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Polynomial–Time Algorithms in Zero–Characteristic for a New Model of Representation of Algebraic Varieties

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Abstract

We suggest a model of representation of algebraic varieties based on representative systems of points of its irreducible components. Deterministic polynomial–time algorithms to substantiate this model are described in zero–characteristic. The main result here is a construction of the intersection of algebraic varieties. As applications we get efficient algorithms for constructing the smooth stratification and smooth cover of an algebraic variety introduced by the author earlier.

Introduction

The present work concludes the series of papers [2], [3], [4], [5], [6], [7], [8], [9] (the correction of Lemma 2 [9], see in [10]), [10], [11], where the polynomial–time algorithms for to algebraic varieties in zero–characteristic are suggested (we do not use the results of [11] in the present paper; but the particular case of [11] from [8] is necessary here). Before formulating our results we describe how to give a quasiprojective algebraic variety using a representative system of points of its irreducible components. The model of representation of algebraic varieties suggested here slightly generalizes the one outlined in [7], see the remarks below. In [7] the description of the algorithms for this representation was postponed. It hase become possible only using the results of four more papers [5], [8], [9], [10].

Let k be a field of zero–characteristic with algebraic closure $\overline{k}$. Let $X_0, X_1, \ldots$ be independent variables over k. Denote by $\mathbb{P}^n(\overline{k})$, $n \geq 0$, the projective space over the field $\overline{k}$ with coordinates $X_0, \ldots, X_n$. We shall suppose that $\mathbb{P}^n(\overline{k})$ is defined over k. For arbitrary homogeneous polynomials $g_1, \ldots, g_m \in \overline{k}[X_0, \ldots, X_n]$ we shall denote by $\mathcal{Z}(g_1, \ldots, g_m)$ the set of all common zeroes of polynomials $g_1, \ldots, g_m$ in $\mathbb{P}^n(\overline{k})$. The similar notations will be used for the sets of zeroes of ideals and polynomials with other fields of coefficients in affine and projective spaces (this will be seen from the context).

Let W be a quasiprojective algebraic variety in $\mathbb{P}^n(\overline{k})$ and W is defined over k. Then we represent

$$W = \bigcup_{1 \leq i \leq b} W^{(i)} \setminus \bigcup_{b+1 \leq i \leq a} W^{(i)}, \tag{1}$$

where $1 \leq b \leq a$ are integers, and all $W^{(i)}$, $1 \leq i \leq a$, are projective algebraic varieties in $\mathbb{P}^n(\overline{k})$ defined over k. Each algebraic variety $W^{(i)}$, $1 \leq i \leq a$, is a union of some irreducible components of the variety $V^{(i)} = \mathcal{Z}(f_1^{(i)}, \ldots, f_{m(i)}^{(i)}) \subset \mathbb{P}^n(\overline{k})$ where homogeneous polynomials $f_1^{(i)}, \ldots, f_{m(i)}^{(i)} \in k[X_0, \ldots, X_n]$ are given, $m(i) \geq 1$. For every $0 \leq s \leq n$ denote by $V^{(i,s)}$ (respectively $W^{(i,s)}$) the union of all irreducible components of dimension $n - s$ of $\mathcal{Z}(f_1^{(i)}, \ldots, f_{m(i)}^{(i)})$ (respectively $W^{(i)}$). Therefore $W^{(i,s)}$ is a union of some irreducible components of $V^{(i,s)}$. For every $0 \leq s \leq n$ the family of linear forms $L_{s+1}^{(i,s)}, \ldots, L_n^{(i,s)} \in k[X_0, \ldots, X_n]$ is given such that the number of points

$$\#V^{(i,s)} \cap \mathcal{Z}(L_{s+1}^{(i,s)}, \ldots, L_n^{(i,s)}) < +\infty \tag{2}$$

is finite, every point $\xi \in V^{(i,s)} \cap \mathcal{Z}(L_{s+1}^{(i,s)}, \ldots, L_n^{(i,s)})$ is a smooth point of the algebraic variety $\mathcal{Z}(f_1^{(i)}, \ldots, f_{m(i)}^{(i)})$, and the intersection of the tangent spaces in the point ξ of $V_s^{(i)}$ and $\mathcal{Z}(L_{s+1}^{(i,s)}, \ldots, L_n^{(i,s)})$ is transversal, i.e.,

$$T_{\xi, V_s^{(i)}} \cap \mathcal{Z}(L_{s+1}^{(i,s)}, \ldots, L_n^{(i,s)}) = \{\xi\} \tag{3}$$

(we consider the tangent space $T_{\xi, V_s^{(i)}}$ as a subspace of $\mathbb{P}^n(\overline{k})$). The set of points

$$\Xi^{(i,s)} = W^{(i,s)} \cap \mathcal{Z}(L_{s+1}^{(i,s)}, \ldots, L_n^{(i,s)})$$

is given. Each point from $\Xi^{(i,s)}$ is represented in form (11), see below. Hence the following property holds. Let $\xi \in V^{(i,s)} \cap \mathcal{Z}(L_{s+1}^{(i,s)}, \ldots, L_n^{(i,s)})$ and E be the uniquely defined irreducible over k component of the algebraic variety $V^{(i,s)}$ such that $\xi \in E$. Then $\xi \in \Xi^{(i,s)}$ if and only if E is a component of $W^{(i)}$. In what follows, unless we state otherwise, we assume that the degrees $\deg_{X_0,\ldots,X_n} f_j^{(i)} < d$ for all i, j.

Thus, formally the suggested in this paper representation of W is a quadruple

$$(f, L, \Xi, b), \tag{4}$$

where f is a family of polynomials

$$f_j^{(i)}, \quad 1 \le j \le m(i),\, 1 \le i \le a, \tag{5}$$

L is a family of linear forms

$$L_w^{(i,s)}, \quad s+1 \le w \le n,\, 0 \le s \le n,\, 1 \le i \le a, \tag{6}$$

and Ξ is a family of finite sets of points

$$\Xi^{(i,s)}, \quad 0 \le s \le n,\, 1 \le i \le a. \tag{7}$$

Denote also

$$\Xi^{(i)} = \bigcup_{0 \le s \le n} \Xi^{(i,s)}. \tag{8}$$

In [7] only the case $b = 1$ is considered. In the present paper in comparison with [7] we replace the low index s by the upper one to avoid an ambiguity of the notation when one consider more than one algebraic variety, see below.

Notice that (2) and (3) can be always verified using the algorithms from [4], see the Introduction of [7] for details. Note also that for a given point $y \in \mathbb{P}^n(\overline{k})$ one can decide whether $y \in W^{(i,s)}$ using the algorithms from [4] and [1], see the Introduction of [7]. Thus, for a given point $y \in \mathbb{P}^n(\overline{k})$ one can decide also within the polynomial time whether $y \in W^{(i)}$, $1 \le i \le a$, and whether $y \in W$.

Let $W = \bigcup_{i \in I} W_i$ be the decomposition of W into the union of defined over k and irreducible over k (respectively irreducible over $\overline{k}$) components and representation (4) is given. Then using Theorem 1 [8] (or more strong Theorem 3 [10]) and Theorem 2 [8] one can construct for every $i \in I$ the representation $(f, L_i, \Xi_i, 1)$ of the irreducible component W_i (in the case when W_i is irreducible over $\overline{k}$ we construct the minimal field of definition k_i of W_i containing k and replace the ground field k by k_i in the representation of W_i, see [8] for details). The working time of this algorithm is polynomial in d^n and the size of input, see [8] (and also the proof of Theorem 1 below for the partial case $\nu = 1$).

Further, see Theorem 3 below, let W_1, W_2 be two quasiprojective algebraic varieties which are similar to W and are given in the similar way (with the same bound d for degrees of the polynomials). Then one can decide whether $W_1 = W_2$ within the time in d^n and the size of input. In [7] this is proved only when W_1 and W_2 are projective algebraic varieties. The general case of quasiprojective algebraic varieties is difficult.

To prove Theorem 3 we need at first describe an algorithm for constructing an intersection of ν quasiprojective algebraic varieties given in the considered model within the time in $d^{n\nu}$ and the size of input, see Theorem 1 below. This algorithm uses the reduction to the diagonal. Here one needs to apply Theorem 1 [10]. The last theorem has a long proof. It is based on [9], [8], [7] and other our papers, see the Introduction of [10]. Besides that in Theorem 1 indices of intersection of algebraic varieties are computed when they are defined. Notice here that at the output of the algorithm from Theorem 1 the intersection of quasiprojective algebraic varieties is not given in the considered model. The irreducible components of the intersections not always can be given using representative systems of points of irreducible components of an algebraic variety $\mathcal{V}$ with good upper bounds for degrees of polynomials giving $\mathcal{V}$. Still we get at the output of the algorithm from Theorem 1 all the information about the intersection. One can consider the representation of the intersection of algebraic varieties from assertion (a) of Theorem 1 as a generalization of representation (4) to the case of intersections of algebraic varieties given in form (4).

Denote $m = m(1)$, and $f_i = f_i^{(1)}$, $1 \le i \le m$. Let $V = \mathcal{Z}(f_1, \dots, f_m) \subset \mathbb{P}^n(\overline{k})$ be an algebraic variety. Recall the definition from [7].

DEFINITION 1 *Smooth cover of the algebraic variety V is a finite family*

$$V_\alpha, \quad \alpha \in A, \tag{9}$$

of quasiprojective smooth algebraic varieties $V_\alpha \subset \mathbb{P}^n(\overline{k})$, $\alpha \in A$ such that V is represented as a union $V = \cup_{\alpha \in A} V_\alpha$. Further, we shall require that all irreducible components of V_α have the same dimension (which depends only on α). Smooth stratification of the algebraic variety V is a smooth cover V_α, $\alpha \in A$, of V such that additionally for any two $\alpha_1, \alpha_2 \in A$ if $\alpha_1 \ne \alpha_2$ then $V_{\alpha_1} \cap V_{\alpha_2} = \emptyset$.

In this paper we assume that the degree of an arbitrary projective algebraic variety is the sum of the degrees of all its irreducible components (of different dimensions). The degree of a quasiprojective algebraic variety $\mathcal{V}$ is by definition the degree of its closure in the corresponding projective space.

In [7] using the construction of local parameters from [5] we prove the existence of smooth cover (respectively smooth stratification) (9) of the algebraic variety V with the bound for the degrees of strata $2^{2^{n^C}} d^n$, and the number of strata $2^{2^{n^C}} d^n$ (respectively the number of strata $2^{2^{n^C}} d^{n(n+1)/2}$) for an absolute constant $0 < C \in \mathbb{R}$, see Theorem 2 [7]. The constructions of [7] are quite explicit. It turns out that it is sufficient to use additionally only Theorem 1 and Theorem 2 (the last only one in the case $(\nu, nn_1) = (3, 2)$) to obtain the algorithms for constructing the smooth cover and smooth stratification from Theorem 2 [7] within the time polynomial in $2^{2^{n^C}} d^n$ (respectively $2^{2^{n^C}} d^{n(n+1)/2}$) and the size of input, see Theorem 4 below.

Now we proceed to the precise statements. Let the integers a, b; $m(i)$, $1 \le i \le b$, the homogeneous polynomials $f_j^{(i)} \in k[X_0, \dots, X_n]$, $1 \le j \le m(i)$, $1 \le i \le a$, the algebraic varieties $V^{(i)}$, $W^{(i)}$, and W be as above.

Let the field $k = \mathbb{Q}(t_1, \dots, t_l, \theta)$ where $t_1, \dots, t_l$ are algebraically independent over the field $\mathbb{Q}$ and θ is algebraic over $\mathbb{Q}(t_1, \dots, t_l)$ with the minimal polynomial $F \in \mathbb{Q}[t_1, \dots, t_l, Z]$ and leading coefficient $\mathrm{lc}_Z F$ of F is equal to 1. We shall represent each polynomial $f = f_j^{(i)}$ in the form

$$f = \frac{1}{a_0} \sum_{i_0, \dots, i_n} \sum_{0 \le j < \deg_Z F} a_{i_0, \dots, i_n, j} \theta^j X_0^{i_0} \cdots X_n^{i_n} \,,$$

where $a_0, a_{i_0,\dots,i_n,j} \in \mathbb{Z}[t_1,\dots,t_l]$, $\mathrm{GCD}_{i_0,\dots,i_n,j}(a_0, a_{i_0,\dots,i_n,j}) = 1$. Define the length $\mathrm{l}(a)$ of an integer a by the formula $\mathrm{l}(a) = \min\{s \in \mathbb{Z} :\ |a| < 2^{s-1}\}$. The length of coefficients $\mathrm{l}(f)$ of the polynomial f is defined to be the maximum of lengths of coefficients from $\mathbb{Z}$ of polynomials $a_0, a_{i_0,\dots,i_n,j}$ and the degree

$$\deg_{t_\gamma}(f) = \max_{i_0,\dots,i_n,j}\{\deg_{t_\gamma}(a_0), \deg_{t_\gamma}(a_{i_0,\dots,i_n,j})\},$$

where $1 \le \gamma \le l$. In the similar way we shall define degrees and lengths of integer coefficients of other polynomials, in particular $\deg_{t_\gamma} F$ and $\mathrm{l}(F)$ are defined. We shall suppose that we have the following bounds

$$\begin{aligned} &\deg_{X_0,\dots,X_n}(f_j^{(i)}) < d,\ \deg_{t_\gamma}(f_j^{(i)}) < d_2,\ \mathrm{l}(f_j^{(i)}) < M, \\ &\deg_Z(F) < d_1,\ \deg_{t_\gamma}(F) < d_1,\ \mathrm{l}(F) < M_1. \end{aligned} \tag{10}$$

for all $1 \le j \le m(i)$, $1 \le i \le a$, $1 \le \gamma \le l$. The size $\mathrm{L}(f)$ of the polynomial f is defined to be the product of $\mathrm{l}(f)$ to the number of all the coefficients from $\mathbb{Z}$ of f in the dense representation. We have

$$\mathrm{L}(f_j^{(i)}) < (\binom{d+n}{n} d_1 + 1) d_2^l M$$

Similarly $\mathrm{L}(F) < d_1^{l+1} M_1$.

