0,n+1}, e_{1,n+1}$ are defined in the Introduction and Section 3). Let $X^{(j)}$ be the family of variables $X_{j,0},\ldots,X_{j,n}$, $1\le j\le 4$. For an arbitrary polynomial p in $n+1$ variables denote for brevity $p(X^{(j)}) = p(X_{j,0},\ldots,X_{j,n})$. Set

$$
\begin{aligned}
&A_0 = \varepsilon_4 - |e_{n+1}(X^{(1)}) - e_{n+1}(X^{(2)})|^2, \quad A_1 = |e_{n+1}(X^{(1)})|^2 - \varepsilon_4^{-1},\\
&B_0 = \varepsilon_4 - |e_{n+1}(X^{(3)}) - e_{n+1}(X^{(4)})|^2, \quad B_1 = |e_{n+1}(X^{(3)})|^2 - \varepsilon_4^{-1},\\
&I_0 = \{1,2\}, \quad I_1 = \{1\}, \quad J_0 = \{3,4\}, \quad J_1 = \{3\}.
\end{aligned}
$$

Let $(w_1,w_2)\in\{(0,0),(0,1),(1,1)\}$. Consider the system of polynomial equations and inequalities with squares of absolute values with respect to the variables $\tau_1,\ldots,\tau_{n_1}$, $X_{j,0},\ldots,X_{j,n}$, $j\in I_{w_1}\cup J_{w_2}$ with coefficients from the field k_4

$$
\begin{cases}
\sum_{1\le i\le n_1}|\tau_i|^2 \le \varepsilon_1,\\
\sum_{\sigma+1\le i\le n-1}|\xi_i|^2 \le \varepsilon_2,\\
\varepsilon_3 \le |\xi_n(X^{(1)}) - \xi_n(X^{(3)})|^2 \le \varepsilon_2,\\
A_{w_1} = 0,\\
B_{w_2} = 0,\\
\xi_i(X^{(j)}) = 0, & j\in I_{w_1}\cup J_{w_2},\ \sigma+1\le i\le n-1,\\
\xi_n(X^{(j_1)}) = \xi_n(X^{(j_2)}), & (j_1,j_2)\in I_{w_1}^2\cup J_{w_2}^2,\\
\xi_0(X^{(j)}) = 1, & j\in I_{w_1}\cup J_{w_2},\\
L_j(X^{(j)}) = 0, & j\in I_{w_1}\cup J_{w_2},\ s+1\le j\le\sigma.
\end{cases}
\tag{47}
$$

LEMMA 21 *The integer $a(c_5,\lambda_5,L^{(5)}) < a(c_1,\lambda_1,L^{(1)})$ if and only if there is a pair $(w_1,w_2)\in\{(0,0),(0,1),(1,1)\}$ such that system (47) has a solution $X_{j,i} = x_{j,i}^*$, $0\le j\le n$, $j\in I_{w_1}\cup J_{w_2}$ satisfying the condition*

$$
(x_{j,0}^* : \ldots : x_{j,n}^*) \in W(\overline{k_5}), \quad j\in I_{w_1}\cup J_{w_2}. \tag{48}
$$

PROOF Suppose that $a(c_5, \lambda_5, L^{(5)}) < a(c_1, \lambda_1, L^{(1)})$. There is an element ε' such that $\varepsilon' > 0$ be an infinitesimal with respect to the field k_2 and ε_3 be an infinitesimal with respect to the field $k_2(\varepsilon')$. By the transfer principle, see [3], it is sufficient to construct a solution (48) of system (47) with the field $\overline{k(\varepsilon', \varepsilon_4)}$ in place of $\overline{k_4}$.

By Lemma 4, Lemma 6, Lemma 7 and the algorithm of reduction to integer coefficients, cf. the proof of the existence of (c_*, λ_*, L^*) in Lemma 17, there are $\tau_i^* \in k(\varepsilon')$ which are infinitesimal with respect to the field k such that for (c_*, λ_*, L^*) satisfies conditions of Lemma 4, Lemma 6, Lemma 7 (for any line l containing (c_*, λ_*, L^*)) in place of $(c_0, \lambda_0, L^{(0)})$, the degrees $\deg W'(c_*, \lambda_*, L^*) = \deg W'(c_1, \lambda_1, L^{(1)})$, $\deg W'(c_*, \lambda_*, L^*, *) = \deg W'(c_1, \lambda_1, L^{(1)}, *)$, and $a(c_*, \lambda_*, L^*) = a(c_1, \lambda_1, L^{(1)})$ (recall that the point (c_*, λ_*, L^*) is defined in Section 3 before system (34)). Let us choose the polynomial $\Phi_{c_*, \lambda_*, L^*}$ (it is uniquely defined up to a factor from $\overline{k(\varepsilon')}$) such that the following property holds. All the coefficients of $\Phi_{c_*, \lambda_*, L^*}$ are not infinitely large with respect to the field k, and some nonzero coefficient φ of $\Phi_{c_*, \lambda_*, L^*}$ is not infinitesimal with respect to the k. Recall that in general situation, if the mapping the standard part $\mathrm{st} : K \setminus \{\text{infinitely large elements}\} \to k$ is defined, then for an arbitrary polynomial f with coefficients from $K \setminus \{\text{infinitely large elements}\}$ the standard part $\mathrm{st}(f)$ is the polynomial with coefficients from k such that all the nonzero coefficients of the polynomial $f - \mathrm{st}(f)$ are infinitesimal with respect to k. Now by Lemma 14 (with the field $k(\varepsilon')$ and $\mathrm{st}_{\varepsilon'}$ in place of k_i and st_i respectively) and since $\deg W'(c_*, \lambda_*, L^*) = \deg W'(c_5, \lambda_5, L^{(5)})$, the standard part $\mathrm{st}_{\varepsilon'}(\Phi_{c_*, \lambda_*, L^*})$ coincides with $\Phi_{c_5, \lambda_5, L^{(5)}}$ up to a nonzero factor from $\overline{k}$. Hence by Lemma 1 applied to $\Phi_{c_5, \lambda_5, L^{(5)}}$ also $\mathrm{st}_{\varepsilon'}(\Phi_{c_*, \lambda_*, L^*, *})$ coincides with $\Phi_{c_5, \lambda_5, L^{(5)}, *}$ up to a nonzero factor from k. We have $\deg_{Z_0, Z_n, Z_{n+1}} \Phi_{c_*, \lambda_*, L^*, *} = \deg_{Z_0, Z_n, Z_{n+1}} \Phi_{c_5, \lambda_5, L^{(5)}, *} = \deg W'(c_1, \lambda_1, L^{(1)}, *)$ and $\deg_{Z_{n+1}} \Phi_{c_*, \lambda_*, L^*, *} = \deg_{Z_{n+1}} \Phi_{c_5, \lambda_5, L^{(5)}, *} = \deg p'_v \deg W'$ by our construction. Hence $\mathrm{st}_{\varepsilon'}(R_{c_*, \lambda_*, L^*, *})) = R_{c_5, \lambda_5, L^{(5)}, *}$. Since $a(c_5, \lambda_5, L^{(5)}) < a(c_*, \lambda_*, L^*)$, the polynomial $R_{c_*, \lambda_*, L^*, *}$ has more pairwise distinct roots than $R_{c_5, \lambda_5, L^{(5)}, *}$. Therefore, there are two distinct infinitely close roots $\xi_{n,0}$, $\xi_{n,1}$ of the polynomial $R_{c_*, \lambda_*, L^*, *}(1, Z_n) \in \overline{k(\varepsilon')}[Z_n]$. Hence $\xi_{n,0} - \xi_{n,1} \in \overline{k(\varepsilon')}$ is an infinitesimal with respect to the field k.

Therefore, for every $i = 0, 1$ one of two cases take place:

1) the polynomial $\Phi_{c_*, \lambda_*, L^*, *}(1, \xi_{n,i}, Z_{n+1})$ has a multiple root $Z_{n+1} = \xi_{n+1,i}$,

2) $(\mathrm{lc}_{Z_{n+1}}(\Phi_{c_*, \lambda_*, L^*, *}))(1, \xi_{n,i}) = 0$ (recall that leading coefficient $\mathrm{lc}_{Z_{n+1}}(\Phi_{c_*, \lambda_*, L^*, *}) \in \overline{k}[Z_0, Z_n]$).

Permuting $i = 0, 1$ we can suppose without loss of generality that if 1) holds for some $i = 0, 1$ then 1) holds for $i = 0$.

For convenience of notation set $I_{1,1} = I_0$, $I_{1,2} = I_1$, $I_{2,1} = J_0$, $I_{2,2} = J_1$. If w) (here w = 1 or w = 2) is fulfilled for i then by the transfer principle there are $\widetilde{\xi}_{n,i} \in \overline{k(\varepsilon', \varepsilon_4)}$ and $\widetilde{\xi}_{n+1,j} \in \overline{k(\varepsilon', \varepsilon_4)}$, $j \in I_{i,w}$, satisfying the following conditions:

(a) the element $\widetilde{\xi}_{n,i}$ is infinitely close to $\xi_{n,i}$ with respect to the field $\overline{k(\varepsilon')}$,

(b) if $w = 1$ then $|\widetilde{\xi}_{n+1,j_1} - \widetilde{\xi}_{n+1,j_2}|^2 = \varepsilon_4$, where $j_1, j_2 \in I_{i,1}$, $j_1 \neq j_2$, and every element $\widetilde{\xi}_{n+1,j}$, $j \in I_{i,w}$ is infinitely close to $\xi_{n+1,i}$,

(c) if $w = 2$ then $|\widetilde{\xi}_{n+1,j}|^2 = \varepsilon_4^{-1}$, where $j \in I_{i,2}$.

Notice that $(1 : \widetilde{\xi}_{n,i} : \widetilde{\xi}_{n+1,j})$ is a generic point of $W'(c_1, \lambda_1, L^{(1)}, *)$ for all considered i, w, j. Further, $q_{c_*,\lambda_*,L^*,*}$ is a dominant morphism of curves and $\deg q_{c_*,\lambda_*,L^*,*} = 1$. Hence

(d) $\#q^{-1}_{c_*,\lambda_*,L^*,*}((1 : \widetilde{\xi}_{n,i} : \widetilde{\xi}_{n+1,j})) = 1$ for all $j \in I_{i,w}$.

Now for $i = 0, 1$ set $w_i = 0$ if 1) holds, and $w_i = 1$ if 2) holds. Put $(x^*_{j,0} : \ldots : x^*_{j,n}) \in q^{-1}_{c_*,\lambda_*,L^*,*}((1 : \widetilde{\xi}_{n,i} : \widetilde{\xi}_{n+1,j}))$, $j \in I_{w_1} \cup I_{w_2}$. Then according to our construction $X_{j,i} = x^*_{j,i}$, $0 \le j \le n$, $j \in I_{w_1} \cup J_{w_2}$ is solution of system (47) satisfying condition (48) (with the field $k(\varepsilon', \varepsilon_4)$ in place of k_4).