REMARK 1 *Unless we state otherwise, in what follows we suppose l to be fixed. The working time of the algorithms from Theorem 1, Theorem 2 and Theorem 4, see below, is essentially the same as for solving systems of polynomial equations with a finite set of solutions in the projective space. So this theorems can be formulated also in the case when l is not fixed, see [1]. Notice that the constants $O(\dots)$, see Theorem 1, Theorem 2 and Theorem 4 below, in the estimate of the lengths of integer coefficients of linear forms L'_j, L_j are absolute; they does not depend on l.*

We shall represent a point $z \in V$ with coordinates from a finite extension of k as follows. An index $0 \le i_0 \le n$ is known such that $X_{i_0}(z) \ne 0$ and an isomorphism of fields

$$k(z) = k((X_1/X_{i_0})(z), \dots, (X_n/X_{i_0})(z)) = k[\eta] \simeq k[Z]/(\Phi) \tag{11}$$

is given where $\eta = \sum_{0 \le i \le n} c_i (X_i/X_{i_0})(z)$, the coefficients $c_i \in \mathbb{Z}$ are given and $\Phi \in k[Z]$ is minimal polynomial of η over k with leading coefficient $\mathrm{lc}_Z \Phi = 1$ (so the point z is defined up to a conjugation over k).

Let $g \in k[\eta]$ be an arbitrary element. Then $g = G(\eta)$ for the uniquely defined polynomial $A \in k[Z]$ such that $\deg_Z G < \deg_Z \Phi$. The length of integer coefficients $\mathrm{l}(g)$, the size $\mathrm{L}(g)$ and the degrees $\deg_{t_\alpha} g$, $1 \le \alpha \le l$, of g are defined by the formulas

$$\mathrm{l}(g) = \mathrm{l}(G), \quad \mathrm{L}(g) = \mathrm{L}(G), \quad \deg_{t_\alpha} g = \deg_{t_\alpha} G.$$

Let us define the size of the point z to be $\mathrm{L}(\Phi) + \sum_{0 \le i \le n} \mathrm{L}(X_i/X_{i_0})$. Now let Ξ be an arbitrary finite set of points defined over k, and every point from Ξ is given in form (11). Hence Ξ gives a zero–dimensional algebraic varieties defined over k. Put the size $\mathrm{L}(\Xi)$ of Ξ to be the sum of sizes of its irreducible over k components.

Recall that in [10] we give the definition of transversality of intersection of algebraic varieties. Now we give the analogous natural definition related to the proper intersections. Let $W_1, \ldots, W_\nu \subset \mathbb{P}^n(\overline{k})$, $\nu \geq 1$, be ν quasiprojective algebraic varieties defined over k. Let E be an arbitrary defined over k and irreducible over k component of $W_1 \cap \ldots \cap W_\nu$. We shall say that the intersection $W_1 \cap \ldots \cap W_\nu$ is proper at E (in the ambient space $\mathbb{P}^n(\overline{k})$) if and only if for every defined over k and irreducible over k component E_i, $1 \leq i \leq \nu$, of W_i such that $E_i \supset W$ the equality $\sum_{1\leq i \leq \nu}(n - \dim E_i) = n - \dim E$ holds (and hence $\dim E_i$ depends only on i). The intersection of $W_1, \ldots, W_\nu$ is proper (in the ambient space $\mathbb{P}^n(\overline{k})$) if and only if it is proper at every its defined over k and irreducible over k component.

If $\nu = 2$ then the index of intersection $i(W_1, W_2; E) = i(W_1, W_2; E')$ where E' is an arbitrary irreducible over $\overline{k}$ component of E, and the index of intersection $i(W_1, W_2; E')$ is defined in the usual way, see, e.g., [12].

For an arbitrary $\nu > 2$ we define the index of intersection $i(W_1, \ldots, W_\nu; E)$ of the algebraic varieties $W_1, \ldots, W_\nu$ at E recursively by the formula $i(W_1, \ldots, W_\nu; E) = \sum_{E''} i(W_1, \ldots, W_{\nu-1}; E'')i(E'', W_\nu; E)$ where E'' runs over all the irreducible over k components of $W_1 \cap \ldots \cap W_{\nu-1}$ such that $E'' \supset E$. For $\nu = 1$ it is natural to put $i(W_1; E) = 1$.

Here all the indices of intersection are considered in $\mathbb{P}^n(\overline{k})$. To specify this we denote $i_{\mathbb{P}^n(\overline{k})}(W_1, \ldots, W_\nu; E) = i(W_1, \ldots, W_\nu; E)$. Assume that all W_j, $1 \leq j \leq \nu$, are subvarieties of an affine space $\mathbb{A}^n(\overline{k})$. One can identify $\mathbb{A}^n(\overline{k}) = \mathbb{P}^n(\overline{k}) \setminus \mathcal{Z}(X_0)$. We shall denote also in this case $i_{\mathbb{A}^n(\overline{k})}(W_1, \ldots, W_\nu; E) = i(W_1, \ldots, W_\nu; E)$ when the last index of intersection is defined.

We shall use the reduction to diagonal for indices of intersection. Namely, let $\nu \geq 1$ be an integer. Let us identify the affine space $\mathbb{A}^{n\nu}(\overline{k}) = (\mathbb{A}^n(\overline{k}))^\nu$. Put $\Delta = \{(x, x, \ldots, x) \in \mathbb{A}^{n\nu}(\overline{k}) : x \in \mathbb{A}^n(\overline{k})\}$ to be the diagonal subvariety. Now we identify

$$\mathbb{A}^n(\overline{k}) = (\mathbb{A}^n(\overline{k}))^\nu \cap \Delta. \tag{12}$$

Let $W_1, \ldots, W_\nu$ be affine algebraic varieties in $\mathbb{A}^n(\overline{k})$ and E be a defined over k and irreducible over k component of $W_1 \cap \ldots \cap W_\nu$ such that the last intersection is proper at E. Then by (12) the variety E is a defined over k and irreducible over k component of $(W_1 \times \ldots \times W_\nu) \cap \Delta$ holds. Obviously the last intersection is proper at E. We have

$$i_{\mathbb{A}^n(\overline{k})}(W_1, \ldots, W_\nu; E) = i_{\mathbb{A}^{n\nu}(\overline{k})}(W_1 \times \ldots \times W_\nu, \Delta; E). \tag{13}$$

This formula of reduction to diagonal is well known for $\nu = 2$, see, e.g., [12]. For an arbitrary ν it is proved by the induction on ν using the general properties of indices of intersection (these general properties also can be found in [12]; we leave the details to the reader).

Let $\nu \geq 1$ be an integer. For every integer $1 \leq \alpha \leq \nu$ let $m_\alpha(i)$, a_α, b_α, $f^{(i)}_{\alpha,j}$, W_α, V_α, $W^{(i)}_\alpha$, $V^{(i)}_\alpha$, $W^{(i)}_\alpha$, $V^{(i,s)}_\alpha$, $W^{(i,s)}_\alpha$, $L^{(i,s)}_{\alpha,\beta}$, $\Xi^{(i,s)}_\alpha$, $(f_\alpha, L_\alpha, \Xi_\alpha, b_\alpha) = \rho_\alpha$ are similar to the introduced above $m(i)$, a, b, $f^{(i)}_j$, W, V, $W^{(i)}$, $V^{(i)}$, $W^{(i)}$, $V^{(i,s)}$, $W^{(i,s)}$, $L^{(i,s)}_\beta$, $\Xi^{(i,s)}$, $(f, L, \Xi, b) = \rho$ respectively. In what follows in this paper we suppose that inequalities (10) with $f^{(i)}_{\alpha,j}$ in place of $f^{(i)}_j$ hold for every $1 \leq \alpha \leq \nu$.

THEOREM 1 *Assume that for every $1 \le \alpha \le \nu$ a representation $(f_\alpha, L_\alpha, \Xi_\alpha, b_\alpha)$ of a quasiprojective algebraic variety W_α is given. Then one can construct linear forms $L_0, \ldots, L_{n+1} \in k[X_0, \ldots, X_n]$ with integer coefficients of length $O(n\nu \log d + \sum_{1\le i\le \nu} \log b_i + \log(\sum_{1\le i\le\nu} a_i))$, the finite set of indices J, and for every $j \in J$ the finite set Ξ_j of points from $W_1 \cap \ldots \cap W_\nu$ (each point is represented in form (11)) such that the following assertions hold.*

(a) For every $j \in J$ there is the unique defined over k and irreducible over k component of $W_1 \cap \ldots \cap W_\nu$ (denote it by E_j) such that $\dim E_j = n - s(j)$ and $\Xi_j = E_j \cap \mathcal{Z}(L_{s(j)+1}, \ldots, L_n)$ in $\mathbb{P}^n(\overline{k})$. Conversely, for every defined over k and irreducible over k component E' of $W_1 \cap \ldots \cap W_\nu$ there is $j' \in J$ such that $E' = E_{j'}$. Hence $W_1 \cap \ldots \cap W_\nu = \bigcup_{j\in J} E_j$ is the decomposition of $W_1 \cap \ldots \cap W_\nu$ into the union of irreducible over k components. Let $\overline{E}_j$ be the closure of E_j with respect to the Zariski topology in $\mathbb{P}^n(\overline{k})$. Then for every $j \in J$

$$\Xi_j \cap \mathcal{Z}(L_0) = \emptyset, \quad \#\Xi_j = \#(L_{n+1}/L_0)(\Xi_j) = \deg \overline{E}_j \tag{14}$$

(here and below $\#(.)$ denote the number of elements of a set), and all the points of Ξ_j are smooth points of $W_1 \cap \ldots \cap W_\nu$. The variety

$$E_j = \overline{E}_j \setminus \Big(\bigcup_{1\le\alpha\le\nu} \ \bigcup_{b_\alpha+1\le i\le a_\alpha} W_\alpha^{(i)} \Big).$$

(b) If the intersection of $W_1, \ldots, W_\nu$ is proper at E_j then one can compute the index of intersection $i_{\mathbb{P}^n(\overline{k})}(W_1, \ldots, W_\nu; E_j)$, and for every point $\xi \in \Xi_j$ the equality $i_{\mathbb{P}^n(\overline{k})}(W_1, \ldots, W_\nu; E_j) = i_{\mathbb{P}^n(\overline{k})}(W_1, \ldots, W_\nu, \mathcal{Z}(L_{s+1}, \ldots, L_n); \xi)$ holds.

(c) For an arbitrary point $z \in \mathbb{P}^n(\overline{k})$ (given in form (11)) one can decide whether $z \in E_j$, and more than that, compute the multiplicity $\mu(z, E_j)$ of the point z at E_j.

(d) Let us identify the set of all $(n+2)$–tuples of linear forms from $\overline{k}[X_0, \ldots, X_n]$ with the affine space $\mathbb{A}^{(n+1)(n+2)}(\overline{k})$. For every $j \in J$ and an arbitrary $\lambda^* = (L_0^*, \ldots, L_{n+1}^*)$, where all $L_j^* \in \overline{k}[X_0, \ldots, X_n]$ are linear forms, put $\Xi_j^* = E_j \cap \mathcal{Z}(L_{s(j)+1}, \ldots, L_n) \subset \mathbb{P}^n(\overline{k})$. Let $\mathfrak{l} \in \mathbb{A}^{(n+1)(n+2)}(\overline{k})$ be a line defined over k' such that $\overline{\lambda} = (L_0, \ldots, L_{n+1}) \in \mathfrak{l}$. Then for all $\lambda^* \in \mathfrak{l}(k')$ (here $\mathfrak{l}(k')$ is the set of all k'–points of $\mathfrak{l}$), except at most a polynomial in $d^{n\nu}(\sum_{1\le i\le\nu} a_i) \prod_{1\le\alpha\le\nu} b_\alpha$ number, assertion (a) holds with λ^*, Ξ_j^*, $j \in J$, in place of $\overline{\lambda}, \Xi_j$, $j \in J$. For every element $\lambda^* \in \mathfrak{l}(k')$ one can decide whether assertions (a) hold with λ^*, Ξ_j^*, $j \in J$, in place of $\overline{\lambda}, \Xi_j$, $j \in J$.