Conversely, suppose that there is solution (48) of system (47). Then as it follows from the description of the algorithm for the considered recursion the inequality $a(c_5, \lambda_5, L^{(5)}) < a(c_1, \lambda_1, L^{(1)})$ holds. The lemma is proved.

Let us return to the description of the general step of the recursion. Notice that system (47) satisfies conditions 1)–4) from Section 2. Hence using Lemma 16 and Remark 11 we decide whether system (47) has a solution satisfying (48), and if such a solution exists we construct it. Suppose that such a solution does not exist. Then by Lemma 21 the integer $a(c_5, \lambda_5, L^{(5)}) = a(c_1, \lambda_1, L^{(1)})$ and the considered step is final. Put $(c_4, \lambda_4, L^{(4)}) = (c_5, \lambda_5, L^{(5)})$. The point $(c_4, \lambda_4, L^{(4)})$ satisfies all the statements from Theorem 2 (e).

Suppose that we have constructed a solution of system (47) satisfying (48), see the statement of Lemma 21. Then from system (47) and (48) we get immediately, cf. the proof of Lemma 21, that $a(c_*, \lambda_*, L^*) > a(c_5, \lambda_5, L^{(5)})$ by this lemma. Similarly to the previous recursions conditions of Lemma 4, Lemma 6 and Lemma 7 hold for (c_*, λ_*, L^*) in place of $(c_0, \lambda_0, L^{(0)})$. By Lemma 4, Lemma 6, Lemma 7 and the observations from the beginning of Section 3 one can apply the algorithm of reduction to integer coefficients, see [12], Proposition 2, and replace subsequently all values τ^*_i by integers τ'_i of length $O(n \log d + (n - \sigma) \log(d'))$. Denote by (c', λ, L') the obtained point. Then $L' \in \mathcal{U}''_0$, the number of elements $\#A_{\lambda,L'} = \deg p'_v \deg W'$, $A_{\lambda,L'} \cap \mathcal{Z}(e'_0) = \emptyset$, and $A_{\lambda,L'} \cap \overline{V \setminus W} = \emptyset$, the dimensions $\dim W_\lambda = n - \sigma$, $\dim(W_\lambda \setminus \mathcal{Z}(e_0, \ldots, e_r)) \cap \mathcal{Z}(e'_{\sigma+1}, \ldots, e'_{n-1}) = 1$, the number of elements $\#(e'_{n+1}/e'_0)(A_{\lambda,L'}) = \#(e_{5,n+1}/e^{(5)}_0)(A_{\lambda_5,L^{(5)}})$, the morphisms $p_{\lambda,L'}$, $p_{\lambda,L',*}$ are dominant, $\deg W'(c', \lambda, L', *) = \deg W'(c_5, \lambda_5, L^{(5)})$, $\deg W'(c', \lambda, L', *) = \deg W'(c_5, \lambda_5, L^{(5)}, *)$, and $a(c', \lambda, L') \ge a(c_*, \lambda_*, L^*)$. Let us apply Lemma 8 to the case when $\iota(\mathbb{A}^1(\overline{k}))$ is the line containing (c', λ, L') and $(c_5, \lambda_5, L^{(5)})$, and $\iota(0) = (c_5, \lambda_5, L^{(5)})$. Then using Theorem 2 (f) [13], Lemma 8 and Lemma 4, Lemma 6, Lemma 7 we construct a point (c'', λ', L''), which is similar to $(c_5, \lambda_5, L^{(5)})$ and such that $a(c'', \lambda', L'') \ge a(c', \lambda, L') > a(c_5, \lambda_5, L^{(5)})$. After that we replace $(c_5, \lambda_5, L^{(5)})$ by (c'', λ', L'') and proceed to the next step of the recursion. The general step and all the recursion are described.

The required estimation for the working time of the present algorithm for assertion (e) of Theorem 2 follows immediately from the estimates for working times of the applied algorithms. Assertion (e) of Theorem 2 is proved. Thus, Theorem 2 is proved, see beginning of this section.

Appendix: a proof of a version of the first Bertini theorem in arbitrary characteristic

To make our presentation in [11], [12], [13] and the present paper self–contained we give an independent proof of the first Bertini theorem (and some other related to it results) for the case of affine algebraic varieties in arbitrary characteristic, see Theorem 4 below. Notice here that in our presentation we have only two references to the first Bertini theorem and only to the partial case of hypersurfaces of affine space in zero–characteristic. Namely, they are in the proofs of Theorem 3 of the present paper and Theorem 3 [11]. Still for convenience of the reader and for possible future applications we decided to consider in Theorem 4 a more general situation of affine algebraic varieties over algebraically closed fields of arbitrary characteristic. Lemma 24 implies the general version of the first Bertini theorem formulated in the Introduction.

In this section we use the notations which may be different from the ones of the Introduction. Let k be a perfect field of arbitrary characteristic with algebraic closure $\overline{k}$. Let B be an arbitrary commutative Noetherian integral ring. We shall denote by B' the integral closure of B in its field of fractions. For a finitely generated B–module M denote by $\mathrm{Ass}_B(M)$ the set of all associated prime ideals of M (they are prime ideals of B). Recall that an ideal of B is called radical if it coincides with its nil–radical. In what follows the morphisms of algebraic varieties are regular morphisms if it is not stated otherwise.

Let $n \geq 2$ be an integer. Let $\mathbb{A}^n(\overline{k})$ be affine space over the field $\overline{k}$ with the coordinate functions $X_1, \ldots, X_n$. Let $x \in \mathbb{A}^n(\overline{k})$ be the point with the coordinates $X_i(x) = 0$, $1 \leq i \leq n$.

Let V be an affine algebraic variety defined over $\overline{k}$ and irreducible over $\overline{k}$. Let $p : V \to \mathbb{A}^n(\overline{k})$ be a dominant separable morphism. We shall identify the ring $\overline{k}[X_1, \ldots, X_n] \subset \overline{k}[V]$. For an element $g \in \overline{k}[X_1, \ldots, X_n]$ we shall denote by $\mathcal{Z}(g)$ the subset of all zeroes of the polynomial g in $\mathbb{A}^n(\overline{k})$. By $V \cap \mathcal{Z}(g)$ we shall denote for convenience of notation the subset $p^{-1}(\mathcal{Z}(g)) \subset V$.

We shall suppose additionally that

$$\dim p^{-1}(x) \leq n - 2. \tag{49}$$

Denote by $B = \overline{k}[V]$ the ring of regular functions of the algebraic variety V. Let $z \in V$ be a point. By $\mathfrak{M}_z$ denote the maximal ideal of B corresponding to the point z.

Let $y \in B$ be a primitive element of the separable algebraic extension $\overline{k}(V) \supset \overline{k}(X_1, \ldots, X_n)$, i.e.,

$$\overline{k}(V) = \overline{k}(X_1, \ldots, X_n)[y]. \tag{50}$$

Let $f \in \overline{k}[X_1, \ldots, X_n, Y]$ be minimal polynomial of y over $\overline{k}(X_1, \ldots, X_n)$ (here Y is a new variable) and f be irreducible in the ring $\overline{k}[X_1, \ldots, X_n, Y]$. So the degree $D = \deg_Y f \geq 1$. Let $\deg_{X_1,\ldots,X_n,Y} f = m_1$ and $f^{(m_1)}$ be the homogeneous with respect to $X_1, \ldots, X_n, Y$ form of degree m_1 of the polynomial f.

The element y is integral over $\overline{k}[X_1, \ldots, X_n]$ if the morphism p is finite. Denote by $\mathrm{lc}_Y f$ the leading coefficient of the polynomial f with respect to Y. We choose f such that $\mathrm{lc}_Y f = 1$ if p is finite.

Besides that, we shall suppose that *if $n = 2$ then p is finite.*

Denote by

$$\Delta_1 = \mathrm{Res}_Y(f, \partial f/\partial Y) \in \overline{k}[X_1, \ldots, X_n] \tag{51}$$

the discriminant of the polynomial f with respect to Y. So $\Delta_1 \neq 0$ since f is separable.

REMARK 13 *By definition we assume that the discriminant Δ_1 is the determinant of the $(2D-1)\times(2D-1)$ Sylvester matrix of the resultant. Although in the case of nonzero characteristic of the ground field the degree $\deg_Y \partial f/\partial Y$ may be less than $D-1$ the partial derivative $\partial f/\partial Y$ is considered formally as a polynomial of degree $D-1$ when one represents this matrix. Some higher coefficients of $\partial f/\partial Y$ may be equal to zero in this representation. Recall once more that this definition of the discriminant is different from the generally accepted by a factor $\mathrm{lc}_Y f$, but it is more convenient for us*

Let us identify the set of all linear forms from $\overline{k}[X_1, \ldots, X_n]$ with $\mathbb{A}^n(\overline{k})$. Let $\mathfrak{P}$ be a prime ideal of $\overline{k}[X_1, \ldots, X_n]$ (respectively B, B'). Suppose that

$$\mathcal{Z}(\mathfrak{P}) \not\subset \mathcal{Z}(X_1, \ldots, X_n),$$

where $\mathcal{Z}(X_1, \ldots, X_n)$ and $\mathcal{Z}(\mathfrak{P})$ are considered as subvarieties of $\mathbb{A}^n(\overline{k})$ (respectively V, the normalization of V). Then the set $S_{\mathfrak{P}}$ of linear forms $L \in \overline{k}[X_1, \ldots, X_n]$ such that $L \notin \mathfrak{P}$ is nonempty and open in the Zariski topology in $\mathbb{A}^n(\overline{k})$ since

$$S_{\mathfrak{P}} = \bigcup_{z \in \mathcal{Z}(\mathfrak{P})} \Big\{ L \in \mathbb{A}^n(\overline{k}) \,:\, L(z) \neq 0 \Big\}.$$

Consider B'/B as a B–module. Set

$$\mathcal{B}_0 = \mathrm{Ass}_B(B'/B), \qquad \mathcal{B}_1 = \{\mathfrak{P} \in \mathcal{B}_0 \,:\, \mathcal{Z}(\mathfrak{P}) \not\subset p^{-1}(x)\},$$
$$\mathcal{B}_2 = \{\mathfrak{P} \in \mathcal{B}_0 \,:\, \dim \mathcal{Z}(\mathfrak{P}) = n-1\}.$$

Consider the set $\mathcal{U}_1$ (respectively $\mathcal{U}_2$) of linear forms L in $X_1, \ldots, X_n$ such that $L \notin \mathfrak{P}$ for every $\mathfrak{P} \in \mathcal{B}_1$ (respectively $\mathcal{B}_2$). Then $\mathcal{U}_1$ (respectively $\mathcal{U}_2$) is a nonempty open in the Zariski topology subset of $\mathbb{A}^n(\overline{k})$ and $\mathcal{U}_1 \subset \mathcal{U}_2$. Note that $\mathcal{U}_1 = \mathcal{U}_2 = \mathbb{A}^n(\overline{k})$ if $\mathcal{B}_1 = \emptyset$, in particular if $B = B'$ (i.e., if V is normal) since in the latter case $\mathrm{Ass}_B(B'/B) = \emptyset$. Note also that if the set of singular points of V has dimension at most $n-2$ then $\mathcal{B}_2 = \emptyset$ and $\mathcal{U}_2 = \mathbb{A}^n(\overline{k})$.