The working time of the algorithm for constructing linear forms $L_0, \ldots, L_n$ and the family of finite sets Ξ_j, $j \in J$, satisfying (a) and also of the algorithm from assertion (b) is polynomial in $d^{n\nu}$, $\prod_{1\le\alpha\le\nu} b_\alpha$, and the sum of sizes $\sum_{1\le\alpha\le\nu} \mathrm{L}((f_\alpha, L_\alpha, \Xi_\alpha, b_\alpha))$. More precisely, this working time is polynomial in $d^{n\nu}$, $\prod_{1\le\alpha\le\nu} b_\alpha$, $\sum_{1\le\alpha\le\nu} a_\alpha$, M, M_1, d_1, d_2, $\sum_{1\le\alpha\le\nu,\, 1\le i\le a_\alpha} m(\alpha, i)$, and

$$\sum_{0\le s\le n,\ 1\le\alpha\le\nu,\ 1\le i\le a_\alpha} \mathrm{L}(\Xi_\alpha^{(i,s)}), \qquad \sum_{0\le s\le n,\ s+1\le w\le n,\ 1\le\alpha\le\nu,\ 1\le i\le a_\alpha} \mathrm{L}(L_{\alpha,w}^{(i,s)}).$$

The working time of the algorithm from (c) (respectively (d)) is polynomial in the same values and the size $\mathrm{L}(z)$ *of the point* z *(respectively the size* $\mathrm{L}(\lambda^*)$ *of the element* λ^**).*

THEOREM 2 *Assume that the conditions of Theorem 1 hold, i.e., for every* $1 \le \alpha \le \nu$ *a representation* $(f_\alpha, L_\alpha, \Xi_\alpha, b_\alpha)$ *of a quasiprojective algebraic variety* W_α *is given. Let* ν_1 *be an integer such that* $1 \le \nu_1 \le \nu$*. Then one can construct linear forms* $L_0, \ldots, L_{n+1} \in k[X_0, \ldots, X_n]$ *with integer coefficients of length* $O(n\nu \log d + \sum_{1\le i\le \nu} \log b_i + \log(\sum_{1\le i\le\nu} a_i))$*, the finite set of indices* $J^{(1)}$ *(respectively* $J^{(2)}$*), and for every* $j \in J^{(1)}$ *(respectively* $j \in J^{(2)}$*) the finite set* Ξ_j *of points from* $W_1 \cap \ldots \cap W_{\nu_1}$ *(respectively* $W_{\nu_1+1} \cap \ldots \cap W_\nu$*) such that the following assertions hold.*

(a) *For every* $j \in J^{(1)}$ *(respectively* $j \in J^{(2)}$*) there is the unique defined over* k *and irreducible over* k *component* E *of* $W_1 \cap \ldots \cap W_{\nu_1}$ *(respectively* $W_{\nu_1+1} \cap \ldots \cap W_\nu$*) such that* $\dim E = n - s(j)$ *and* $\Xi_j = E \cap \mathcal{Z}(L_{s(j)+1}, \ldots, L_n)$ *in* $\mathbb{P}^n(\overline{k})$*. Denote* $E = E_j$*. Conversely, for every defined over* k *and irreducible over* k *component* E' *of* $W_1 \cap \ldots \cap W_{\nu_1}$ *(respectively* $W_{\nu_1+1} \cap \ldots \cap W_\nu$*) there is* $j' \in J^{(1)}$ *(respectively* $j' \in J^{(2)}$*) such that* $E' = E_{j'}$*. Hence*

$$W_1 \cap \ldots \cap W_{\nu_1} = \bigcup_{j \in J^{(1)}} E_j\,, \qquad W_{\nu_1+1} \cap \ldots \cap W_\nu = \bigcup_{j\in J^{(2)}} E_j$$

are the decompositions of $W_1 \cap \ldots \cap W_{\nu_1}$ *and* $W_{\nu_1+1} \cap \ldots \cap W_\nu$ *into the unions of irreducible over* k *components. Let* $\overline{E}_j$ *be the closure of* E_j *with respect to the Zariski topology in* $\mathbb{P}^n(\overline{k})$*. Then for every* $j \in J^{(1)}$ *(respectively* $j \in J^{(2)}$*)*

$$\Xi_j \cap \mathcal{Z}(L_0) = \emptyset, \quad \#\Xi_j = \#(L_{n+1}/L_0)(\Xi_j) = \deg \overline{E}_j, \tag{15}$$

and all the points of Ξ_j *are smooth points of* $W_1 \cap \ldots \cap W_{\nu_1}$ *(respectively* $W_{\nu_1+1} \cap \ldots \cap W_\nu$*). Put* $A^{(1)} = \{1, \ldots, \nu_1\}$*,* $A^{(2)} = \{\nu_1+1, \ldots, \nu\}$*. Then for every* $j \in J^{(i)}$*,* $i = 1, 2$ *the variety*

$$E_j = \overline{E}_j \setminus \Big(\bigcup_{\alpha \in A^{(i)}} \; \bigcup_{b_\alpha + 1 \le i \le a_\alpha} W_\alpha^{(i)} \Big).$$

(b) *One can decide for every* $j_1 \in J^{(1)}$ *for every* $j_2 \in J^{(2)}$ *whether* $E_{j_1} \subset \overline{E}_{j_2}$*. More precisely, for the constructed linear forms* $L_0, \ldots, L_{n+1}$ *the inclusion* $E_{j_1} \subset \overline{E}_{j_2}$ *holds if and only if* $\Xi_{j_1} \subset \overline{E}_{j_2}$*.*

(c) *One can decide for every* $j_1 \in J^{(1)}$ *for every* $j_2 \in J^{(2)}$ *whether* $E_{j_1} \subset E_{j_2}$*.*

The working time of each of the algorithms from this theorem is polynomial in $d^{n\nu}$*,* $\prod_{1\le\alpha\le\nu} b_\alpha$*, and the sum of sizes* $\sum_{1\le\alpha\le\nu} \mathrm{L}((f_\alpha, L_\alpha, \Xi_\alpha, b_\alpha))$*. More precisely, the considered working time is polynomial in* $d^{n\nu}$*,* $\prod_{1\le\alpha\le\nu} b_\alpha$*,* $\sum_{1\le\alpha\le\nu} a_\alpha$*,* M*,* M_1*,* d_1*,* d_2*,* $\sum_{1\le\alpha\le\nu,\ 1\le i\le a_\alpha} m(\alpha, i)$*, and*

$$\sum_{0\le s\le n,\ 1\le\alpha\le\nu,\ 1\le i\le a_\alpha} \mathrm{L}(\Xi_\alpha^{(i,s)}), \qquad \sum_{0\le s\le n,\ s+1\le w\le n,\ 1\le\alpha\le\nu,\ 1\le i\le a_\alpha} \mathrm{L}(L_{\alpha,w}^{(i,s)}).$$

As an immediate consequence of Theorem 2 we get the following result.

THEOREM 3 *Assume that representations $(f_\alpha, L_\alpha, \Xi_\alpha, b_\alpha)$, $\alpha = 1, 2$ of two quasiprojective algebraic variety W_α are given. Then one can decide whether $W_1 \subset W_2$. Hence one can decide also whether $W_2 \subset W_1$ and whether $W_1 = W_2$. The working time of this algorithm is polynomial in d^n and the sizes $\mathrm{L}((f_\alpha, L_\alpha, \Xi_\alpha, b_\alpha))$ of the given representations of W_α, $\alpha = 1, 2$.*

PROOF Indeed by Theorem 2 one can decide whether $W_1 \subset W_2$ and $W_2 \subset W_1$. The theorem is proved (modulo Theorem 2).

THEOREM 4 *One can construct a smooth cover (respectively a smooth stratification) V_α, $\alpha \in A$, of the algebraic variety V such that every quasiprojective algebraic variety V_α is defined over k, irreducible over k, and represented in the accepted way. Let $\dim V_\alpha = n - s$ where $0 \le s \le n$, and $s = s(\alpha)$ depends on α. Let (4) be the constructed representation of V_α (it depends on α). Denote $h_{\alpha,j} = f_j^{(1)}$, $1 \le j \le m(1)$, and $\Delta_\alpha = f_1^{(2)}$ if $a \ge 2$, see (5). Then the constructed representation of V_α satisfies the following properties.*

(i) The integer $b = 1$. For the case of smooth cover $a = 2$ if $s < n$, and $a = 1$ if $s = n$. For the case of smooth stratification $a \le 2^{2^{n^C}} d^{n(n+1)/2}$ for an absolute constant $0 < C \in \mathbb{R}$.

(ii) $m(1) = s$ and if $a \ge 2$ then $m(2) = 1$. Hence if $a \ge 2$ then V_α is an irreducible component of the algebraic variety $\mathcal{Z}(h_{\alpha,1}, \ldots, h_{\alpha,s}) \setminus \mathcal{Z}(\Delta_\alpha)$ in the case of the smooth cover (respectively an open in the Zariski topology subset of an irreducible component of the latter algebraic variety in the case of smooth stratification).

(iii) There are linearly independent linear forms $Y_0, \ldots, Y_n \in k[X_0, \ldots, X_n]$ such that $X_i = \sum_{0 \le j \le n} x_{i,j} Y_j$, all the coefficients $x_{i,j} \in k$, all $x_{i,j}$ are integers with lengths $O(2^{n^C} + n \log d)$ for an absolute constant $C > 0$, and

$$\Delta_\alpha = \det(\partial h_{\alpha,i}/\partial Y_j)_{1 \le i,j \le s} = \det(\sum_{0 \le v \le n} x_{v,j} \partial h_{\alpha,i}/\partial X_v)_{1 \le i,j \le s}.$$

Hence V_α is a smooth algebraic variety by the implicit function theorem. Besides that, in the case of smooth cover one can take $Y_i = X_{\sigma(i)}$ for some permutation σ of the set $0, \ldots, n$.

(iv) The lengths of integer coefficients of all linear forms from the family L is $O(2^{n^C} + n \log d)$ for an absolute constant $0 < C \in \mathbb{R}$.

(v) For all $\alpha \in A$, $1 \le j \le s(\alpha)$ degrees $\deg_{X_0,\ldots,X_n} h_{\alpha,j}$ are less than $n^{2^{s(\alpha)^C}} d$ for an absolute constant $0 < C \in \mathbb{R}$. In the case of smooth stratification for all $i > 2$, j degrees $\deg_{X_0,\ldots,X_n} f_j^{(i)}$ are less than $2^{2^{n^C}} d$.

(vi) For all $\alpha \in A$, $1 \le j \le s(\alpha)$ lengths of coefficients of polynomials $h_{\alpha,j}$ are bounded from above by a polynomial in $n^{2^{s(\alpha)^C}} d^n$, d_1, d_2, M, M_1, m for an absolute constant $0 < C \in \mathbb{R}$. Further, in the case of smooth stratification lengths of coefficients of all polynomials from the family f are bounded from above by a polynomial in $2^{2^{n^C}} d^n$, d_1, d_2, M, M_1, m for

an absolute constant $0 < C \in \mathbb{R}$. The similar estimation holds for the size $L(\xi)$ of each point $\xi \in \Xi^{(i,s)}$ for all i, s, see (7).

(vii) The number of elements $\#A$ of A is bounded from above by $2^{2^{n^C}} d^n$ for the case of smooth cover (respectively $2^{2^{n^C}} d^{n(n+1)/2}$ for the case of smooth stratification) for an absolute constant $0 < C \in \mathbb{R}$.

The working time of the algorithm for constructing the smooth cover (respectively the smooth stratification) is polynomial in $2^{2^{n^C}} d^n$, d_1, d_2, M, M_1, m (respectively $2^{2^{n^C}} d^n$, d_1, d_2, M, M_1, m), where $0 < C \in \mathbb{R}$ is an absolute constant.

PROOF In Theorem 2 [7] we prove the existence of all the objects from the statement of Theorem 2. According to Sections 2–5 of [7] it is sufficient to use Theorem 1 and Theorem 2 of the present paper to obtain the algorithms for the constructions from proof of Theorem 2 [7]. More precisely, using Theorem 1 and Theorem 2 one can construct immediately the algebraic varieties arising from the proof of Theorem 2 [7]. We leave to the reader to repeat everything in details.

The required bounds for the working time of the algorithms from Theorem 4 follow immediately from the ones for the applied algorithms. The theorem is proved (modulo Theorem 1 and Theorem 2).

In the case when k is a subfield of $\mathbb{R}$ Theorem 4 gives as an immediate consequence an algorithm for computing dimension of a real algebraic variety $V(\mathbb{R})$ within the time polynomial in the size of input and $2^{2^{n^C}} d^n$, see Theorem 3 [7]. So we obtain one more proof of the last theorem.

1 Proof of Theorem 1

LEMMA 1 *Let V be a projective algebraic variety in $\mathbb{P}^n(\overline{k})$ such that all the irreducible components of V have the same dimension $n-s$ where $0 \le s \le n$. Let $L_0, \dots, L_{n+1} \in \overline{k}[X_0, \dots, X_n]$ be linear forms. Denote $\Xi = V \cap \mathcal{Z}(L_{s+1}, \dots, L_n)$ in $\mathbb{P}^n(\overline{k})$. Assume that for $\overline{\lambda} = (L_0, \dots, L_{n+1})$ the following condition holds.*

$$\mathcal{Z}(L_s) \cap \Xi = \emptyset. \tag{16}$$

Let us identify the set of all $(n+2)$–tuples of linear forms from $\overline{k}[X_0, \dots, X_n]$ with the affine space $\mathbb{A}^{(n+1)(n+2)}(\overline{k})$. Let $\lambda^* = (L_0^*, \dots, L_{n+1}^*) \in \mathbb{A}^{(n+1)(n+2)}(\overline{k})$, where all $L_j^* \in \overline{k}[X_0, \dots, X_n]$ are linear forms. Put $\Xi^* = V \cap \mathcal{Z}(L_{s+1}, \dots, L_n) \subset \mathbb{P}^n(\overline{k})$. Let $\mathfrak{l} \in \mathbb{A}^{(n+1)(n+2)}(\overline{k})$ be a line such that $\overline{\lambda} \in \mathfrak{l}$. Then for all $\lambda^* \in \mathfrak{l}$, except at most a polynomial in $2^{n-s+1} \deg V$ number, condition (16) holds for λ^* in place of $\overline{\lambda}$, i.e., for L_s^*, Ξ^* in place of L_s, Ξ.

PROOF Let $\overline{\lambda} \ne \lambda^{(1)} = (L_0^{(1)}, \dots, L_{n+1}^{(1)}) \in \mathfrak{l}$. Then the Bézout theorem implies that for all $t \in \overline{k}$, except at most polynomial in $2^{n-s+1} \deg V$ number, the intersection $V \cap \mathcal{Z}(L_j + tL_j^{(1)},\ s \le j \le n) = \emptyset$ in $\mathbb{P}^n(\overline{k})$, cf., e.g., the proof of Lemma 19 [10]. The lemma is proved.