Let $L \in \overline{k}[X_1, \ldots, X_n]$ be a nonzero linear form. Consider B/LB as a B–module. Set

$$\mathcal{B}_{1,L} = (\mathcal{B}_0 \setminus \mathcal{B}_1) \cap \mathrm{Ass}_B(B/LB), \quad \mathcal{B}_{2,L} = (\mathcal{B}_0 \setminus \mathcal{B}_2) \cap \mathrm{Ass}_B(B/LB).$$

Let us show that there is a nonzero element $\Delta_3 \in \overline{k}[X_1, \ldots, X_n]$ such that the extensions of rings

$$B[\Delta_3^{-1}] \supset \overline{k}[X_1, \ldots, X_n,\, \Delta_3^{-1}][y] \supset \overline{k}[X_1, \ldots, X_n][\Delta_3^{-1}] \tag{52}$$

are integral. Indeed, by (50) one can choose Δ_3 to be the product of $\mathrm{lc}_Y f$ and all the leading coefficients of minimal polynomials over $\overline{k}[X_1, \ldots, X_n]$ of the elements of a finite system of generators of the ring B over $\overline{k}[X_1, \ldots, X_n]$. In the case when p is finite we choose $\Delta_3 = 1$. Fix such an element Δ_3.

Set $\mathcal{B}_3 = \mathrm{Ass}_{B'}(B'/\Delta_3 B')$ where $B'/\Delta_3 B'$ is considered as a B'–module. Hence the elements of $\mathcal{B}_3$ are prime ideals of B'. By $\mathcal{U}_3$ denote the set of all

linear forms $L \in \overline{k}[X_1, \dots, X_n]$ such that $L \notin \mathfrak{P}$ for every $\mathfrak{P} \in \mathcal{B}_3$. Since B' is integrally closed, the dimension $\dim \mathcal{Z}(\mathfrak{P}) = n-1$ for every $\mathfrak{P} \in \mathcal{B}_3$, see [2]. Therefore, $\mathcal{U}_3$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^n(\overline{k})$. Note that if p is finite then $\mathcal{U}_3 = \mathbb{A}^n(\overline{k})$. Note also that if $L \in \mathcal{U}_3$ then L does not divide Δ_3 in the ring $\overline{k}[X_1, \dots, X_n]$.

THEOREM 4 *Let $n \geq 2$. Let V, B, $p : V \to \mathbb{A}^n(\overline{k})$, y, f, Δ_1, Δ_3, m_1, $f^{(m_1)}$, $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$, $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_{1,L}$ and $\mathcal{B}_{2,L}$ be as above.*

(i) Then there is a nonzero linear form $L \in \overline{k}[X_1, \dots, X_n]$ in $X_1, \dots, X_n$ such that the ideal $(f, L) \subset \overline{k}[X_1, \dots, X_n, Y]$ is radical and if $n \geq 3$ it is prime. The element $\Delta_1 \bmod L \in \overline{k}[X_1, \dots, X_n]/(L)$ is nonzero (i.e., L does not divide Δ_1) and

$$f^{(m_1)} \bmod L \in (\,\overline{k}[X_1, \dots, X_n]/(L)\,)[Y]$$

is also nonzero.

(ii) The set $\mathcal{U}_4$ of linear forms L satisfying the properties from (i) is a nonempty open in the Zariski topology subset of $\mathbb{A}^n(\overline{k})$.

(iii) Let $L \in \mathcal{U}_1 \cap \mathcal{U}_3 \cap \mathcal{U}_4$ (respectively $L \in \mathcal{U}_2 \cap \mathcal{U}_3 \cap \mathcal{U}_4$). Then the ideal

$$LB = \mathfrak{p} \cap \mathfrak{q},$$

where the ideal $\mathfrak{p}$ is the radical of the ideal LB and if $n \geq 3$ it is prime. The ideal $\mathfrak{q}$ is an intersection of $\mathfrak{P}$-primary ideals for all prime ideals $\mathfrak{P} \in \mathcal{B}_{1,L}$ (respectively $\mathfrak{P} \in \mathcal{B}_{2,L}$), herewith we suppose that the intersection over the empty set (when $\mathcal{B}_{1,L} = \emptyset$ or $\mathcal{B}_{2,L} = \emptyset$) is B. Hence if $L \in \mathcal{U}_1 \cap \mathcal{U}_3 \cap \mathcal{U}_4$ and p is finite then $\mathfrak{q}$ is an intersection of $\mathfrak{M}_z$-primary ideals for some $z \in p^{-1}(x)$ or $\mathfrak{q} = B$. Besides that, for every $\mathfrak{P} \in \mathcal{B}_{1,L}$ (respectively $\mathfrak{P} \in \mathcal{B}_{2,L}$) the dimension $\dim \mathcal{Z}(\mathfrak{P}) \leq n-2$ and therefore, the $\mathfrak{P}$–primary component of the ideal LB is embedded.

(iv) Let $L \in \mathcal{U}_2 \cap \mathcal{U}_3 \cap \mathcal{U}_4$. Then the algebraic variety $V \cap \mathcal{Z}(L)$ has dimension $\dim V - 1$ and if $n \geq 3$ then $V \cap \mathcal{Z}(L)$ is irreducible over $\overline{k}$. Further, the projection induced by p

$$V \cap \mathcal{Z}(L) \to \mathcal{Z}(L) \tag{53}$$

is a dominant and separable morphism of affine algebraic varieties (here $\mathcal{Z}(L) \subset \mathbb{A}^n(\overline{k})$). Besides that if p is a finite morphism then morphism (53) is also finite.

Denote by $\overline{k}(V \cap \mathcal{Z}(L))$ the total quotient ring of the ring of regular functions of the algebraic variety $V \cap \mathcal{Z}(L)$; hence it is the field of rational functions of this algebraic variety if $n \geq 3$. Then

$$[\overline{k}(V \cap \mathcal{Z}(L)) : \overline{k}(\mathcal{Z}(L))] = [\overline{k}(V) : \overline{k}(X_1, \dots, X_n)], \tag{54}$$

herewith the left part of this equality is the dimension of the total quotient ring over the field $\overline{k}(\mathcal{Z}(L))$ and the right part is the degree of the extension of fields of rational functions $\overline{k}(V) \supset \overline{k}(\mathbb{A}^n(\overline{k}))$. Denote by $y \bmod L$ the image of the element y in $\overline{k}[V \cap \mathcal{Z}(L)]$. Then minimal polynomial of the element $y \bmod L$ over $\overline{k}(\mathcal{Z}(L))$ is $f \bmod L$ and $y \bmod L$ is a primitive element of the separable algebra $\overline{k}(V \cap \mathcal{Z}(L))$ over $\overline{k}(\mathcal{Z}(L))$.

(v) *Let $n \geq 3$. Let $\mathcal{L}$ be a linear subspace of the space of all linear forms from $\overline{k}[X_1, \ldots, X_n]$ with $\dim \mathcal{L} \geq 3$. Suppose that for every $\mathfrak{P} \in \mathcal{B}_3$ such that $\dim \mathcal{Z}(\mathfrak{P}) = n-1$ the dimension of the closure*

$$\dim \overline{p(\mathcal{Z}(\mathfrak{P}))} \geq n-2. \tag{55}$$

Then $\mathcal{L} \cap \mathcal{U}_2 \cap \mathcal{U}_3 \neq \emptyset$.

(vi) *Let $n \geq 3$. Let $\mathcal{L}$ be a linear subspace of the space of all linear forms from $\overline{k}[X_1, \ldots, X_n]$ with $\dim \mathcal{L} \geq 3$. Then $\mathcal{L} \cap \mathcal{U}_4 \neq \emptyset$.*

PROOF Let $n = 2$. Then we define $\mathcal{U}_4$ to be the set of all linear forms $L \in \overline{k}[X_1, X_2]$ such that L does not divide $\Delta_1 f^{(m_1)}$. Hence $\mathcal{U}_4$ is a nonempty open in the Zariski topology subset of $\mathbb{A}^2(\overline{k})$ in this case. Recall that by our assumption the leading coefficient $\mathrm{lc}_Y f = 1$ if $n = 2$. Hence if $L \in \mathcal{U}_4$ then $\Delta_1 \bmod L \neq 0$ and therefore, the polynomial $f \bmod L$ is separable. Therefore, the ideal (f, L) is radical. Also $f^{(m_1)} \bmod L \neq 0$. Assertions (i) and (ii) are proved for $n = 2$.

Now our aim is to reduce (iii), (iv) and (v) to (i) and (ii) for $n \geq 3$ (assertion (i) and (ii) for $n \geq 3$ will be proved later). So we assume now that (i) and (ii) hold for $n \geq 3$. Let $\widetilde{V}$ be the normalization of V. Hence $\overline{k}[\widetilde{V}] = B'$.

Denote by $S_1 = \{\Delta_3^N : 0 \leq N \in \mathbb{Z}\}$ the multiplicatively closed subset of $\overline{k}[X_1, \ldots, X_n]$. Denote for brevity $A = \overline{k}[X_1, \ldots, X_n]$. By V_1 denote the affine algebraic variety with the ring of regular functions $A[Y]/(f)$.

LEMMA 22 *Let $L \in \mathcal{U}_3 \cap \mathcal{U}_4$. Then the ideal LB' of the ring B' is radical (respectively prime if $n \geq 3$) and the natural homomorphisms*

$$B'/LB' \to S_1^{-1}(B'/LB'), \tag{56}$$

$$A[Y]/(f, L) \to S_1^{-1}A[Y]/(f, L) \tag{57}$$

are monomorphisms. The natural homomorphism $A[Y]/(f, L) \to B'/LB'$ is a monomorphism. Hence one can identify $A[Y]/(f, L) \subset B'/LB'$. Finally under this identification the algebras $A[Y]/(f, L)$ and B'/LB' have the same total quotient ring, i.e., one can identify the total quotient rings $\overline{k}(\widetilde{V} \cap \mathcal{Z}(L))$ and $\overline{k}(V_1)$ of the rings of regular functions of the algebraic varieties $\widetilde{V} \cap \mathcal{Z}(L)$ and V_1.

PROOF Let us prove that (56) is a monomorphism. It is sufficient to prove that the multiplication to $\Delta_3 : B'/LB' \to B'/LB'$ is a monomorphism, see [2]. Let $z \in B'$ and $\Delta_3 z = Lz_1$ for some $z_1 \in B'$. Since $L \in \mathcal{U}_3$, the multiplication to $L : B'/\Delta_3 B' \to B'/\Delta_3 B'$ is a monomorphism. Therefore, there is $z_2 \in B'$ such that $z_1 = \Delta_3 z_2$. Since B' is integral, the element $z = Lz_2$. Hence (56) is a monomorphism.

Since $L \in \mathcal{U}_3$, the linear form L does not divide Δ_3. Therefore, the homomorphism (57) is a monomorphism.

The extensions of rings (52) are integral, and the element y is separable over $\overline{k}(X_1, \ldots, X_n)$. Hence, see, e.g., [2], in this case one can identify

$$S_1^{-1}B' \subset \frac{1}{\Delta_1} S_1^{-1}A[Y]/(f, L).$$

Let us show that under this identification

$$L\,S_1^{-1}B' = \Big(\frac{L}{\Delta_1}S_1^{-1}A[Y]/(f,L)\Big) \cap S_1^{-1}B'. \tag{58}$$

Indeed, let $(L/\Delta_1)a_1 = a_2$ where $a_1 \in S_1^{-1}A[Y]/(f,L)$ and $a_2 \in S_1^{-1}B'$. Set $a = a_2/L$. Then $La \in S_1^{-1}B'$ and $\Delta_1 a \in S_1^{-1}A[Y]/(f,L)$. Consider the minimal polynomial $\psi \in A[Z]$ (here Z is a new variable) of the element a over $\overline{k}(X_1,\dots,X_n)$ such that ψ is an irreducible element of the ring $A[Z]$. Then the leading coefficient $\mathrm{lc}_Z\psi$ divides $\mathrm{G\,C\,D}\,(L^D,\Delta_1^D)$ in the ring $S_1^{-1}A$ Hence $\mathrm{lc}_Z\psi = \lambda\Delta_3^N$ where $0 \neq \lambda \in \overline{k}$ and an integer $N \geq 0$. Therefore, $a \in S_1^{-1}B'$, since the last ring is integrally closed. The required assertion is proved.

Let us show that $(\,LS_1^{-1}B'\,) \cap (\,S_1^{-1}A[Y]/(f,L)\,) = LS_1^{-1}A[Y]/(f,L)$. Indeed, L does not divide Δ_1 and $S_1^{-1}A[Y]/(f,L)$ is a free $S_1^{-1}(A/(L))$–module. Hence the required assertion follows from (58). Now one can identify

$$S_1^{-1}A[Y]/(f,L) \subset S_1^{-1}(B'/LB') \subset \frac{1}{\Delta_1 \bmod L}S_1^{-1}A[Y]/(f,L) \subset \overline{k}(V_1 \cap \mathcal{Z}(L)). \tag{59}$$

Since (56) and (57) are monomorphisms, (59) implies that $A[Y]/(f,L) \to B'/LB'$ is also a monomorphism. Therefore, the first and the last statements of the lemma again follows from (59). The lemma is proved.

Now we are going to prove (iii) and (iv). By Lemma 22 the ideal LB' of B' is radical and if $n \geq 3$ it is prime. Hence the ideal $LB' \cap B$ is radical in B and if $n \geq 3$ it is prime. Since B' is integral over B the ideal $LB' \cap B$ is the radical of the ideal LB of B. Consider the morphism $\gamma : B/LB \to B'/LB'$ induced by the inclusion $B \subset B'$. Denote $E_1 = \mathrm{Ker}\gamma$ and $E_2 = \mathrm{Im}\gamma$. So we have the exact sequence

$$0 \to E_1 \to B/LB \to E_2 \to 0.$$

Hence $\mathrm{Ass}_B(B/LB) \subset \mathrm{Ass}_B(E_1) \cup \mathrm{Ass}_B(E_2)$. We have $\mathrm{Ass}_B(E_2) = \mathrm{Ass}_B(B/(LB' \cap B))$ since $E_2 \simeq B/(LB' \cap B)$.

The inclusion $B \subset B'$ and the multiplication to $L : B' \to B'$ induce the commutative diagram of homomorphisms of B–modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & B' & \longrightarrow & B'/B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} & & \\ 0 & \longrightarrow & B & \longrightarrow & B' & \longrightarrow & B'/B & \longrightarrow & 0 \end{array}$$

with the exact lines. Considering Ker–Coker sequence corresponding to this diagram we get the exact sequence

$$0 \longrightarrow E_1 \longrightarrow B'/B \xrightarrow{L} B'/B \longrightarrow 0,$$

i.e., E_1 can be identified with the kernel of the homomorphism of multiplication to $L : B'/B \to B'/B$. But $L \in \mathcal{U}_1$ (respectively $L \in \mathcal{U}_2$). Hence $\mathrm{Ass}_B(E_1) \subset \mathcal{B}_{1,L}$ (respectively $\mathrm{Ass}_B(E_1) \subset \mathcal{B}_{2,L}$). Further, $\mathrm{Ass}_B(B/(LB' \cap B)) \cap (\mathcal{B}_0 \setminus \mathcal{B}_2) = \emptyset$ since for every $\mathfrak{P} \in \mathrm{Ass}_B(B/(LB' \cap B))$ the dimension $\dim \mathcal{Z}(\mathfrak{P}) = n-1$ and for every $\mathfrak{P} \in \mathcal{B}_0 \setminus \mathcal{B}_2$ the dimension $\dim \mathcal{Z}(\mathfrak{P}) \leq n-2$. Note that $\mathcal{B}_0 \setminus \mathcal{B}_1 \subset \mathcal{B}_0 \setminus \mathcal{B}_2$. Therefore, $\mathrm{Ass}_B(E_1) = \mathcal{B}_{1,L}$ (respectively $\mathrm{Ass}_B(E_1) = \mathcal{B}_{2,L}$) and also for every $\mathfrak{P} \in \mathrm{Ass}_B(E_1)$ the set $\mathcal{Z}(\mathfrak{P})$ is not an irreducible component of $\mathcal{Z}(LB' \cap B)$.

Thus,

$$\overline{k}[V \cap \mathcal{Z}(L)] \simeq B/(B \cap LB') \subset B'/LB' \simeq \overline{k}[\widetilde{V} \cap \mathcal{Z}(L)]. \tag{60}$$

Now (60) and Lemma 22 imply that one can identify

$$A[Y]/(f, L) \subset B/(B \cap LB') \subset B'/LB',$$

and the rings $A[Y]/(f, L)$, $B/(B \cap LB')$, B'/LB' have the same total quotient ring.

Finally we need to prove (54). It is sufficient to show that

$$[\overline{k}(V_1 \cap \mathcal{Z}(L)) : \overline{k}(\mathcal{Z}(L))] = [\overline{k}(V_1) : \overline{k}(X_1, \ldots, X_n)]. \tag{61}$$

But it is equivalent to the equality $\deg_Y f = \deg_Y (f \bmod L)$ which is satisfied since L does not divide Δ_1.

Thus assertions (iii) and (iv) (modulo (i) and (ii) for $n \geq 3$) are proved. Let us prove (v). If $\mathfrak{P} \in \mathcal{B}_2 \setminus \mathcal{B}_3$ then $\dim p(\mathcal{Z}(\mathfrak{P})) = \dim \mathcal{Z}(\mathfrak{P}) = n - 1$, since extensions of rings (52) are integral. Therefore, for every $\mathfrak{P} \in \mathcal{B}_2 \cup \mathcal{B}_3$ the dimension $\dim p(\mathcal{Z}(\mathfrak{P})) \geq n - 2$. Hence $\dim(\mathfrak{P} \cap \mathcal{L}) \leq 2$ for every $\mathfrak{P} \in \mathcal{B}_2 \cup \mathcal{B}_3$. Thus $\mathcal{L} \setminus \mathfrak{P} \neq \emptyset$ for every $\mathfrak{P} \in \mathcal{B}_2 \cup \mathcal{B}_3$. This implies (v) (modulo (i) and (ii) for $n \geq 3$).

Now our aim is to prove assertions (i), (ii) and (vi) of Theorem 4 for $n \geq 3$. Recall that V_1 is the affine algebraic variety with ring of regular functions $A[Y]/(f)$ and the ring of polynomials $A = \overline{k}[X_1, \ldots, X_n]$.

Denote by $\mathcal{U}'_n$ the set of linear forms $L = \sum_{1 \leq i \leq n} l_i X_i \in \overline{k}[X_1, \ldots, X_n]$ such that all $l_1, \ldots, l_n \in \overline{k}$ and $l_n \neq 0$. For every $L \in \mathcal{U}'_n$ the isomorphism

$$\overline{k}[X_1, \ldots, X_n]/(L) \simeq \overline{k}[X_1, \ldots, X_{n-1}] \tag{62}$$

induced by the substitution $X_n = -(l_1 X_1 + \ldots + l_n X_n)/l_n$ holds. In what follows for every linear form $L \in \mathcal{U}'_n$ we shall identify the rings in the left and right parts of (62) using this isomorphism.

Recall that the set of linear forms $\mathcal{U}_4$ is defined in assertion (ii) of Theorem 4. If $f \bmod L$ is irreducible over $\overline{k}$ then the condition $\Delta_1 \bmod L \neq 0$ (i.e., that L does not divide Δ_1) is equivalent to $(\partial f/\partial Y) \bmod L \neq 0$ and $\deg_Y f \bmod L = \deg_Y f$. Hence $\mathcal{U}_4 \cap \mathcal{U}'_n$ is the subset of all linear forms from $\mathcal{U}'_n$ such that the element $f \bmod L$ is irreducible in the ring $\overline{k}[X_1, \ldots, X_{n-1}, Y]$, the degrees

$$\deg_{Y, X_1, \ldots, X_n}(f \bmod L) = \deg_{Y, X_1, \ldots, X_n} f, \quad \deg_Y(f \bmod L) = \deg_Y f$$

and $(\partial f/\partial Y) \bmod L \neq 0$.

Due to symmetry of indices $1, \ldots, n$ assertions (i) and (ii) of Theorem 4 follow immediately from the following lemma.