Further, let $f_1, \dots, f_m$, V, W, s, σ, $x^{(0)}$, $e_0, \dots, e_r$, p, $\mathcal{U}'_0, \mathcal{U}''_0, \mathcal{U}'''_0$ are similar to the ones from the statement of Theorem 1 [10] and Theorem 2 [10], but we do not require at present that $\dim(W \cap \mathcal{Z}(e_0, \dots, e_r)) < \dim W$.

Notice that $\dim(W \cap \mathcal{Z}(e_0, \dots, e_r)) = \dim W$ is equivalent to $W \subset \mathcal{Z}(e_0, \dots, e_r)$. In this case $\sigma = n+1$, and $p : \emptyset \to \mathbb{P}^r(\overline{k})$ is a trivial morphism. Obviously one can decide (directly or using Theorems 1 and 2 [8]) whether $W \subset \mathcal{Z}(e_0, \dots, e_r)$.

Let $L'_w \in \overline{k}[X_0, \dots, X_r]$, $w \in \{0, \sigma+1, \dots, n\}$, be a family of linear forms. Put $L'_{n+1} = L'_0$. Set $L' = (L'_0, L'_{\sigma+1}, \dots, L'_n)$.

Recall that in [10] for V, W, p for every $\sigma \le i \le n+1$ the algebraic varieties $W_i(L') = W(L'_{\sigma+1}, \dots, L'_i)$, $W_i^{(\beta)}(L') = W^{(\beta)}(L'_{\sigma+1}, \dots, L'_i)$, $\beta = 1, 2, 3$, are defined (for $i = \sigma$ they do not depend on L').

If $W \subset \mathcal{Z}(e_0, \dots, e_r)$ then by definition $\mathcal{U}'_0 = \mathcal{U}''_0 = \mathcal{U}''_0$ is the set consisting of one element $L' = ()$, and $W_{n+1}(L') = \emptyset$, $W_{n+1}^{(1)}(L') = W$, $W_{n+1}^{(2)}(L') = \emptyset$, $W_{n+1}^{(3)}(L') = \emptyset$.

Put $\widetilde{e}_i = e'_i = L'_i(e_0, \dots, e_r)$, $i \in \{\sigma+1, \dots, n+1\}$, cf. the Introduction of [10]. For convenience of notation set $\widetilde{e}_\sigma = 0$, the variety $W_{\sigma-1}(L') = W$. Hence now $W = W_{\sigma-1}(L') \cap \mathcal{Z}(e'_\sigma)$. Hence

$$W_{i-1}(L') \cap \mathcal{Z}(\widetilde{e}_i) = W_i(L') \cup W_i^{(1)}(L') \cup W_i^{(2)}(L') \cup W_i^{(3)}(L'), \quad \sigma \le i \le n+1. \tag{17}$$

When $i \ge \sigma+1$ and $W \not\subset \mathcal{Z}(e_0, \dots, e_r)$ equality (17) follows from the Introduction of [10].

Assertions (a) and (b) of Theorem 1 [10] are simplified in the case when $W \subset \mathcal{Z}(e_0, \dots, e_r)$. Namely, let $\overline{\lambda} = (L_0, \dots, L_{n+1})$ where all $L_w \in k[X_0, \dots, X_n]$ are linear forms. *If $W \subset \mathcal{Z}(e_0, \dots, e_r)$ then by definition we shall say that assertions (a) and (b) of Theorem 1 [10] hold for (V, W and) $\overline{\lambda}$ if and only if $\overline{\lambda}$ satisfies assertion of Theorem 3 [10] for V, W.* If $W \subset \mathcal{Z}(e_0, \dots, e_r)$ then one can construct $\overline{\lambda}$ satisfying assertions (a) and (b) of Theorem 1 [10] using Theorem 3 [10]. Obviously, if $W \subset \mathcal{Z}(e_0, \dots, e_r)$ then assertion (g) of Theorem 1 [10] remains true (its proof is just simplified).

Now let $N > 0$ be an integer. Suppose that $f_1^{(j)}, \dots, f_{m(j)}^{(j)} \in k[X_0, \dots, X_n]$, $V^{(j)}$, $W^{(j)}$, $s(j)$, $\sigma(j)$, $x^{(0,j)}$, $e_0^{(j)}, \dots, e_r^{(j)}$, $p^{(j)}$, $\mathcal{U}'''_{j,0}$, for every $1 \le j \le N$ are similar to the considered $f_1, \dots, f_m$, V, W, s, σ, $x^{(0)}$, $e_0, \dots, e_r$, p, $\mathcal{U}'''_0$ Hence we do not exclude that $W^{(j)} \subset \mathcal{Z}(e_0^{(j)}, \dots, e_r^{(j)})$ for some j.

Let $f_1^{(j)}, \dots, f_{m(j)}^{(j)}$, $0 \le j \le N$, satisfy the same estimations for degrees and lengths of coefficients as $f_1, \dots, f_m$ from Theorem 1 [10] do. Assume that $e_\alpha^{(j)} = e_\alpha$, $0 \le \alpha \le r$, are the same for all $1 \le j \le N$. Further, suppose that if $e_\alpha \ne 0$ then

$$\deg_{X_0, \dots, X_n}(e_\alpha) = d', \ \deg_{t_\gamma}(e_\alpha) < d'_2, \ \mathrm{l}(e_\alpha) < M',$$

for all $1 \le \gamma \le l$, $0 \le \alpha \le r$, i.e., $e_0, \dots, e_r$ satisfy the estimations for degrees and lengths of coefficients from the statement of Theorem 1 [10].

Let us replace in the definitions of $W(L'_{\sigma+1}, \dots, L'_i)$, $W^{(\beta)}(L'_{\sigma+1}, \dots, L'_i)$, $\beta = 1, 2, 3$, $\sigma \le i \le n+1$, the triple (V, W, p) by $(V^{(j)}, W^{(j)}, p^{(j)})$ (hence the σ is replaced by $\sigma(j)$; and $L'_{\sigma(j)+1}, \dots, L'_{n+1} \in \overline{k}[X_0, \dots, X_r]$ are linear forms, $L'_{n+1} = L'_0$). Denote the obtained varieties by $W^{(j)}(L'_{\sigma(j)+1}, \dots, L'_i)$,

$W^{(j,\beta)}(L'_{\sigma(j)+1}, \ldots, L'_i)$, $\beta = 1, 2, 3$, $\sigma(j)+1 \le i \le n+1$ respectively (since we shall not use V, W, p in what follows the first notation $W^{(j)}(L'_{\sigma(j)+1}, \ldots, L'_i)$ will not lead to an ambiguity for $j \le 3$).

Put $e'_{j,i} = \widetilde{e}_i = L'_i(e_0, \ldots, e_r)$ for all $i \in \{\sigma(j)+1, \ldots, n+1\}$, $1 \le j \le N$. For convenience of notation set $e'_{j,\sigma(j)} = 0$, for every $1 \le j \le N$. Hence now (17) holds for $V^{(j)}, W^{(j)}, p^{(j)}$ in place of V, W, p. (note that here $e'_{j,i}$ corresponds to $\widetilde{e}_i$). Set $\iota = \iota(j,i) = i - \sigma(j) + s(j)$.

Put $\sigma = \min\{\sigma(j) \,:\, 1 \le j \le N\}$ (we shall not use the old σ defined for V, W, p, L' in what follows, and, hence, this new definition of σ will not lead to an ambiguity of notation). Put $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n)$. Further, denote for brevity $W_i^{(j)}(L') = W^{(j)}(L'_{\sigma(j)+1}, \ldots, L'_i)$ and $W_i^{(j,\beta)}(L') = W^{(j,\beta)}(L'_{\sigma(j)+1}, \ldots, L'_i)$, for all $\beta = 1, 2, 3$, $\sigma(j) \le i \le n+1$.

LEMMA 2 *Under considered assumptions for $V^{(j)}$, $W^{(j)}$, $p^{(j)}$, $1 \le j \le N$, one can compute an element $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n)$ such that $(L'_0, L'_{\sigma(j)+1}, \ldots, L'_n) \in \mathcal{U}'''_{j,0}$ for every $1 \le j \le N$, and all linear forms L'_w, $w \in \{0, \sigma+1, \ldots, n\}$ have integer coefficients of length $O(n \log d + (n-\sigma) \log d' + \log N)$. Further, for this L' one can construct linear forms $L_0, \ldots, L_{n+1}$ with integer coefficients of length $O(n \log d + (n-\sigma) \log d' + \log N)$ such that assertions (a) and (b) of Theorem 1 [10] holds for $V^{(j)}$ and $(L'_0, L'_{\sigma(j)+1}, \ldots, L'_n)$ in place of V and $(L'_0, L'_{\sigma+1}, \ldots, L'_n)$ for all $0 \le j \le N$ (i.e., linear forms $L'_0, L'_{\sigma(j)+1}, \ldots, L'_n$ and $L_0, \ldots, L_{n+1}$ do not depend on j; other objects from the statement of this theorem for $V^{(j)}$ depend on j). Further, the linear forms $L_0, \ldots, L_{n+1}$ satisfy additionally the following conditions. For all $x \in W_{i-1}^{(j)}(L') \cap \mathcal{Z}(e'_{j,i}) \cap \mathcal{Z}(L_{\iota(j,i)+1}, \ldots, L_n)$ for all $1 \le j, \beta \le N$ for all $\sigma(j) \le i \le n+1$, $\sigma(\beta) \le \alpha \le n+1$ the multiplicity*

$$\begin{aligned} &\mu(x, W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha})) = \\ &\min\{\mu(x', W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha})) \,:\, x' \in W_{i-1}^{(j)}(L') \cap \mathcal{Z}(e'_{j,i})\}. \end{aligned} \tag{18}$$

In particular if E_1 (respectively E_2) is a defined over k and irreducible over k component of $W_{i-1}^{(j)}(L') \cap \mathcal{Z}(e'_{j,i})$ (respectively of $W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha})$) such that $E_1 \not\subset E_2$ then $x \notin E_2$. The working time of this algorithm is polynomial in $d^n (d')^{n-\sigma}$, d_1, d_2, M, M_1, m, N and the sum of sizes $\sum_{0 \le i \le N} \mathrm{L}(x^{(0,i)})$ of the points $x^{(0,i)}$.

PROOF Applying Theorem 2 (a) [10] we construct for every $1 \le j \le N$ an element $L^{(j)} \in \mathcal{U}'''_{j,0}$. Further, using Theorem 2 (f) [10] and the algorithm of reduction to integer coefficients, see Proposition 2 [9], we construct an element $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n)$ such that all linear forms L'_α, $\alpha \in \{0, \sigma+1, \ldots, n\}$ have integer coefficients of length $O(n \log d + (n-\sigma) \log d' + \log N)$ and $(L'_0, L'_{\sigma(j)+1}, \ldots, L'_n) \in \mathcal{U}'''_{j,0}$ for every $1 \le j \le N$.

Applying Theorem 1 [10] (and the remarks given above in the case when $W^{(j)} \in \mathcal{Z}(e_0, \ldots, e_r)$) we construct for every $1 \le j \le N$ an element $\overline{\lambda}^{(j)} = (L_0^{(j)}, \ldots, L_{n+1}^{(j)})$ (similar $\overline{\lambda}$) satisfying assertions (a) and (b) of Theorem 1 [10] for $(V^{(j)}, W^{(j)}, p^{(j)})$ in place of (V, W, p). Further, using Theorem 1 (g) [10] and the algorithm of reduction to integer coefficients, see Proposition 2 [9], we construct an element $\overline{\lambda} = (L_0, \ldots, L_{n+1})$ such that for this $\overline{\lambda}$ the assertions of Theorem 1 [10] holds for $(V^{(j)}, W^{(j)}, p^{(j)})$ in place of (V, W, p) for all $0 \le j \le N$.

Now we are going to describe a recursion with respect to $(L_0, \dots, L_{n+1})$. The recursive assumption is that for all $0 \le j \le N$ the assertions of Theorem 1 [10] hold for $(\overline{\lambda}, V^{(j)}, W^{(j)}, p^{(j)})$ in place of $(\overline{\lambda}, V, W, p)$, and an integer $1 \le w \le n$ is known (the base of the recursion $w = n$) such that for every $1 \le j \le N$ for every $\sigma(j) \le i \le n+1$ if $\iota = \iota(i,j) \ge w+1$ then

$$W_{i-1}^{(j)}(L') \cap \mathcal{Z}(e'_{j,i}) \cap \mathcal{Z}(L_\iota, \dots, L_n) = \emptyset \tag{19}$$

in $\mathbb{P}^n(\overline{k})$. At the next step of the recursion using Theorem 1 (g) [10], Lemma 1 and the algorithm of reduction to integer coefficients, see Proposition 2 [9], we construct the minimal possible integer $t_w > 0$ such that the formulated recursive assumption holds for $w-1$ and $(L_0, \dots, L_w + t_w L_0, L_{w+1}, \dots, L_{n+1})$ in place of $(L_0, \dots, L_{n+1})$. By the estimates from Theorem 1 (g) [10] and Lemma 1 the integer t_w is of length $O(n \log d + (n-\sigma)\log d' + \log N)$. After that we replace $L_0, \dots, L_{n+1}$ by $(L_0, \dots, L_w + t_w L_0, L_{w+1}, \dots, L_{n+1})$. If $w - 1 \ge 1$ we proceed to the next step of the recursion. Thus, at the end of the recursion the recursive assumption holds with $w = 0$. Obviously (19) is satisfied also if $\iota(j,i) = 0$. Hence at the end of the recursion (19) is fulfilled for every $1 \le j \le N$ for every $\sigma(j) \le i \le n+1$.