LEMMA 23 *Let $n \geq 3$ and $f \in \overline{k}[X_1, \ldots, X_n, Y]$ be an irreducible polynomial such that $\partial f/\partial Y \neq 0$. Then $\mathcal{U}_4 \cap \mathcal{U}'_n$ is a nonempty open in the Zariski topology subset of $\mathcal{U}'_n$.*

PROOF At first our aim is to show that $\mathcal{U}_4 \cap \mathcal{U}'_n \neq \emptyset$, i.e., to construct some linear form L from $\mathcal{U}_4 \cap \mathcal{U}'_n$. Denote $\deg_Y f = D$ and $\deg_{Y, X_1, \ldots, X_n} f = m_1$.

Let us represent $f = \sum_{i,j\geq 0} f_{i,j}Y^j$ where $f_{i,j} \in \overline{k}[X_1, \ldots, X_n]$ are homogeneous polynomials with $\deg f_{i,j} = i$ for all i, j. Let $\alpha, \beta \in \overline{k}$ and

$$(\Delta_1 \sum_{0\leq i\leq m_1} f_{i,m_1-i})(X_1, X_2, \ldots, X_{n-2}, \alpha X_1, \beta X_1) \neq 0. \tag{63}$$

Then $\Delta_1 \sum_{0\leq i\leq m_1} f_{i,m_1-i} \notin (X_{n-1}-\alpha X_1, X_n-\beta X_1)$ where $(X_{n-1}-\alpha X_1, X_n-\beta X_1)$ is the ideal of the ring $\overline{k}[X_1, \ldots, X_n]$ generated by the elements $X_{n-1} - \alpha X_1$, $X_n - \beta X_1$. Hence

$$f \notin (X_{n-1} - \alpha X_1, X_n - \beta X_1). \tag{64}$$

Denote by $\widetilde{f} \in \overline{k}[X_1, \ldots, X_n, Y]$ the polynomial such that

$$\widetilde{f}(X_1, X_2, \ldots, X_{n-2}, X_{n-1} - \alpha X_1, X_n - \beta X_1, Y) = f.$$

Then $\widetilde{f}$ is an irreducible element of the ring $\overline{k}[X_1, \ldots, X_n, Y]$. Since $\widetilde{f}$ is irreducible the polynomial $\widetilde{f}(X_1, X_2, \ldots, X_{n-1}, X_{n-1}X_n, Y)$ is reducible if and only if X_{n-1} divides it, i.e., if and only if $\widetilde{f} \in (X_{n-1}, X_n)$. This is impossible by (64). Hence $\widetilde{f}(X_1, X_2, \ldots, X_{n-1}, X_{n-1}X_n, Y)$ is also an irreducible element of the ring $\overline{k}[X_1, \ldots, X_n, Y]$. Suppose that

$$\widetilde{f}(X_1, X_2, \ldots, X_{n-1}, X_{n-1}X_n, Y) \in \overline{k}[X_1, \ldots, X_{n-1}, Y]. \tag{65}$$

Then we can take $L = X_n - \alpha X_1$. So we shall suppose in what follows in the proof that (65) does not hold.

Suppose that there is $\lambda \in \overline{k}$ such that the polynomial $\widetilde{f}(X_1, \ldots, X_{n-1}, \lambda X_{n-1}, Y)$ is irreducible in the ring $\overline{k}[X_1, \ldots, X_{n-1}, Y]$. Then obviously the linear form $L = \lambda(X_1 - \beta X_n) - (X_1 - \alpha X_{n-1})$ satisfies the statement of the theorem.

Consider the lexicographic order on the n–tuples $(i_1, \ldots, i_n)$, where $0 \leq i_1, \ldots, i_n \in \mathbb{Z}$. Namely, set $(i_1, \ldots, i_n) > (j_1, \ldots, j_n)$ if and only if there is $1 \leq r \leq n$ such that $i_1 = j_1, \ldots, i_{r-1} = j_{r-1}$ but $i_r > j_r$. Let a field $K \supset \overline{k}$. Let

$$F = \sum_{i_1,\ldots,i_n\geq 0} F_{i_1,\ldots,i_n} X_1^{i_1} \ldots X_{n-1}^{i_{n-1}} Y^{i_n} \in K[X_1, \ldots, X_{n-1}, Y]$$

be an arbitrary nonzero polynomial where all $F_{i_1,\ldots,i_n} \in K$. Set

$$\begin{aligned}
&(j_1, \ldots, j_n) = \max\{(i_1, \ldots, i_n) \,:\, F_{i_1,\ldots,i_n} \neq 0\},\\
&\mathrm{lex}(F) = X_1^{j_1} \ldots X_{n-1}^{j_{n-1}} Y^{j_n},\\
&\mathrm{lexc}(F) = F_{j_1,\ldots,j_n}.
\end{aligned}$$

Note that the condition $\mathrm{lex}(F) \neq 1$ is equivalent to $F \notin \overline{k}$. For every polynomials F_1, F_2 similar to F the equalities $\mathrm{lex}(F_1)\mathrm{lex}(F_2) = \mathrm{lex}(F_1F_2)$, $\mathrm{lexc}(F_1)\mathrm{lexc}(F_2) = \mathrm{lexc}(F_1F_2)$ hold.

Suppose that

(B) for every $\lambda \in \overline{k}$ the polynomial $\widetilde{f}_\lambda = \widetilde{f}(X_1, \ldots, X_{n-1}, \lambda X_{n-1}, Y)$ is not irreducible in the ring $\overline{k}[X_1, \ldots, X_{n-1}, Y]$.

Let $\widetilde{f}_\lambda = g_\lambda h_\lambda$ where $g_\lambda, h_\lambda \in \overline{k}[X_1, \ldots, X_{n-1}, Y] \setminus \overline{k}$. Hence there is $\lambda_0 \in \overline{k}$ such that $\mathrm{lex}(g_\lambda) = \mathrm{lex}(g_{\lambda_0})$, $\mathrm{lex}(h_\lambda) = \mathrm{lex}(h_{\lambda_0})$ for infinitely many $\lambda \in \overline{k}$. Note that $\mathrm{lex}(g_{\lambda_0}) \neq 1$ and $\mathrm{lex}(h_{\lambda_0}) \neq 1$ since $g_{\lambda_0}, h_{\lambda_0} \notin \overline{k}$. Now consider the polynomials G, H in $X_1, \ldots, X_{n-1}, Y$ with indeterminate coefficients such that $\mathrm{lex}(G) = \mathrm{lex}(g_{\lambda_0})$, $\mathrm{lex}(H) = \mathrm{lex}(h_{\lambda_0})$ and

$$\deg_{Y,X_1,\ldots,X_n} G, \deg_{Y,X_1,\ldots,X_n} H \leq$$
$$\deg_{Y,X_1,\ldots,X_n} \widetilde{f}(X_1, X_2, \ldots, X_{n-1}, X_{n-1}X_n, Y).$$

In particular $\mathrm{lexc}(G), \mathrm{lexc}(H)$ are independent variables. Denote for brevity $\widetilde{f}_{X_n} = \widetilde{f}(X_1, X_2, \ldots, X_{n-1}, X_{n-1}X_n, Y)$. The equality $\widetilde{f}_{X_n} = GH$ and the conditions $\mathrm{lexc}(G) \neq 0, \mathrm{lexc}(H) \neq 0$ are equivalent to a system of polynomial equations

$$\Psi_1 = \ldots = \Psi_a = 0,\ Z\mathrm{lexc}(G)\mathrm{lexc}(H) = 1 \tag{66}$$

in the indeterminate coefficients of G and H, X_n and Z (here Z is a new variable) with coefficients in the field $\overline{k}$. Now the condition that $f_\lambda = g_\lambda h_\lambda$, $\mathrm{lex}(g_\lambda) = \mathrm{lex}(g_{\lambda_0})$ and $\mathrm{lex}(h_\lambda) = \mathrm{lex}(h_{\lambda_0})$ is equivalent to the existence of a solution of system (66) with $X_n = \lambda$. Hence there are infinitely many $\lambda \in \overline{k}$ such that system (66) has a solution with $X_n = \lambda$. Therefore there is an irreducible component of variety of solutions of system (66) which is not contained in any hyperplane $\mathcal{Z}(X_n - c)$, $c \in \overline{k}$. Hence there is an irreducible curve C which is contained in the variety of solutions of system (66) and is not contained in any hyperplane $\mathcal{Z}(X_n - c)$, $c \in \overline{k}$. Then the field of rational functions $\overline{k}(C)$ is algebraic over the field $\overline{k}(X_1)$. Thus the polynomial $\widetilde{f}_{X_n}$ is reducible in the ring $\overline{k(X_n)}[X_1, \ldots, X_{n-1}, Y]$ where $\overline{k(X_n)}$ is algebraic closure of the field $\overline{k}(X_n)$.

Let $\widetilde{f}_{X_n} = \widetilde{g}\,\widetilde{h}$ where

$$\widetilde{g},\ \widetilde{h} \in \overline{k(X_1)}[X_1, \ldots, X_{n-1}, Y] \setminus \overline{k(X_n)},$$

$\deg_Y \widetilde{g} = n_1$, $\deg_Y \widetilde{h} = n_2$. Let us show that $n_1 \geq 1$ and $n_2 \geq 1$. Indeed, suppose that, say, $n_2 = 0$. Then $\widetilde{h} \in \overline{k(X_n)}[X_1, \ldots, X_{n-1}] \setminus \overline{k(X_n)}$ and $n_1 = \deg_Y f > 0$. Represent

$$\widetilde{f}_{X_n} = \sum_{0 \leq i \leq n_1} \widetilde{f}_i Y^i, \qquad \widetilde{g} = \sum_{0 \leq i \leq n_1} \widetilde{g}_i Y^i, \qquad \text{where}$$
$$\widetilde{f}_i \in \overline{k}[X_1, \ldots, X_n], \qquad \widetilde{g}_i \in \overline{k(X_n)}[X_1, \ldots, X_{n-1}], \qquad 0 \leq i \leq n_1.$$

The subrings

$$\overline{k}[X_1, \ldots, X_n, \widetilde{h}], \tag{67}$$
$$\overline{k}[X_1, \ldots, X_n, \widetilde{g}_0, \ldots, \widetilde{g}_{n_1}, \widetilde{h}] \tag{68}$$

of the ring $\overline{k(X_n)}[X_1, \ldots, X_{n-1}]$ are of dimension n and integral. Denote by $S' \subset \mathbb{A}^{n+1}(\overline{k})$ and $S'' \subset \mathbb{A}^{n+n_1+2}(\overline{k})$ the affine algebraic varieties with the rings of regular functions (67) and (68) respectively. There is a polynomial $0 \neq Q \in \overline{k}[X_n]$ such that all the coefficients of the polynomials $Q\widetilde{g}_0, \ldots, Q\widetilde{g}_{n_1}$, $Q\widetilde{h}$ (considered as polynomials in $X_1, \ldots, X_{n-1}$ with coefficients from $\overline{k(X_n)}$) are integral over $\overline{k}[X_n]$.

Since $\dim S' = n$ and ring (67) is integral there is an irreducible polynomial (uniquely defined up to a nonzero constant factor from $\overline{k}$)

$$H = \sum_{0 \le i \le N} H_i Z^i \in \overline{k}[X_1, \ldots, X_n, Z]$$

which is a minimal polynomial of $\widetilde{h}$ over the field $\overline{k}(X_1, \ldots, X_n)$, herewith all $H_i \in \overline{k}[X_1, \ldots, X_n]$, $H_N \neq 0$ and the ring

$$\overline{k}[S'] = \overline{k}[X_1, \ldots, X_n, \widetilde{h}] \simeq \overline{k}[X_1, \ldots, X_n, Z]/(H). \tag{69}$$

The polynomial $H_0 \notin \overline{k}[X_n]$ since $\deg_{X_1,\ldots,X_{n-1}} H_0 \ge \deg_{X_1,\ldots,X_{n-1}} \widetilde{h} \ge 1$.

The embeddings of the rings of regular functions $\overline{k}[X_1, \ldots, X_n] \subset \overline{k}[S'] \subset \overline{k}[S'']$ induce the finite dominant morphisms

$$S'' \setminus \mathcal{Z}(Q) \xrightarrow{\rho_2} S' \setminus \mathcal{Z}(Q) \xrightarrow{\rho_1} \mathbb{A}^n(\overline{k}) \setminus \mathcal{Z}(Q).$$

The polynomial Q does not belong to the radical of the ideal (H_0) in the ring $\overline{k}[X_1, \ldots, X_n]$ since $Q \in \overline{k}[X_n]$ and $\deg_{X_1,\ldots,X_{n-1}} H_0 \ge 1$. Hence Q does not belong to the radical of the ideal (H_0, Z) in the ring $\overline{k}[X_1, \ldots, X_n, Z]$. But the ideal $(H_0, Z) = (H, Z)$ in this ring. Therefore $\emptyset \neq \mathcal{Z}(\widetilde{h}) \setminus \mathcal{Z}(Q)$ in $S' \setminus \mathcal{Z}(Q)$. Since the morphism ρ_2 is finite and dominant $\emptyset \neq \mathcal{Z}(\widetilde{h}) \setminus \mathcal{Z}(Q)$ in $S'' \setminus \mathcal{Z}(Q)$. Further, $\widetilde{f}_i = \widetilde{g}_i \widetilde{h}$ for all $1 \le i \le n_1$. Hence

$$\emptyset \neq \mathcal{Z}(\widetilde{h}) \setminus \mathcal{Z}(Q) \subset \mathcal{Z}(f_0, \ldots, f_{n_1}) \setminus \mathcal{Z}(Q)$$

in $S'' \setminus \mathcal{Z}(Q)$. Hence $\dim \mathcal{Z}(f_0, \ldots, f_{n_1}) \setminus \mathcal{Z}(Q) \ge n-1$ in $S'' \setminus \mathcal{Z}(Q)$. Therefore $\dim \mathcal{Z}(f_0, \ldots, f_{n_1}) \setminus \mathcal{Z}(Q) \ge n-1$ in $\mathbb{A}^n(\overline{k}) \setminus \mathcal{Z}(Q)$ since the morphism $\rho_1 \circ \rho_2$ is finite and dominant. This means that $\mathrm{GCD}\{f_0, \ldots, f_{n_1}\} \neq 1$ in the ring $\overline{k}[X_1, \ldots, X_n]$, i.e., $\widetilde{f}_{X_n}$ is reducible in the ring $k[X_1, \ldots, X_n, Y]$. This contradiction proves the required assertion that $n_1 \ge 1$ and $n_2 \ge 1$.

Now

$$f = \widetilde{f}\left(X_1, \ldots, X_{n-2}, X_{n-1} - \alpha X_1, \left(\frac{X_n - \beta X_1}{X_{n-1} - \alpha X_1}\right)(X_{n-1} - \alpha X_1), Y\right)$$

is a reducible element of the ring

$$\overline{k\left(\frac{X_n - \beta X_1}{X_{n-1} - \alpha X_1}\right)}(X_1, \ldots, X_{n-1})[Y] \tag{70}$$

Namely, represent $f/\mathrm{lc}_Y f = gh$ where g and h are polynomials from the ring (70) with leading coefficients $\mathrm{lc}_Y g = 1$, $\mathrm{lc}_Y h = 1$ and $\deg_Y g \ge 1$, $\deg_Y h \ge 1$.

Denote for brevity $k_{\alpha,\beta} = \overline{k}((X_n - \beta X_1)/(X_{n-1} - \alpha X_1))$. Denote by k' the field generated by coefficients of the polynomials g, h from ring (70) over the field

$$k_{\alpha,\beta}(X_1, \ldots, X_{n-1}) = \overline{k}(X_1, \ldots, X_n).$$

Hence $k' \subset \overline{k}_{\alpha,\beta}(X_1, \ldots, X_{n-1})$ and the extension $k' \supset k_{\alpha,\beta}(X_1, \ldots, X_{n-1})$ is finite separable. Set $k'_{\alpha,\beta} = k' \cap \overline{k}_{\alpha,\beta}$. Hence the extension $k'_{\alpha,\beta} \supset k_{\alpha,\beta}$ is finite separable.

Let us show that $k' = k'_{\alpha,\beta}(X_1, \ldots, X_{n-1})$. The definition of k' implies that $k' \subset k''_{\alpha,\beta}(X_1, \ldots, X_{n-1})$, where $k''_{\alpha,\beta} \supset k_{\alpha,\beta}$ is a finite (and separable) Galois extension. Now the required assertion follows from the Galois theory.

The degree of the extension $k'_{\alpha,\beta} \supset k_{\alpha,\beta}$ is greater than 1 since, otherwise, the polynomial $f \in \overline{k}[X_1, \ldots, X_n, Y]$ is also reducible by the Gauss lemma. It is known that in this case there is a discrete valuation $v : k_{\alpha,\beta} \to \mathbb{Z} \cup \{+\infty\}$ of the field $k_{\alpha,\beta}$ over $\overline{k}$ (or which is the same v is zero on $\overline{k} \setminus \{0\}$) and a discrete valuation w which is an extension of v to $k'_{\alpha,\beta}$ such that w is ramified over v. The valuation v is defined by a uniformizing element

$$\pi = \frac{\mu_1(X_n - \beta X_1) + \mu_2(X_{n-1} - \alpha X_1)}{\mu_3(X_n - \beta X_1) + \mu_4(X_{n-1} - \alpha X_1)}$$

such that $\mu_1, \mu_2, \mu_3, \mu_4 \in \overline{k}$, $\mu_1\mu_4 - \mu_3\mu_2 \neq 0$ and $v(\pi) = 1$.

Let $y_1, \ldots, y_D$ be all the roots of the polynomial f in the algebraic closure of the field $\overline{k}(X_1, \ldots, X_n)$. Then the field $\overline{k}(X_1, \ldots, X_n)[y_1, \ldots, y_D]$ contains $k'_{\alpha,\beta}$ since the coefficients of g and h are polynomials in $y_1, \ldots, y_D$. Let us continue w till a discrete valuation w_1 of the field $k'_{\alpha,\beta}(X_1, \ldots, X_{n-1})$ such that $w_1(z) \geq 0$ for every $z \in \overline{k}[X_1, \ldots, X_n]$ and

$$w_1(\mu_1(X_n - \beta X_1) + \mu_2(X_{n-1} - \alpha X_1)) = 1.$$

Denote the field $k'' = \overline{k}(X_1, \ldots, X_n)[y_1, \ldots, y_D]$. Hence k'' is a finite separable extension of $\overline{k}(X_1, \ldots, X_n)$ and of $k'_{\alpha,\beta}(X_1, \ldots, X_{n-1})$. Further, let us continue w_1 till a discrete valuation w_2 of the field k''. By our construction there is an element $z_1 \in k'_{\alpha,\beta}(X_2, \ldots, X_n)$ such that $w_2(z_1) = 1/e$ where $e > 1$ is an integer.

By w_3 (respectively w_4) denote the restriction of w_2 to the field $\overline{k}(X_1, \ldots, X_n)[y]$, (respectively $\overline{k}(X_1, \ldots, X_n)$) herewith we suppose without loss of generality that $y = y_1$. Since k'' is a minimal Galois extension of $\overline{k}(X_1, \ldots, X_n)$ containing $\overline{k}(X_1, \ldots, X_n)[y]$ and w_2 is ramified over w_4, then, see, e.g., [2], the discrete valuation w_3 is also ramified over w_4. Therefore $\mu_1(X_n - \beta X_1) + \mu_2(X_{n-1} - \alpha X_1)$ divides the discriminant Δ_1 of the polynomial f with respect to Y.

Now for all $\beta, \alpha \in \overline{k}$ satisfying (63) there are $\mu_1, \mu_2 \in \overline{k}$ such that $(\mu_1, \mu_2) \neq (0, 0)$ and

$$(\mu_1(X_n - \beta X_1) + \mu_2(X_{n-1} - \alpha X_1)) \quad \text{divides} \quad \Delta_1 \tag{71}$$

in the ring of polynomials $\overline{k}[X_1, \ldots, X_n]$. This is possible only in the case when $\Delta_1 = 0$ which contradicts to the separability of the polynomial f. Thus the assumption (B) leads to a contradiction. The existence of the required linear form L is proved.

Now we are going to prove that the set $\mathcal{U}_4 \cap \mathcal{U}'_n$ is open in the Zariski topology. By definition $L \in \mathcal{U}_4$ if and only if the following two conditions hold:

(a) for every $L \in \mathcal{U}_4$ the ideal (f, L) is prime in $\overline{k}[X_1, \ldots, X_n, Y]$,

(b) for every $L \in \mathcal{U}_4$ the polynomial $\Delta_1 \sum_{0 \leq i \leq m_1} f_{i,m_1-i} \notin (L)$.