Further, for every $\sigma \le i \le n+1$ we define the linear projection $\varphi^{(i)}$: $\mathbb{P}^n(\overline{k}) \setminus \mathcal{Z}(L_0, L_{\iota+1}, \dots, L_n) \to \mathbb{P}^{n-\iota}(\overline{k})$, $(X_0, \dots, X_n) \mapsto (L_0 : L_{\iota+1} : \dots : L_n)$. For every $\sigma(j) \le i \le n+1$ for every $1 \le j \le N$ put $\varphi_i^{(j)}$ to be the restriction of $\varphi^{(i)}$ to $W_{i-1}^{(j)}(L') \cap \mathcal{Z}(e'_{j,i})$. Hence $\varphi_i^{(j)}$ is a finite dominant morphism by assertions (a) and (b) of Theorem 1 [10] for $V^{(j)}, W^{(j)}, p^{(j)}$ in place of V, W, p. Let $z = (1 : z_1 : \dots : z_n) \in \mathbb{P}^n(\overline{k})$ be a point. Denote for brevity $P_i^{(j)}(z) = (\varphi_i^{(j)})^{-1}((1 : z_{\iota+1} : \dots : z_n))$.

Notice that $W_{i-1}^{(j)}(L')$ is a union of some irreducible component of $V_{i-1}^{(j)}(L') = \mathcal{Z}(f_1^{(a)}, \dots, f_{m(j)}^{(j)}, L'_{\sigma(j)+1}, \dots, L'_{i-1})$. Hence applying Theorem 2 [9] to the set $U'' = \mathcal{Z}(e'_{j,i}, L_{\iota+1} - z_{\iota+1}L_0, \dots, L_n - z_n L_0)$ and after that Theorem 2 [8] one can construct $P_i^{(j)}(z)$ (cf. also [10] Section 5 where the algorithm for computing $\varphi_i^{-1}((1 : \nu_{\iota+1} : \dots : N_n))$ is described). Set

$$\mu(z) = \sum_{1 \le j, \beta \le N} \ \sum_{\sigma(j) \le i \le n+1} \ \sum_{\sigma(\beta) \le \alpha \le n+1} \ \sum_{x \in P_i^{(j)}(z)} \mu(x, W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha})).$$

Therefore, according to Theorem 1 (e) [10] one can compute $\mu(z)$ for a given z. Our aim now is to construct a point z with minimal possible $\mu(z)$.

Let us describe a new recursion with respect to $\overline{\lambda}$ and $z = (1 : z_1 : \dots : z_n)$. We take the constructed $\overline{\lambda} = (L_0, \dots, L_{n+1})$ and $z = (1 : 0 : \dots : 0)$ as the base of the new recursion. The recursive assumption is that linear forms $L_0, L_1 - z_1 L_0, \dots, L_n - z_n L_0, L_{n+1}$ (in place of $L_0, \dots, L_{n+1}$) satisfy assertions (a) and (b) of Theorem 1 [10] for $(V^{(j)}, W^{(j)}, p^{(j)})$ in place of (V, W, p) for all $0 \le j \le N$ and, besides that (put $z_0 = 0$), for every $1 \le j \le N$ for every $\sigma(j) \le i \le n+1$

$$W_{i-1}^{(j)}(L') \cap \mathcal{Z}(e'_{j,i}) \cap \mathcal{Z}(L_{\iota(j,i)} - z_{\iota(j,i)}L_0, \dots, L_n - z_n L_0) = \emptyset \tag{20}$$

in $\mathbb{P}^n(\overline{k})$. Notice that (20) for the base of the recursion follows from (19). Further, (20) implies that for every (i, j, β, α, x) such that $1 \le j, \beta \le N$,

$\sigma(j) \le i \le n+1$, $\sigma(\beta) \le \alpha \le n+1$, $x \in P_i^{(j)}(z)$, and $\dim W_{i-1}^{(j)}(L') \cap \mathcal{Z}(e'_{j,i}) > \dim W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha})$ for every irreducible component E_1 of $W_{i-1}^{(j)}(L') \cap \mathcal{Z}(e'_{j,i})$ for every irreducible component E_2 of $W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha})$ the intersection

$$(E_1 \cap \mathcal{Z}(L_{\iota(j,i)+1} - z_{\iota(j,i)}L_0, \dots, L_n - z_nL_0)) \cap E_2 = \emptyset.$$

Let us describe the general step of this recursion. We enumerate the 5–tuples (i, j, β, α, x) such that $1 \le j,\ \beta \le N$, $\sigma(j) \le i \le n+1$, $\sigma(\beta) \le \alpha \le n+1$, $x \in P_i^{(j)}(z)$, and $\dim W_{i-1}^{(j)}(L') \cap \mathcal{Z}(e'_{j,i}) \le \dim W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha})$ (the last condition is equivalent to $\iota(j,i) \ge \iota(\beta,\alpha)$).

For the considered (i, j, β, α, x) we compute the set $P_i^{(j)}(z)$, see above, and after that the integer $\mu(z)$. Recall that for an arbitrary projective algebraic variety $E \subset \mathbb{P}^n(\overline{k})$ we denote by $\mathrm{con}(E) \subset \mathbb{A}^{n+1}(\overline{k})$ the affine algebraic variety which is the set of all zeroes of the homogeneous ideal of E in $\mathbb{A}^{n+1}(\overline{k})$.

Applying Theorem 2 [9] to the variety $(\mathrm{con}(V_{i-1}^{(j)}(L')) \times \mathrm{con}(V_{\alpha-1}^{(\beta)}(L')))(\overline{k_2})$ we decide whether there is a pair $(x'', x''') \in (W_{i-1}^{(j)}(L') \times W_{\alpha-1}^{(\beta)}(L'))(\overline{k_2})$ such that

$$\left\{ \begin{array}{l} \sum_{0 \le w \le n} |(X_w/L_0)(x'') - (X_w/L_0)(x)|^2 \le \varepsilon_1, \\ \varepsilon_2 \le \sum_{0 \le w \le n} |(X_w/L_0)(x'') - (X_w/L_0)(x''')|^2 \le \varepsilon_1, \\ \varphi^{(\alpha)}(x'') = \varphi^{(\alpha)}(x'''), \\ e'_{j,i}(x'') = e'_{\beta,\alpha}(x''') = 0, \end{array} \right. \tag{21}$$

cf. system (2) from [4]. If there does not exist such a pair (x'', x'''), and not all (i, j, β, α, x) are enumerated then we proceed to the next (i, j, β, α, x). Suppose that there exists a pair $(x'', x''') \in (W_{i-1}^{(j)}(L') \times W_{\alpha-1}^{(\beta)}(L'))(\overline{k_2})$ satisfying (21). Then we compute $\varphi_i^{(j)}(x'') = (1 : z''_{\iota+1} : \ldots : z''_n)$ where all $z''_w \in \overline{k_2}$. Put $z''_w = z'_w$ for $1 \le \alpha \le \iota$ and $z'' = (1 : z''_1 : \ldots : z''_n)$. Then $\mu(z'') < \mu(z')$.

It is not difficult to see, by e.g., [4], that for every β for every α for an arbitrary line $\mathfrak{l} \subset \mathbb{P}^n(\overline{k_2})$ and any point $x^* \in \mathfrak{l}$ the number of elements $x^{**} \in \mathfrak{l}$ with $\mu(x^{**}, W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha})) > \mu(x^*, W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha}))$ is bounded from above by a polynomial in $d^n(d')^{n-\sigma}$, cf. also the proof of Lemma 1 [7]. Therefore, applying Theorem 1 (g) [10], Lemma 1 and the algorithm of reduction to integer coefficients, see Proposition 2 [9], to the points z'' and $(1 : 0 : \ldots : 0)$ one can construct a point $\widetilde{z} = (1 : \widetilde{z}_1 : \ldots : \widetilde{z}_n) \in \mathbb{P}^n(\overline{k})$ with integer $\widetilde{z}_w$ of length $O(n \log d + (n-\sigma)\log d' + \log N)$ such that for $\overline{\lambda}$ and $\widetilde{z}$ (in place of $\overline{\lambda}$ and z) the recursive assumption holds. Then we replace $(\overline{\lambda}, z)$ by $(\overline{\lambda}, \widetilde{z})$ and proceed to the next step of the recursion.

Finally we shall come to the case when for all enumerated 5–tuples (i, j, β, α, x) there is no pair (x'', x''') satisfying all the considered conditions. Then $\mu(x, W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha})) = \min\{\mu(x', W_{\alpha-1}^{(\beta)}(L') \cap \mathcal{Z}(e'_{\beta,\alpha})) \,:\, x' \in W_{i-1}^{(j)}(L') \cap \mathcal{Z}(e'_{j,i})\}$ for all $x \in P_i^{(j)}(z)$ for all $1 \le j,\ \beta \le N$, $\sigma(j) \le i \le n+1$, $\sigma(\beta) \le \alpha \le n+1$. This step is final. We replace $L_0, \dots, L_{n+1}$ by $L_0, L_1 - z_1L_0, \dots, L_n - z_nL_0, L_{n+1}$ respectively. Now the new $L_0, \dots, L_{n+1}$ satisfy all the required conditions.

The required bound for the working time of the described algorithm follows immediately from the ones for the working times of the applied algorithms. The lemma is proved.

Let us proceed to the proof of Theorem 1. Let us show that the case $\nu = 1$ is reduced to $\nu = 2$. Indeed, it is sufficient to define $m_\alpha(i)$, a_α, b_α, $f^{(i)}_{\alpha,j}$, W_α, V_α, $W^{(i)}_\alpha$, $V^{(i)}_\alpha$, $W^{(i)}_\alpha$, $V^{(i,s)}_\alpha$, $W^{(i,s)}_\alpha$, $L^{(i,s)}_{\alpha,\beta}$, $\Xi^{(i,s)}_\alpha$, $(f_\alpha, L_\alpha, \Xi_\alpha, b_\alpha) = \rho_\alpha$ for $\alpha = 2$ to be the corresponding objects with $\alpha = 1$, and apply Theorem 1 for $\nu = 2$. Thus, we shall assume in what follows without loss of generality that $\nu \geq 2$. Besides that, we shall assume also that $n \geq 1$.

Let $\rho = (f, L, \Xi, b)$ be representation (4). We shall use the following notations. Put $n(\rho) = n$, $a(\rho) = a$, $b(\rho) = b$, $q_1(i, \rho) = V^{(i)}$, and $q_2(i, \rho) = W^{(i)}$ for every $1 \leq i \leq a = a(\rho)$. These functorial notations $n(.)$, $a(.)$, $b(.)$, $q_1(.,.)$ and $q_2(.,.)$ are used also for other representations similar to (4). Put the set

$$B_1 = \{(i_1, \ldots, i_\nu) \,:\, 1 \leq i_\alpha \leq b_\alpha \;\&\; i_\alpha \in \mathbb{Z} \;\&\; 1 \leq \alpha \leq \nu \;\&\; \alpha \in \mathbb{Z}\}.$$

Notice that

$$\bigcap_{1\leq\alpha\leq\nu} W_\alpha = \Big(\bigcup_{(i_1,\ldots,i_\nu)\in B_1} \; \bigcap_{1\leq\alpha\leq\nu} W^{(i_\alpha)}_\alpha \Big) \setminus \Big(\bigcup_{1\leq\alpha\leq\nu} \; \bigcup_{b_\alpha+1\leq i\leq a_\alpha} W^{(i)}_\alpha \Big). \tag{22}$$

Enumerating $c = 1, 2, \ldots$ let us construct the set of integers $\mathcal{C}$ such that for every $c \in \mathcal{C}$ the intersection

$$\Big(\bigcup_{1\leq\alpha\leq\nu} \; \bigcup_{1\leq j\leq a_\alpha} \Xi^{(j)}_\alpha \Big) \cap \mathcal{Z}\Big(\sum_{0\leq i\leq n} c^i X_i \Big) = \emptyset \tag{23}$$

in $\mathbb{P}^n(\overline{k})$, the number of elements

$$\#\mathcal{C} = n\Big(d^{n\nu} \prod_{1\leq\alpha\leq\nu} b_\alpha + d^n \sum_{1\leq\alpha\leq\nu} a_\alpha \Big) + n + 1, \tag{24}$$

and all the elements of $\mathcal{C}$ are minimal possible, i.e., the sum $\sum_{c\in\mathcal{C}} c$ is minimal possible. By the Bézout theorem the maximal element $\max \mathcal{C}$ of $\mathcal{C}$ is bounded from above by a polynomial in $d^{n\nu}$, $(\sum_{1\leq\alpha\leq\nu} a_\alpha)$, $\prod_{1\leq\alpha\leq\nu} b_\alpha$. Put

$$\mathcal{H} = \Big\{ \sum_{0\leq i\leq n} c^i X_i \,:\, c \in \mathcal{C} \Big\}.$$

Hence $\mathcal{H}$ is a set of linear forms with integer coefficients.

For every $H \in \mathcal{H}$ we perform the following construction. The defined below objects depends on H although we do not indicate it explicitly in the notations. Let us identify $\mathbb{P}^n(\overline{k}) \setminus \mathcal{Z}(H) = \mathbb{A}^n(\overline{k})$, where $\mathbb{A}^n(\overline{k})$ has the coordinate functions $Y_i = X_i/H$, $1 \leq i \leq n$. For every quasiprojective variety $E \subset \mathbb{P}^n(\overline{k})$ put $\widetilde{E} = E \setminus \mathcal{Z}(H) \subset \mathbb{A}^n(\overline{k})$. Below for convenience we denote $\widetilde{W}_\alpha = \widetilde{W_\alpha}$ and so on, i.e., we use tilde without taking into account upper and low indices. This will not lead to an ambiguity. Now (22) implies

$$\bigcap_{1\leq\alpha\leq\nu} \widetilde{W}_\alpha = \Big(\bigcup_{(i_1,\ldots,i_\nu)\in B_1} \; \bigcap_{1\leq\alpha\leq\nu} \widetilde{W}^{(i_\alpha)}_\alpha \Big) \setminus \Big(\bigcup_{1\leq\alpha\leq\nu} \; \bigcup_{b_\alpha+1\leq i\leq a_\alpha} \widetilde{W}^{(i)}_\alpha \Big). \tag{25}$$

Let us identify $\mathbb{A}^{n\nu}(\overline{k}) = (\mathbb{A}^n(\overline{k}))^\nu$, where for every $1 \leq i \leq \nu$ the i-th factor $\mathbb{A}^n(\overline{k})$ of $\mathbb{A}^{n\nu}(\overline{k})$ in the left part of the last equality has the coordinate functions $Y_{i,j}$, $1 \leq j \leq n$. Hence $\mathbb{A}^{n\nu}(\overline{k})$ has the coordinate functions $Y_{i,j}$, $1 \leq i \leq \nu$,

$1 \le j \le n$. Put $\Delta = \{(x, x, \dots, x) \in \mathbb{A}^{n\nu}(\overline{k}) \,:\, x \in \mathbb{A}^n(\overline{k})\}$ to be the diagonal subvariety, cf. the Introduction.

Let us identify $\mathbb{A}^{n\nu}(\overline{k}) \subset \mathbb{P}^{n\nu}(\overline{k})$, where $\mathbb{P}^{n\nu}(\overline{k})$ has the homogeneous coordinate functions $X_{0,0}$, $X_{i,j}$, $1 \le i \le \nu$, $1 \le j \le n$, and all $Y_{i,j} = X_{i,j}/X_{0,0}$. Put $\overline{\Delta}$ to be the closure of Δ in $\mathbb{P}^{n\nu}(\overline{k})$ with respect to the Zariski topology.

Further, we assume that the identification (12) holds. Hence (12) implies $\bigcap_{1\le\alpha\le\nu} \widetilde{W}_\alpha^{(i_\alpha)} = (\prod_{1\le\alpha\le\nu} \widetilde{W}_\alpha^{(i_\alpha)}) \cap \Delta$ for every $(i_1, \dots, i_\nu) \in B_1$, and $\widetilde{W}_\alpha^{(i)} = (\widetilde{W}_\alpha^{(i)})^\nu \cap \Delta$ for all $b_\alpha + 1 \le i \le a_\alpha$, $1 \le \alpha \le \nu$. Notice that the intersection $\bigcap_{1\le\alpha\le\nu} \widetilde{W}_\alpha^{(i_\alpha)}$ is transversal if and only if the intersection $(\prod_{1\le\alpha\le\nu} \widetilde{W}_\alpha^{(i_\alpha)}) \cap \Delta$ is transversal, and in this case $\deg \bigcap_{1\le\alpha\le\nu} \widetilde{W}_\alpha^{(i_\alpha)} = \deg \prod_{1\le\alpha\le\nu} \widetilde{W}_\alpha^{(i_\alpha)} = \prod_{1\le\alpha\le\nu} \deg \widetilde{W}_\alpha^{(i_\alpha)}$. In the general case the inequality $\deg \bigcap_{1\le\alpha\le\nu} \widetilde{W}_\alpha^{(i_\alpha)} \le \deg \prod_{1\le\alpha\le\nu} \widetilde{W}_\alpha^{(i_\alpha)}$ holds.

For every $(i_1, \dots, i_\nu) \in B_1$ denote

$$V_{i_1,\dots,i_\nu} = \prod_{1\le\alpha\le\nu} \widetilde{V}_\alpha^{(i_\alpha)} \subset \mathbb{A}^{n\nu}(\overline{k}),$$

$$W_{i_1,\dots,i_\nu} = \prod_{1\le\alpha\le\nu} \widetilde{W}_\alpha^{(i_\alpha)} \subset V_{i_1,\dots,i_\nu}.$$

Hence $V_{i_1,\dots,i_\nu}$ is an affine algebraic variety given by a system of polynomial equations with respect to $Y_{i,j}$, $1 \le i \le \nu$, $1 \le j \le n$, and $W_{i_1,\dots,i_\nu}$ is a union of some irreducible components of $V_{i_1,\dots,i_\nu}$. Denote by $\overline{V}_{i_1,\dots,i_\nu}$ (respectively $\overline{W}_{i_1,\dots,i_\nu}$) the closure of $V_{i_1,\dots,i_\nu}$ (respectively of $W_{i_1,\dots,i_\nu}$) with respect to the Zariski topology in $\mathbb{P}^{n\nu}(\overline{k})$. Hence $\overline{V}_{i_1,\dots,i_\nu}$ is a projective algebraic variety given by a system of polynomial equations with respect to $X_{0,0}$, $X_{i,j}$, $1 \le i \le \nu$, $1 \le j \le n$, and $\overline{W}_{i_1,\dots,i_\nu}$ is a union of some irreducible components of $V_{i_1,\dots,i_\nu}$.

Let $\overline{V}_{i_1,\dots,i_\nu} \cup \mathcal{Z}(X_{0,0}) = \bigcup_{j \in J'_{i_1,\dots,i_\nu}} E_j$ be the decomposition into the union of defined over k and irreducible over k components E_j, $j \in J'_{i_1,\dots,i_\nu}$.

For every $(i_1, \dots, i_\nu) \in B_1$ we apply Theorem 1 [8] to the algebraic variety $\overline{V}_{i_1,\dots,i_\nu} \cup \mathcal{Z}(X_{0,0})$ (in place of V) and construct for every $j \in J'_{i_1,\dots,i_\nu}$ a representation ρ_j similar to (4) of the irreducible component E_j such that $n(\rho_j) = n\nu$, $a(\rho_j) = b(\rho_j) = 1$, $q_1(1, \rho_j) = \overline{V}_{i_1,\dots,i_\nu} \cup \mathcal{Z}(X_{0,0})$ and $q_2(1, \rho_j) = E_j$. Put $\Xi_{i_1,\dots,i_\nu} = \prod_{1\le\alpha\le\nu} \Xi_\alpha^{(i_\alpha)} \subset \mathbb{A}^{n\nu}(\overline{k}) \subset \mathbb{P}^{n\nu}(\overline{k})$. By (23) for every irreducible component E of $\overline{V}_{i_1,\dots,i_\nu} \cup \mathcal{Z}(X_{0,0})$ the intersection $E \cap \Xi_{i_1,\dots,i_\nu} \ne \emptyset$ if and only if E is an irreducible component of $\overline{W}_{i_1,\dots,i_\nu}$, i.e. the finite set $\Xi_{i_1,\dots,i_\nu}$ gives $\overline{W}_{i_1,\dots,i_\nu}$ as the union of some irreducible components of $\overline{V}_{i_1,\dots,i_\nu} \cup \mathcal{Z}(X_{0,0})$. Now using Theorem 2 [8] we construct the subset $J_{i_1,\dots,i_\nu} \subset J'_{i_1,\dots,i_\nu}$ such that

$$\overline{W}_{i_1,\dots,i_\nu} = \bigcup_{j \in J_{i_1,\dots,i_\nu}} E_j.$$

Notice that $E_j \not\subset \mathcal{Z}(X_{0,0})$ for every $j \in J_{i_1,\dots,i_\nu}$ for every $(i_1, \dots, i_\nu) \in B_1$.

Set $a'_1 = a_1 + \#\mathcal{C}$. Let H_j, $a_1 + 1 \le j \le a'_1$, be the family of linear forms such that the set $\{H_j \,:\, a_1 + 1 \le j \le a'_1\} = \mathcal{H}$. Put $V_1^{(j)} = W_1^{(j)} = \mathcal{Z}(H_j)$ for every $a_1 + 1 \le j \le a'_1$.

Put

$$B_2 = \{(\alpha, i) \,:\, 1 \le \alpha \le \nu \;\&\; \alpha \in \mathbb{Z} \;\&\; b_\alpha + 1 \le i \le a_\alpha \;\&\; i \in \mathbb{Z}\}$$

and $B'_2 = B_2 \cup \{(1,i) \,:\, a_1+1 \le i \le a'_1\}$. Hence $B_2 \subset B'_2$. Set for every $(\alpha,i) \in B'_2$

$$V_{\alpha;i} = (\widetilde{V}^{(i)}_{\alpha})^{\nu} \subset \mathbb{A}^{n\nu}(\overline{k})$$
$$W_{\alpha;i} = (\widetilde{W}^{(i)}_{\alpha})^{\nu} \subset V_{\alpha;i}.$$

Hence $V_{\alpha;i}$ is an affine algebraic variety given by a system of polynomial equations with respect to $Y_{i,j}$, $1 \le i \le \nu$, $1 \le j \le n$, and $W_{\alpha;i}$ is a union of some irreducible components of $V_{\alpha;i}$. Denote by $\overline{V}_{\alpha;i}$ (respectively $\overline{W}_{\alpha;i}$) the closure of $V_{\alpha;i}$ (respectively of $W_{\alpha;i}$) with respect to the Zariski topology in $\mathbb{P}^{n\nu}(\overline{k})$. Hence $\overline{V}_{\alpha;i}$ is a projective algebraic variety given by a system of polynomial equations with respect to $X_{0,0}$, $X_{i,j}$, $1 \le i \le \nu$, $1 \le j \le n$, and $\overline{W}_{\alpha;i}$ is a union of some irreducible components of $\overline{V}_{\alpha;i}$.

Let $\overline{V}_{\alpha;i} \cup \mathcal{Z}(X_{0,0}) = \bigcup_{j \in J'_{\alpha;i}} E_j$ be the decomposition into the union of defined over k and irreducible over k components E_j, $j \in J'_{\alpha;i}$.

For every $(\alpha,i) \in B'_2$ we apply Theorem 1 [8] to the algebraic variety $\overline{V}_{\alpha;i} \cup \mathcal{Z}(X_{0,0})$ (in place of V) and construct for every $j \in J'_{\alpha;i}$ a representation ρ_j similar to (4) of the irreducible component E_j such that $n(\rho_j) = n\nu$, $a(\rho_j) = b(\rho_j) = 1$, $q_1(1,\rho_j) = \overline{V}_{\alpha;i} \cup \mathcal{Z}(X_{0,0})$ and $q_2(1,\rho_j) = E_j$. Since $\Xi^{(j)}_{\alpha}$ for all $1 \le \alpha \le \nu$, $1 \le j \le a_{\alpha}$, are known we are able to construct the set $\Xi_{\alpha;i}$ of smooth points of $\overline{V}_{\alpha;i} \cup \mathcal{Z}(X_{0,0})$ satisfying the following property. For every irreducible component E of $\overline{V}_{\alpha;i} \cup \mathcal{Z}(X_{0,0})$ the intersection $\Xi_{\alpha;i} \cap E \ne \emptyset$ if and only if E is an irreducible component of $\overline{W}_{\alpha;i}$. After that using Theorem 2 [8] we construct a subset $J_{\alpha;i} \subset J'_{\alpha;i}$ such that

$$\overline{W}_{\alpha;i} = \bigcup_{j \in J_{\alpha;i}} E_j.$$

Notice that $E_j \not\subset \mathcal{Z}(X_{0,0})$ for every $j \in J_{\alpha;i}$ for every $(\alpha,i) \in B'_2$.

Let us construct also the representation $\rho_{j_{0,0}}$ such that $n(\rho_{j_{0,0}}) = n\nu$, $a(\rho_{j_{0,0}}) = b(\rho_{j_{0,0}}) = 1$, $q_1(\rho_{j_{0,0}}) = q_2(\rho_{j_{0,0}}) = \mathcal{Z}(X_{0,0})$.

To simplify the notation we shall assume without loss of generality that

- for all $(i_1,\dots,i_\nu) \in B_1$, $(i'_1,\dots,i'_\nu) \in B_1$ if $(i_1,\dots,i_\nu) \ne (i'_1,\dots,i'_\nu)$ then $J_{i_1,\dots,i_\nu} \cap J_{i'_1,\dots,i'_\nu} = \emptyset$,
- for all $(i_1,\dots,i_\nu) \in B_1$, $(\alpha,i) \in B'_2$ the intersection $J_{i_1,\dots,i_\nu} \cap J_{\alpha;i} = \emptyset$,
- for all $(\alpha,i) \in B'_2$, $(\alpha',i') \in B'_2$ if $(\alpha,i) \ne (\alpha',i')$ then $J_{\alpha;i} \cap J_{\alpha';i'} = \emptyset$,

replacing if it is necessarily the considered sets of indices for new ones. Put

$$J_1 = \bigcup_{(i,\dots,i_\nu) \in B_1} J_{i_1,\dots,i_\nu}, \quad J'_2 = \bigcup_{(\alpha,i) \in B'_2} J_{\alpha;i}, \quad J'_0 = \{j_{0,0}\} \cup J_1 \cup J'_2,$$
$$J_2 = \bigcup_{(\alpha,i) \in B_2} J_{\alpha;i}, \quad J_0 = J_1 \cup J_2,$$

Besides that, we shall suppose in what follows without loss of generality that $J'_0 \cap \mathbb{Z} = \emptyset$, i.e., that each element of J'_0 is not an integer. Denote $V^{(j)} = q_1(1,\rho_j)$, $W^{(j)} = q_2(1,\rho_j)$ for all $j \in J'_0$.