Obviously the set of linear forms satisfying (b) is a open in the Zariski topology subset of $\mathbb{A}^n(\overline{k})$. Hence it is sufficient to prove that the set of all $L \in \mathcal{U}'_n$ satisfying (a) is a open in the Zariski topology subset of $\mathbb{A}^n(\overline{k})$. Let X_0 be

a new variable. Then the homogeneous polynomial in $X_0, \ldots, X_{n-1}$, Y with coefficients in the field $\overline{k}$

$$\overline{f} = X_0^{\deg_{X_1,\ldots,X_n,Y} f} f(X_1/X_0, \ldots, X_n/X_0, Y/X_0)$$

is irreducible over $\overline{k}$. Let $L = \sum_{1 \le i \le n} l_i X_i$ satisfy (a), where all $l_i \in \overline{k}$, $l_n \ne 0$. Let $\sum_{1 \le i \le n} u_i X_i$ be a linear form with coefficients $u_i \in \overline{k}$. Suppose that $l_n + u_n \ne 0$. Then the linear form $L + \sum_{1 \le i \le n} u_i X_i$ satisfies (a) if and only if the polynomial

$$\overline{f}\left(X_0, \ldots, X_{n-1}, -\sum_{1 \le i \le n-1} (l_n + u_n)^{-1}(l_i + u_i)X_i, Y\right) \tag{72}$$

is irreducible over the field $\overline{k}$. Consider the set P' of all homogeneous polynomials in $X_0, \ldots, X_{n-1}, Y$ of degree $\deg_{X_1,\ldots,X_{n-1},Y} f$. Let n_0 be the number of coefficients of such polynomials in dense representation. Let us identify the set P' with the affine space $\mathbb{A}^{n_0}(\overline{k})$. Then it is known, see e.g., [15], that the set P'' of all irreducible over $\overline{k}$ polynomials is a nonempty (recall that $n \ge 3$) open in the Zariski topology subset of $\mathbb{A}^{n_0}(\overline{k})$. Hence there is a polynomial $\Phi \in \overline{k}[X_1, \ldots, X_n]$ such that $\Phi(0, \ldots, 0) \ne 0$ and if $\Phi(u_1, \ldots, u_n) \ne 0$ then polynomial (72) is irreducible over $\overline{k}$. Thus, $\mathcal{U}_4 \cap \mathcal{U}'_n$ is open in the Zariski topology subset of $\mathbb{A}^n(\overline{k})$. The lemma is proved.

Let us prove (vi). We can suppose without loss of generality performing a nondegenerate linear transformation of $X_1, \ldots, X_n$ that $X_1, X_{n-1}, X_n \in \mathcal{L}$. We have proved that in this case $\mathcal{L} \cap \mathcal{U}_4 \ne \emptyset$. This implies (vi). The theorem is proved.

REMARK 14 *The sets $\mathcal{U}_i$, $1 \le i \le 4$, from the statement of Theorem 4 satisfy the following property. For all $0 \ne \lambda \in \overline{k}$ and $L \in \mathcal{U}_i$ the linear form $\lambda L \in \mathcal{U}_i$.*

COROLLARY 2 *Under conditions of Theorem 4 let $L \in \mathcal{U}_2 \cap \mathcal{U}_3 \cap \mathcal{U}_4$ be a linear form. Let W be the set of all smooth point of the algebraic variety $V \cap \mathcal{Z}(L) \setminus p^{-1}(x)$. Then W is a dense open in the Zariski topology subset of the algebraic variety $V \cap \mathcal{Z}(L)$, every point $z \in W$ is a smooth point of V, the local ring $\mathcal{O}_{z,V\cap\mathcal{Z}(L)}$ of the algebraic variety $V \cap \mathcal{Z}(L)$ at the point z is isomorphic to $\mathcal{O}_{z,V}/L\mathcal{O}_{z,V}$. In particular if $V \subset \mathbb{A}^N(\overline{k})$ for some integer N and p is the linear projection to the first n coordinates then the intersection of the tangent spaces*

$$T_{z,V} \cap \mathcal{Z}(L) \tag{73}$$

of V and $\mathcal{Z}(L)$ (we identify $T_{z,\mathcal{Z}(L)}$ with $\mathcal{Z}(L)$) at the point z is transversal, i.e., $\dim T_{z,V} \cap \mathcal{Z}(L) = n-1$ and there is a system of local parameters $h_1, \ldots, h_{N-n}$ of V at the point z such that $L, h_1, \ldots, h_{N-n}$ is a system of local parameters of $V \cap \mathcal{Z}(L)$ at the point z.

PROOF We can suppose without loss of generality that $V \subset \mathbb{A}^N(\overline{k})$ and p is the linear projection to the first n coordinates. Obviously W is a dense open in the Zariski topology subset of $V \cap \mathcal{Z}(L)$. Let $z \in W$ be a smooth point of $V \cap \mathcal{Z}(L)$. Denote by $\mathcal{O}_z$ the local ring at the point z of the algebraic variety $\mathbb{A}^N(\overline{k})$. Denote by $I(V)$ the ideal of the affine algebraic variety $V \subset \mathbb{A}^N(\overline{k})$. So $\overline{k}[V] = A_1/I(V)$ where $A_1 = \overline{k}[X_1, \ldots, X_N]$. Then by assertion (iii) the equality

$$\mathcal{O}_{z,V\cap\mathcal{Z}(L)} = \mathcal{O}_z/(I(V)\mathcal{O}_z + L\mathcal{O}_z)$$

holds. Hence L is one of local parameters at the point z of the algebraic variety $V \cap \mathcal{Z}(L)$ (considered as a subvariety of $\mathbb{A}^N(\overline{k})$) and other such local parameters can be chosen from $I(V)$. Hence there is a system of local parameters $L, h_1, \ldots, h_{N-n}$ in the point z of $V \cap \mathcal{Z}(L)$ such that $h_1, \ldots, h_{N-n} \in I(V)$.

This implies that the point z is a smooth point of V, the tangent space $T_{z,V} = \mathcal{Z}(d_z h_1, \ldots, d_z h_{N-n})$ and the intersection of tangent spaces (73) is transversal. The corollary is proved.

COROLLARY 3 *Let $n \geq 3$. Let $f \in \overline{k}[X_1, \ldots, X_n]$ be an irreducible polynomial with $\deg f \geq 1$. Suppose that f is not a homogeneous polynomial relative to $X_1, \ldots, X_n$ or $n \geq 4$. Let $\mathcal{L}$ be a linear subspace of the space of all linear forms in $X_1, \ldots, X_n$ with coefficients in $\overline{k}$. Suppose that the dimension $\dim \mathcal{L} \geq 3$. Then there is a nonempty open in the Zariski topology subset $\mathcal{U}'''$ of $\mathcal{L}$ such that for every $L \in \mathcal{U}'''$ the element $f \bmod L \in \overline{k}[X_1, \ldots, X_n]/(L)$ is irreducible.*

PROOF Suppose that f is not a homogeneous polynomial relative to $X_1, \ldots, X_n$. Consider the polynomial

$$F = Y^{\deg_{X_1,\ldots,X_n} f} f(X_1/Y, \ldots, X_n/Y).$$

Then $F \in \overline{k}[X_1, \ldots, X_n, Y]$ is irreducible. The polynomial $F \bmod L$ is irreducible if and only if $f \bmod L$ is irreducible. If characteristic $\mathrm{char}(\overline{k}) = 0$ then set $F_1 = F$. If $\mathrm{char}(\overline{k}) = p > 0$ then set $s \geq 0$ to be maximal integer such that $F \in \overline{k}[X_1, \ldots, X_n, Y^{p^s}]$, and denote $F_1 = F(X_1, \ldots, X_n, Y^{p^{-s}})$. Then $F_1 \in \overline{k}[X_1, \ldots, X_n, Y]$ is irreducible and $\partial F/\partial Y \neq 0$. If $\mathrm{char}(\overline{k}) > 0$ and $s \geq 1$ then there is $1 \leq i \leq n$ such that $\partial F/\partial X_i \neq 0$ since f is irreducible and, therefore, separable. We can suppose without loss of generality performing a linear transformation of $X_1, \ldots, X_n$ that $i = n$.

Applying Theorem 4 (vi) to the algebraic variety $\mathcal{Z}(F_1)$ and the polynomial F_1 (in place of V and f) we conclude that there is a nonempty open in the Zariski topology subset $\mathcal{U}' \subset \mathcal{L}$ such that for every $L \in \mathcal{U}'$ the polynomial $F_1 \bmod L$ is irreducible and $\partial(F_1 \bmod L)/\partial Y \neq 0$. Let $L = l_1 X_1 + \ldots + l_n X_n$ where all $l_i \in \overline{k}$ and $l_n \neq 0$. Then, see above, $F_1 \bmod L \in \overline{k}[X_1, \ldots, X_{n-1}]$. It is known, see, e.g., [5], that in the considered case the polynomial $F \bmod L$ is irreducible if and only if $F \bmod L$ is separable with respect to $X_1, \ldots, X_{n-1}, Y$. If $\mathrm{char}(\overline{k}) = 0$ or $s = 0$ the last condition holds. If $\mathrm{char}(\overline{k}) > 0$ and $s \geq 1$ then $\partial F/\partial X_n \neq 0$. Hence in this case the condition $\partial(F \bmod L)/\partial X_{n-1} \neq 0$ defines an open in the Zariski topology subset $\mathcal{U}'' \subset \mathcal{L}$. Set $\mathcal{U}''' = \mathcal{U}' \cap \mathcal{U}''$. Thus $\mathcal{U}'''$ is the required open in the Zariski topology subset of $\mathcal{L}$.

In the case when $n \geq 4$ one can perform a nondegenerate linear transformation of $X_1, \ldots, X_n$ and suppose without loss of generality that leading coefficient $\mathrm{lc}_{X_n} f = 1$ and $\partial F/\partial X_1 \neq 0$. Besides that, we can assume without loss of generality that $X_n \notin \mathcal{L}$ if $\dim \mathcal{L} < n$. After that we set $X_n = Y$ and applying Lemma 23 obtain a linear form L satisfying the required conditions. Now similarly to the proof of Lemma 23 we get the existence of the required open in the Zariski topology subset $\mathcal{U}'''$ of $\mathcal{L}$ in the considered case. The corollary is proved.

LEMMA 24 *Let k be a perfect field of arbitrary characteristic. In the notations from the Introduction for every $\sigma + 1 \leq \alpha \leq n - 1$ the set $\mathcal{U}_\alpha$ is nonempty and open in the Zariski topology in $\mathbb{A}^{(r+1)(\alpha-\sigma)}(\overline{k})$.*

PROOF Using the Veronese embedding we shall suppose without loss of generality that $d' = 1$. By Theorem 4 there exists $\lambda = (L_{s+1}, \ldots, L_\sigma)$, see the

Introduction, such that the irreducible over $\overline{k}$ components of the algebraic varieties W and $W \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma)$ are in the one–to–one correspond by the rule $E \mapsto E \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma)$, and the intersection $W \cap \mathcal{Z}(L_{s+1}, \dots, L_\sigma)$ is transversal (all these are conditions of general position for λ). Replacing W by W_λ we shall assume in what follows without loss of generality that $s = \sigma$. In the case of the nonzero characteristic replacing k by its purely transcendental extension $k(t)$ we shall suppose without loss of generality that the field k is infinite.