Set $r = n(\nu-1)-1$, and $e_0, \dots, e_r$ to be the family of polynomials $X_{1,i} - X_{w,i}$, $2 \le w \le \nu$, $1 \le i \le n$ (we choose an arbitrary linear order on the elements of the last family to obtain $e_0, \dots, e_r$). Notice that $\overline{\Delta} = \mathcal{Z}(e_0, \dots, e_r)$. More than that, $e_0, \dots, e_r$ generate the homogeneous ideal of $\overline{\Delta} \subset \mathbb{P}^{n\nu}(\overline{k})$, and $\dim \overline{\Delta} = n\nu - r - 1$. Denote by $\delta : \mathbb{P}^{n\nu}(\overline{k}) \setminus \mathcal{Z}(e_0, \dots, e_r) \to \mathbb{P}^r(\overline{k})$, $(X_0 : \ldots : X_n) \mapsto (e_0 : \ldots : e_r)$ the morphism of the linear projection. Denote by $p^{(j)} : W^{(j)} \setminus \mathcal{Z}(e_0, \dots, e_r) \to \mathbb{P}^r(\overline{k})$ the restriction of δ to $W^{(j)} \setminus \mathcal{Z}(e_0, \dots, e_r)$ for every $j \in J_0'$. Note that if $W^{(j)} \subset \overline{\Delta}$ then $\dim W^{(j)} = 0$ (but, of course, not conversely).

Let $j \in J_0'$ and $\sigma(j) \le i \le n\nu + 1$. Now $W_i^{(j)}(L'')$ (respectively $W_i^{(j,\beta)}(L'')$, $\beta = 1, 2, 3$) is a projective algebraic subvariety of $\mathbb{P}^{n\nu}(\overline{k})$, and the dimension of every irreducible component of $W_i^{(j)}(L'')$ (respectively $W_i^{(j,\beta)}(L'')$) is $n\nu - \iota(j,i) = n\nu - i + \sigma(j) - s(j)$, see the Introduction of [10] and the beginning of the section.

Notice that at present for all $j \in J_0'$ and $\sigma(j) \le i \le n\nu + 1$ the algebraic variety $W_i^{(j,3)}(L') = \emptyset$ since $L'' \in \mathcal{U}_{j,0}''$, see [10]. Further, for every $j \in J_0'$

$$\bigcup_{\sigma(j) \le i \le n\nu+1} W_i^{(j,1)}(L'') = W^{(j)} \cap \overline{\Delta}, \tag{26}$$

again since $L'' \in \mathcal{U}_{j,0}''$ (recall that $\mathcal{U}_{j,0}''' \subset \mathcal{U}_{j,0}''$), see the Introduction of [10]. We denote in this case $W_i^{(j,1)}(L'') = W_i^{(j,1)}$.

Now we are going to apply Lemma 2 with $N = \#J_0'$ and the set of indices $j \in J_0'$ in place of $j \in \{1, \dots, N\}$. We replace also (n, σ) by $(n\nu, \sigma_1)$. At present all the other objects from the statement of this lemma are defined in the natural way using the representations ρ_j, $j \in J_0'$.

Thus, applying Lemma 2 we get linear forms $L_0'', L_{\sigma_1+1}'', \dots, L_{n\nu}'' \in k[X_0, \dots, X_r]$ in place of $L_0', L_{\sigma+1}', \dots, L_n'$. Set $L'' = (L_0'', L_{\sigma_1+1}'', \dots, L_{n\nu}'')$ and $e_{j,w}'' = e_w'' = L_w''(e_0, \dots, e_r)$ (recall that $L_{n\nu+1}'' = L_0''$, see the Introduction of [10]), and $e_{j,\sigma(j)}'' = 0$ for all $w \in \{0, \sigma(j)+1, \dots, n\nu+1\}$ and $j \in J_0'$. Further, we construct linear forms and $L_0''', \dots, L_{n\nu+1}''' \in k[X_{0,0}, X_{i,j}, 1 \le j \le \nu, 1 \le i \le n]$ in place of $L_0, \dots, L_{n+1} \in k[X_0, \dots, X_n]$. We have replaced the notation L_j' by L_j'' (respectively L_w by L_w''') to avoid an ambiguity in what follows.

Applying Theorem 1 [10] let us construct for every $j \in J_0'$ for every $\sigma(j) - 1 \le i \le n\nu + 1$ the finite sets

$$\begin{aligned} A_i^{(j)} &= W_{i-1}^{(j)}(L'') \cap \mathcal{Z}(e_{j,i}'') \cap \mathcal{Z}(L_{\iota(j,i)+1}''', \dots, L_{n\nu}'''), \\ A_i^{(j,1)} &= W_i^{(j,1)} \cap \mathcal{Z}(L_{\iota(j,i)+1}''', \dots, L_{n\nu}'''). \end{aligned}$$

Let E be a defined over k and irreducible over k component of $W_i^{(j,1)}$ with $j \in J_0'$ and $\sigma(j) \le i \le n\nu + 1$. According to Lemma 2 an irreducible component $E \subset \mathcal{Z}(X_{0,0})$ if and only if $E \cap \mathcal{Z}(L_{\iota+1}''', \dots, L_{n\nu}''') \subset \mathcal{Z}(X_{0,0})$. Put for every $j \in J_0$

$$\begin{aligned} &D(H, j) = \deg(W^{(j)} \cap \overline{\Delta} \setminus \mathcal{Z}(X_{0,0})) = \\ &\quad \sum_{\sigma(j) \le i \le n\nu+1} \deg(W_i^{(j,1)} \setminus \mathcal{Z}(X_{0,0})) = \sum_{\sigma(j) \le i \le n\nu+1} \#(A_i^{(j,1)} \setminus \mathcal{Z}(X_{0,0})). \end{aligned}$$

Let us compute $D(H, j)$ by the previous formula for every $j \in J_0$, and after that $D(H) = \sum_{j \in J_0} D(H, j)$.

We construct the subset $\mathcal{H}_0 \subset \mathcal{H}$ such that $D(H') = \max\{D(H) : H \in \mathcal{H}\}$ for every $H' \in \mathcal{H}_0$. We claim that

(a) for every $H \in \mathcal{H}_0$ for every $(i_1, \ldots, i_\nu) \in B_1$ for each defined over k and irreducible over k component E_α of $V_\alpha^{(i_\alpha)}$, $1 \leq \alpha \leq \nu$, the inclusion $\bigcap_{1\leq\alpha\leq\nu} \widetilde{E}_\alpha \subset \bigcap_{1\leq\alpha\leq\nu} E_\alpha$ induces the one–to–one correspondence between the defined over k and irreducible over k components of the algebraic varieties $\bigcap_{1\leq\alpha\leq\nu} \widetilde{E}_\alpha$ and $\bigcap_{1\leq\alpha\leq\nu} E_\alpha$,

(b) the number of elements $\#\mathcal{H}_0 \geq n + 1$.

Indeed, the required assertion follows from the Bézout theorem and the fact that for every nonempty projective algebraic variety $\mathcal{V} \subset \mathbb{P}^n(\overline{k})$ there are at most n distinct linear forms $H \in \mathcal{H}$ such that $\mathcal{V} \subset \mathcal{Z}(H)$. Let us construct a subset $\mathcal{H}_1 \subset \mathcal{H}_0$ such that the number of elements $\#\mathcal{H}_1 = n + 1$.

Further, for every $H \in \mathcal{H}_1$ we perform the following construction. Let E be a defined over k and irreducible over k component of $W_i^{(j,1)}$ with $j \in J_0$ and $\sigma(j) \leq i \leq n\nu + 1$. Let $\dim E = n\nu - \iota$. Then

$$E \cap \mathcal{Z}(L'''_{\iota+1}, \ldots, L'''_{n\nu}) = \Xi_E \subset A_i^{(j,1)},$$

and $\iota = i - \sigma(j) + s(j)$. Notice that $E \not\subset \mathcal{Z}(X_{0,0})$ by property (a). We identify

$$E \setminus \mathcal{Z}(X_{0,0}) \subset \mathbb{A}^{n\nu}(\overline{k}) \cap \Delta = \mathbb{A}^n(\overline{k}) = \mathbb{P}^n(\overline{k}) \setminus \mathcal{Z}(H) \subset \mathbb{P}^n(\overline{k}) \qquad (27)$$

using isomorphism (12). For all $j \in J_0$ let us construct all such subsets Ξ_E applying assertions (a) and (b) of Theorem 1 [10]. Note also that under identifying (27) the intersection $\Xi_E \cap \left(\bigcup_{H\in\mathcal{H}_1} \mathcal{Z}(H)\right) = \emptyset$ by the choice of $(V_{1;i}, W_{1;i})$, $a_1 + 1 \leq i \leq a'_1$, and Lemma 2.

Let $J' \subset J_0$ be a subset. We shall say that a defined over k and irreducible over k component E of $W^{(j)} \cap \overline{\Delta}$ with $j \in J'$ is maximal with respect to J' if and only if for every $j' \in J'$ for every defined over k and irreducible over k component E' of $W^{(j')} \cap \overline{\Delta}$ the inclusion $E \subset E'$ implies the equality $E = E'$.

Let E_γ, $\gamma = 1, 2$, be a defined over k and irreducible over k component of $W^{(j_\gamma)} \cap \overline{\Delta}$ with $j_\gamma \in J_0$ and $E_\gamma \not\subset \mathcal{Z}(X_{0,0})$. According to Lemma 2 an irreducible component $E_1 \subset E_2$ if and only if $\Xi_{E_1} \subset E_2$.

Hence applying Theorem 1 [10] one can construct the sets Ξ_j, $j \in I_1$ (respectively Ξ_j, $j \in I_2$), such that

- for every $j \in I_1$ (respectively $j \in I_2$) the set $\Xi_j = \Xi_E$ for a defined over k and irreducible over k component E of $W^{(j')} \cap \overline{\Delta}$ with $j' \in J_1$ (respectively $j' \in J_2$) such that E is maximal with respect to J_0 (respectively with respect to J_2) and $E \not\subset \mathcal{Z}(X_{0,0})$.

- conversely, for every $j' \in J_1$ (respectively $j' \in J_2$) for every defined over k and irreducible over k component E of $W^{(j')} \cap \overline{\Delta}$ such that E is maximal with respect to J_0 (respectively with respect to J_2) and $E \not\subset \mathcal{Z}(X_{0,0})$ there is $j \in I_1$ (respectively $j \in I_2$) with $\Xi_j = \Xi_E$,

- for all $j_1, j_2 \in I_1$ (respectively $j_1, j_2 \in I_2$) if $j_1 \neq j_2$ then $\Xi_{j_1} \neq \Xi_{j_2}$.

For every $j \in I_1 \cup I_2$ put $\overline{E}_j = \overline{E}$, where $\Xi_j = \Xi_E$ and $\overline{E}$ is the the closure of $E \setminus \mathcal{Z}(X_{0,0})$ in $\mathbb{P}^n(\overline{k})$ with respect to the Zariski topology by inclusion (27). We identify $\Xi_j \subset \mathbb{P}^n(\overline{k})$ by (27). Now $\overline{E}_j$, $j \in I_1$ (respectively $\overline{E}_j$, $j \in I_2$) is the family of all the defined over k and irreducible over k components of $\bigcup_{(i_1,\ldots,i_\nu)\in B_1} \bigcap_{1\le\alpha\le\nu} W_\alpha^{(i_\alpha)}$ (respectively $\bigcup_{1\le\alpha\le\nu} \bigcup_{b_\alpha+1\le i\le a_\alpha} W_\alpha^{(i)}$) by Lemma 2, by property (a) and since $H \in \mathcal{H}_1 \subset \mathcal{H}_0$.

Let $j_1 \in I_1$, $j_2 \in I_2$. Let $\mathbb{P}^{n\nu}(\overline{k}) \supset E_\gamma \setminus \mathcal{Z}(X_{0,0}) = E_{j_\gamma} \setminus \mathcal{Z}(H)$, $\gamma = 1, 2$, according to (27). Again applying Theorem 1 [10] we decide for every $j_1 \in I_1$ for every $j_2 \in I_2$ whether $\Xi_{E_1} \subset E_2$. It is equivalent to $\Xi_{j_1} \subset \overline{E}_{j_2}$ and also to $\overline{E}_{j_1} \subset \overline{E}_{j_2}$ by Lemma 2. Thus, we construct the subset

$$J = \{j \in I_1 \,:\, \Xi_j \not\subset \overline{E}_{j_2} \quad \text{for all} \quad j_2 \in I_2\}.$$

Hence by Lemma 2 for every $j \in J$ for every $j_2 \in J_2$ the set $\Xi_j \not\subset W^{(j_2)} \cap \overline{\Delta}$. For every $j \in J$ denote

$$E_j = \overline{E} \setminus \Big(\bigcup_{1\le\alpha\le\nu} \bigcup_{b_\alpha+1\le i\le a_\alpha} W_\alpha^{(i)} \Big),$$

where E is a defined over k and irreducible over k component of $W^{(j_1)} \cap \overline{\Delta}$ for some $j_1 \in J_1$ such that the set $\Xi_j = \Xi_E$, and $\overline{E}$ is defined above. Now E_j, $j \in J$ is the family of all the defined over k and irreducible over k components of $W_1 \cap \ldots \cap W_\nu$ by property (a) and since $H \in \mathcal{H}_1 \subset \mathcal{H}_0$.

Set

$$(L'_0, L'_{\sigma+1}, \ldots, L'_n) = (L''_0, L''_{\sigma_1+1}, \ldots, L''_{n\nu})$$

and $L' = (L'_0, L'_{\sigma+1}, \ldots, L'_n)$ (hence $\sigma = \sigma_1 - n(\nu-1)$). We construct L'. Consider the homomorphism $\tau : \overline{k}[X_{0,0}, X_{i,j}, 1 \le i \le \nu, 1 \le j \le n] \to \overline{k}[X_0, \ldots, X_n]$ of $\overline{k}$–algebras such that $\tau(X_{0,0}) = H$, $\tau(X_{i,j}) = X_j$ for all $1 \le i \le \nu$, $1 \le j \le n$. Put

$$(L_0, \ldots, L_{n+1}) = (\tau(L'''_0), \tau(L'''_{n(\nu-1)+1}), \tau(L'''_{n(\nu-1)+2}), \ldots, \tau(L'''_{n\nu+1}))$$

and construct linear forms $L_0, \ldots, L_{n+1}$. Using isomorphism (12) we identify $\Xi_j \subset \mathbb{A}^n(\overline{k}) \subset \mathbb{P}^n(\overline{k})$ for every $j \in J$ (respectively $j \in I_1$, $j \in I_2$). According to the described construction assertion (a) of Theorem 1 holds for the obtained E_j, Ξ_j, $j \in J$, linear forms $L'_0, L'_{\sigma+1}, \ldots, L'_n$ and $L_0, \ldots, L_{n+1}$ for every $H \in \mathcal{H}_1$. Recall that these objects depend on H.