Let us consider at first the case of zero–characteristic (at the end of the proof we shall return to the case of arbitrary characteristic). Let us show that $\mathcal{U}_\alpha \neq \emptyset$. Let $L'' \in \mathcal{U}_0''$. Denote $p_{L''} = p_{\lambda, L''}$ (now $\lambda = ()$ since $s = \sigma$). Suppose that for an integer $\sigma \leq i < \alpha$

(*) linear forms $L'_{\sigma+1}, \dots, L'_i$, $\sigma+1 \leq i < \alpha$ exist such that all L'_j, $s+1 \leq j \leq i$ are linear combinations of $L_0'', L''_{\sigma+1}, \dots, L''_n$ with coefficients from k, and $(L'_{\sigma+1}, \dots, L'_i) \in \mathcal{U}_i$.

Then by Lemma 11 [12] and Lemma 14 [12] the morphism $W \cap \mathcal{Z}(L'_{\sigma+1}, \dots, L'_i) \to \mathcal{Z}(X_1, \dots, X_{i-\sigma}) = \mathbb{P}^{n-i}(\overline{k})$ induced by $p_{L''}$ is dominant. It induces the morphism of affine algebraic varieties $\mathrm{con}(W) \cap \mathcal{Z}(L'_{\sigma+1}, \dots, L'_i) \to \mathbb{A}^{n-i+1}(\overline{k})$. Let us represent

$$\mathrm{con}(W) \cap \mathcal{Z}(L'_{\sigma+1}, \dots, L'_i) \setminus \mathcal{Z}(L''_0, L''_{\sigma+1}, \dots, L''_n) = \bigcup_{j \in J} E_j,$$

where E_j, $J \in J$, is a finite family of affine nonempty open in the Zariski topology subsets of $\mathrm{con}(W) \cap \mathcal{Z}(L'_{\sigma+1}, \dots, L'_i) \setminus \mathcal{Z}(L''_0, L''_{\sigma+1}, \dots, L''_n)$. Hence by Theorem 4 applied to every morphism $E_j \to \mathbb{A}^{n-i+1}(\overline{k})$, $j \in J$, there is a linear form L''_{i+1} (from the intersection of open in the Zariski topology subsets from the statement of Theorem 4 over all $j \in J$) such that (*) holds with $i+1$ in place of i. Thus, we have proved the existence of $(L'_{\sigma+1}, \dots, L'_\alpha) \in \mathcal{U}_\alpha$ by the induction on i.

Let us prove that $\mathcal{U}_\alpha$ is open in the Zariski topology subset of $\mathbb{A}^{(n-\alpha)(n+1)}(\overline{k})$. At first we shall show that $\mathcal{U}_\alpha$ is a constructive subset of $\mathbb{A}^{(n-\alpha)(n+1)}(\overline{k})$. Probably this can be proved using the first order theory of algebraically closed fields. But we give the direct proof.

Recall that by definition a subset of an algebraic variety F is constructive if and only if it is a union $\cup_{\gamma \in \Gamma}(F'_\gamma \cap F''_\gamma)$, where the number of elements $\#\Gamma < +\infty$, all F'_γ are open in the Zariski topology subsets of F, and all F''_γ are closed in the Zariski topology subsets of F. We shall use the following criterion. Suppose $E \subset F$, and for every closed subvariety $E' \subset F$ such that $E' \cap E$ is dense with respect to the Zariski topology in E' there exists a nonempty open in the Zariski topology subset $E'' \subset E'$ such that $E'' \subset E$. Then E is a constructive subset of F. The proof of this criterion is by the induction on $\dim \overline{E}$ (here $\overline{E}$ is the closure of E with respect to Zariski topology). Indeed, there is a dense open in the Zariski topology subset $E''' \subset \overline{E}$ such that $E''' \subset E$. One can replace E by $E \setminus E'''$. Obviously $\dim \overline{E \setminus E'''} < \dim \overline{E}$. In the given criterion one can assume additionally that E' is irreducible over the algebraic closure of the ground field.

Let us show that $\mathcal{U}_\alpha$ satisfies the formulated criterion. Let E' be a closed irreducible over $\overline{k}$ subvariety of $\mathbb{A}^{(n-\alpha)(n+1)}(\overline{k})$ and $(L'_{\sigma+1}, \dots, L'_\alpha) \in E' \cap \mathcal{U}_\alpha$. Let us extend the ground field k till k_u, see the Introduction. We shall assume

without loss of generality that the all the coefficients of the linear forms U_i, $\sigma+1 \le i \le \alpha$ are infinitely close to the corresponding coefficients of the linear forms L'_i. Suppose that $(L''_{\sigma+1}, \dots, L''_\alpha) \in E' \cap \mathcal{U}_\alpha$, all linear forms $L''_i - L'_i$, $\sigma+1 \le i \le \alpha$, have coefficients from $\overline{k_u}$, which are infinitely small with respect to k (or zeroes). Then by Lemma 13 [13] $(L''_{\sigma+1}, \dots, L''_\alpha) \in \mathcal{U}''_\alpha$. By Lemma 15 [13] $\mathrm{st}(W(L''_{\sigma+1}, \dots, L''_\alpha)) = W(L'_{\sigma+1}, \dots, L'_\alpha)$. Hence by Lemma 5 [13] the degree of every irreducible over $\overline{k_u}$ component of $W(L''_{\sigma+1}, \dots, L''_\alpha)$ is at least $\deg W(L'_{\sigma+1}, \dots, L'_\alpha)$. Therefore, $W(L''_{\sigma+1}, \dots, L''_\alpha)$ is irreducible over $\overline{k_u}$.

Further, one can choose $(L''_{\sigma+1}, \dots, L''_\alpha)$ such that additionally $\xi = (L''_{\sigma+1}, \dots, L''_\alpha)$ is a generic point of E'. Denote by $\overline{k}(\xi)$ the field generated over $\overline{k}$ by the coordinates of the point ξ. We shall identify it with $\overline{k}(E')$. There exists a linear projection $\pi : W(L''_{\sigma+1}, \dots, L''_\alpha) \to \mathbb{P}^{n-\alpha+1}(\overline{k})$, $(X_0 : \dots : X_n) \mapsto (Z_0 : \dots : Z_{n-\alpha+1})$ such that all Z_j, $0 \le j \le n-\alpha+1$ are linear forms with coefficients from k, the image $\pi(W(L''_{\sigma+1}, \dots, L''_\alpha)) = \mathcal{Z}(H)$, $H \in \overline{k}(\xi)[X_0, \dots, X_{n-\alpha+1}]$ is separable, the morphism $W(L''_{\sigma+1}, \dots, L''_\alpha) \to \mathcal{Z}(H)$ induced by π is finite and $\deg W(L''_{\sigma+1}, \dots, L''_\alpha) = \deg H$ (these are conditions of general position for the linear projection). Hence the number of irreducible over $\overline{k(\xi)}$ factors of H is the number of irreducible over $\overline{k}$ components of W. Replacing H and all L''_j by cH and cL''_j with an appropriate $0 \ne c \in \overline{k}[E']$ we shall assume without loss of generality that $H \in \overline{k}[E'][X_0, \dots, X_{n-\alpha+1}]$ and all $L''_j \in \overline{k}[E'][X_0, \dots, X_n]$. Obviously the polynomial $H(Z_0, \dots, Z_{n-\alpha+1})$ vanishes on $W(L''_{\sigma+1}, \dots, L''_\alpha)$. Now consider the homomorphism $\overline{k}[E'] \to \overline{k}$ giving a point $\widetilde{\xi} \in E'$. This homomorphism is extended in the natural way till the homomorphisms of the rings of polynomials with coefficients from $\overline{k}[E']$. Denote by $\widetilde{H}$ and $\widetilde{L}_j$ the images of H and L'_j under these homomorphisms. There is a nonempty open in the Zariski topology subset $E^{(1)} \subset E'$ such that if $\widetilde{\xi} \in E^{(1)}$ then $(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_\alpha) \in \mathcal{U}''_\alpha$, the image $\pi(W(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_\alpha)) = \mathcal{Z}(\widetilde{H})$, the morphism $W(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_\alpha) \to \mathcal{Z}(\widetilde{H})$ induced by π is finite, and $\deg W(\widetilde{L}_{\sigma+1}, \dots, \widetilde{L}_\alpha) = \deg \widetilde{H}$, $\widetilde{H} \in \overline{k}[X_0, \dots, X_{n-\alpha+1}]$ is separable. This assertion is straightforward (we leave the details to the reader). Finally there exists a nonempty open in the Zariski topology subset $E^{(2)} \subset E^{(1)}$ such that for every $\widetilde{\xi} \in E^{(2)}$ the number of the irreducible over $\overline{k}$ factors of the polynomial $\widetilde{H}$ is equal to the number of irreducible over $\overline{k(\xi)}$ factors of H. Indeed, in Lemma 23 we have proved that $\mathcal{U}_4 \cap \mathcal{U}'_n$ is open in the Zariski topology, and the last assertion is proved in the similar way.

Thus, the set $\mathcal{U}_\alpha$ is constructive. Finally, as we have seen in the proof for every point of $L' \in \mathcal{U}_\alpha$ there exists an infinitely small neighborhood $\mathcal{W}$ of L' in $\mathcal{U}''_\alpha$ such that $\mathcal{W} \subset \mathcal{U}_\alpha$. Obviously it is possible only in the case if $\mathcal{U}_\alpha$ is an open in the Zariski topology. The lemma is proved for zero–characteristic.

In the case of nonzero characteristic it is sufficient to generalize all the lemmas from [12], [13] which are used in the proof to the case of infinite ground field of nonzero characteristic under condition that the extension $k(W) \supset k(W')$ is separable. It is possible without difficulties. We leaves the details to the reader. One can define infinitesimals in the case of nonzero characteristic using valuations, see [2]. For example, in the most simple case for a transcendental over k element ε we have the valuation $v : k(\varepsilon) \to \mathbb{Z} \cup \{+\infty\}$ over k such that $v(\varepsilon) = 1$. It can be extended till the valuation $\overline{v} : \overline{k(\varepsilon)} \to \mathbb{Q} \cup \{+\infty\}$. By definition a nonzero element $a \in \overline{k(\varepsilon)}$ is infinitesimal if and only if $\overline{v}(a) > 0$. The lemma is proved.

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