Now we slightly change the notations. *In what follows in the general case to take into account the dependence on H we shall denote $J(H)$ in place of J, and $\Xi_j(H)$ in place of Ξ_j for every $j \in J(H)$. Let us choose $H_0 \in \mathcal{H}_1$. Put $J = J(H_0)$, $\Xi_j = \Xi_j(H_0)$ for every $j \in J(H_0)$, and $L'_0, L'_{\sigma+1}, \ldots, L'_n$, $L_0, \ldots, L_{n+1}$ to be the constructed linear forms corresponding to H_0.* Thus, assertion (a) is proved.

Let us prove (c). Let us enumerate linear forms $H \in \mathcal{H}_1$. Suppose that $z \notin \mathcal{Z}(H)$. Then we enumerate $j \in J(H)$. Let us choose a defined over k and irreducible over k component E of $W^{(j_1)} \cap \overline{\Delta}$ for some $j_1 \in J_1$ such that $\Xi_j(H) = \Xi_E$, see above. Using isomorphism (12) and applying Theorem 1 [10] we decide whether $z \in E$ (respectively $z \in \bigcup_{j_2\in J_2}(W^{(j_2)} \cap \overline{\Delta})$). Now $z \in E_j$ if and only if $z \in E$ and $z \notin \bigcup_{j_2\in J_2}(W^{(j_2)} \cap \overline{\Delta})$. If $z \in E_j$ then the multiplicity $\mu(z, E_j) = \mu(z, E)$ is computed by Theorem 1 (c) [10] applied to $V^{(j_1)}, W^{(j_1)}, p^{(j_1)}$ in place of V, W, p.

Notice that by Lemma 2 and our construction for every $H \in \mathcal{H}_1$ for every $j \in J(H)$ the intersection $\Xi_j \cap (\bigcup_{H' \in \mathcal{H}_1} \mathcal{Z}(H')) = \emptyset$. Hence using the considered case when $z \notin \mathcal{Z}(H_0)$ and applying Theorem 1 [10] one can decide for every $H \in \mathcal{H}_1$ for every $j \in J(H)$ for every $j_0 \in J$ whether $\Xi_j \subset E_{j_0}$. By Lemma 2 the set $\Xi_j \subset E_{j_0}$ if and only if $E_j = E_{j_0}$. Thus, we are able to construct the bijection $\gamma_H : J(H) \to J$ such that for every $j \in J(H)$ the irreducible component $E_{\gamma_H(j)} = E_j$. In what follows replacing $J(H)$ by J using γ_H we shall suppose without loss of generality that $J(H) = J$ and γ_H is the identity mapping for every $H \in \mathcal{H}_1$.

Finally, for every point $z \in \mathbb{P}^n(\overline{k})$ there is $H \in \mathcal{H}_1$ such that $H(z) \neq 0$ since $\mathcal{H}_1 \subset \mathcal{H}$ and $\#\mathcal{H}_1 = n+1$. Thus, one can decide whether $z \in E_j$ and compute the multiplicity $\mu(z, E_j)$ for every $j \in J$. Assertion (c) is proved.

Let us prove (b). Let E_j, $j \in J$, be a defined over k and irreducible over k component of $W_1 \cap \ldots \cap W_\nu$, and the last intersection is proper at E_j. Then $\widetilde{E}_j$ is a irreducible over k component of $\widetilde{W}_1 \cap \ldots \cap \widetilde{W}_\nu$ and $i_{\mathbb{P}^n(\overline{k})}(W_1, \ldots, W_\nu; E_j) = i_{\mathbb{A}^n(\overline{k})}(\widetilde{W}_1, \ldots, \widetilde{W}_\nu; \widetilde{E}_j)$. As above we identify $\widetilde{E}_j$ with the defined over k and irreducible over k component of $(\widetilde{W}_1 \times \ldots \times \widetilde{W}_\nu) \cap \Delta$. Now using the reduction to the diagonal, see the Introduction (13), we get $i_{\mathbb{A}^n(\overline{k})}(\widetilde{W}_1, \ldots, \widetilde{W}_\nu; \widetilde{E}_j) = i_{\mathbb{A}^{n\nu}(\overline{k})}(\widetilde{W}_1 \times \ldots \times \widetilde{W}_\nu, \Delta; \widetilde{E}_j)$. Let $\widetilde{E}$ (respectively E) be the closure of $\widetilde{E}_j$ in $\mathbb{A}^{n\nu}(\overline{k})$ (respectively $\mathbb{P}^{n\nu}(\overline{k})$) with respect to the Zariski topology.

Set $i(Q^{(1)}, Q^{(2)}; Q^{(3)}) = 0$ for any projective (respectively affine) algebraic varieties $Q^{(1)}, Q^{(2)}, Q^{(3)}$ such that $Q^{(3)}$ is defined over k and irreducible over k and $Q^{(3)}$ is not an irreducible component of $Q^{(1)} \cap Q^{(2)}$.

Let $\gamma \in J_{i_1,\ldots,i_\nu}$ for some $(i_1, \ldots, i_\nu) \in B_1$. Suppose that E is an irreducible component of $\overline{W}_{i_1,\ldots,i_\nu} \cap \overline{\Delta}$. Then the intersection $\overline{W}_{i_1,\ldots,i_\nu} \cap \overline{\Delta}$ is proper at E. Recall that $e_0, \ldots, e_r$ generate the ideal of $\overline{\Delta} \subset \mathbb{P}^{n\nu}(\overline{k})$ and $\dim \overline{\Delta} = n\nu - r - 1$. Hence each e_w, $0 \leq w \leq r$, is a linear combination of $e''_0, e''_{\sigma_1+1}, \ldots, e''_{n\nu}$. Therefore, $n\nu - \sigma_1 = r$, the variety E is an irreducible component of $W^{(\gamma)}_{n\nu}(L'') \cap \mathcal{Z}(e''_{n\nu+1})$, and the last intersection is proper at E. Recall that $e''_0 = e''_{n\nu+1}$, see the Introduction of [10]. Hence by the property of indices of intersection corresponding to the associativity of intersection

$$
\begin{aligned}
&i_{\mathbb{P}^{n\nu}(\overline{k})}(W^{(\gamma)}, \overline{\Delta}; E) = i_{\mathbb{P}^{n\nu}(\overline{k})}(W^{(\gamma)}, \mathcal{Z}(e''_{\sigma_1+1} \ldots, e''_{n\nu+1}); E) = \\
&\sum_{E'} i_{\mathbb{P}^{n\nu}(\overline{k})}(W^{(\gamma)}, \mathcal{Z}(e''_{\sigma_1+1}, \ldots, e''_{n\nu}); E') i_{\mathbb{P}^{n\nu}(\overline{k})}(E', \mathcal{Z}(e''_{n\nu+1}); E),
\end{aligned}
$$

where E' runs over all the defined over k and irreducible over k components of $W^{(\gamma)}_{n\nu}(L'')$. But, see the Introduction of [10], for every E' the index of intersection $i(W^{(\gamma)}, \mathcal{Z}(e''_{\sigma_1+1}, \ldots, e''_{n\nu}); E') = 1$ since $L'' \in \mathcal{U}'_{j,0}$, see the notation before Lemma 2. Hence $i(W^{(\gamma)}, \overline{\Delta}; E) = \sum_{E'} i(E', \mathcal{Z}(e''_{n\nu+1}); E) = i(W^{(\gamma)}_{n\nu}(L''), \mathcal{Z}(e''_{n\nu+1}); E)$.

Now by the general properties of indices of intersection

$$
i_{\mathbb{A}^{n\nu}(\overline{k})}(\widetilde{W}_1 \times \ldots \times \widetilde{W}_\nu, \Delta; \widetilde{E}_j) = \sum_{(i, \ldots, i_\nu) \in B_1} i_{\mathbb{A}^{n\nu}(\overline{k})}(W_{i_1,\ldots,i_\nu}, \Delta; \widetilde{E}) =
$$

$$\sum_{(i,\dots,i_\nu)\in B_1} i_{\mathbb{P}^{n\nu}(\overline{k})}(\overline{W}_{i_1,\dots,i_\nu},\overline{\Delta};E) = \sum_{(i,\dots,i_\nu)\in B_1}\ \sum_{\gamma\in J_{i_1,\dots,i_\nu}} i_{\mathbb{P}^{n\nu}(\overline{k})}(E_\gamma,\overline{\Delta};E)$$
$$= \sum_{(i,\dots,i_\nu)\in B_1}\ \sum_{\gamma\in J_{i_1,\dots,i_\nu}} i_{\mathbb{P}^{n\nu}(\overline{k})}(W^{(\gamma)}_{n\nu}(L''),\mathcal{Z}(e''_{n\nu+1});E).$$

We compute each index of intersection $i_{\mathbb{P}^{n\nu}(\overline{k})}(W^{(\gamma)}_{n\nu}(L''),\mathcal{Z}(e''_{n\nu+1});E)$ applying Theorem 1 (f) [10]. After that one can calculate by the given formula $i_{\mathbb{A}^{n\nu}(\overline{k})}(\widetilde{W}_1\times\ldots\times\widetilde{W}_\nu,\Delta;\widetilde{E}_j) = i_{\mathbb{P}^n(\overline{k})}(W_1,\dots,W_\nu;E_j)$. The equality $i_{\mathbb{P}^n(\overline{k})}(W_1,\dots,W_\nu;E_j) = i_{\mathbb{P}^n(\overline{k})}(W_1,\dots,W_\nu,\mathcal{Z}(L_{s+1},\dots,L_n);\xi)$ follows from the property of indices of intersection corresponding to the associativity of intersection. Assertion (b) is proved.

Using isomorphism (12) assertion (d) follows immediately from assertion (g) of Theorem 1 [10] applied for every $j_0\in J_0$ to $V^{(j_0)},W^{(j_0)},p^{(j_0)}$ in place of V,W,p.

The required bounds for the working time of the algorithms from Theorem 1 follow immediately from the ones for the applied algorithms. The theorem is proved.

2 Proof of Theorem 2

The proof of Theorem 2 is similar to the proof of Theorem 1. More than that, we reduce Theorem 2 to Theorem 1. If $\nu=\nu_1$ then all the assertions of Theorem 2 follow from Theorem 1 immediately. Thus, we shall suppose in what follows without loss of generality that $\nu>\nu_1$, and hence $\nu\geq 2$. Besides that, we shall assume also that $n\geq 1$.

Put $\mathcal{V}^{(1)}_\alpha=\mathcal{W}^{(1)}_\alpha=\mathbb{P}^n(\overline{k})$, and $\mathcal{V}^{(j)}_\alpha=V^{(j-1)}_\alpha$, $\mathcal{W}^{(j)}_\alpha=W^{(j-1)}_\alpha$ for every $1\leq\alpha\leq\nu$, $2\leq j\leq a_\alpha+1$. For every $1\leq\alpha\leq\nu$ let us replace the pair (a_α,b_α) by $(a_\alpha+1,b_\alpha+1)$. Now the conditions from the statement of Theorem 1 are fulfilled for $\mathcal{V}^{(j)}_\alpha$, $\mathcal{W}^{(j)}_\alpha$, $1\leq j\leq a_\alpha+1$, in place of $V^{(j)}_\alpha$, $W^{(j)}_\alpha$, $1\leq j\leq a_\alpha$ in the natural way. We apply the construction from the proof of Theorem 1 to $\mathcal{V}^{(j)}_\alpha$, $\mathcal{W}^{(j)}_\alpha$, $1\leq j\leq a_\alpha+1$, in place of $V^{(j)}_\alpha$, $W^{(j)}_\alpha$, $1\leq j\leq a_\alpha$.

In what follows $\mathcal{H}$, B_1 B_2, $J_{i_1,\dots,i_\nu}$ for $(i_1,\dots,i_\nu)\in B_1$, $J_{\alpha;i}$ for $(\alpha,i)\in B_2$, J_0, J_1, J_2, the sets of points Ξ_E for all defined over k and irreducible over k components E of $W^{(j)}\cap\overline{\Delta}$ for all $j\in J_0$, $\mathcal{H}_1$, linear forms $L_0,\dots,L_{n+1}$, $L'_0,L'_{\sigma+1},\dots,L'_n$ and other objects constructed in the proof of Theorem 1 correspond to $\mathcal{V}^{(j)}_\alpha$, $\mathcal{W}^{(j)}_\alpha$, $1\leq j\leq a_\alpha+1$. At present we assume that Ξ_E, and linear forms $L_0,\dots,L_{n+1}$, $L'_0,L'_{\sigma+1},\dots,L'_n$ correspond to $H\in\mathcal{H}_1$, and hence they depend on H, see Section 1 (we do not fix H_0 at this stage). For every $H\in\mathcal{H}_1$ we perform the following construction. Some objects defined below depends on H although we do not indicate it explicitly in the notations. Put

$$B_{1,1}=\{(i_1,\dots,i_{\nu_1},1,\dots,1)\in B_1\,:\,2\leq i_\alpha\leq b_\alpha+1\,\forall\,1\leq\alpha\leq\nu_1\}\,,$$
$$\mathcal{B}_{1,1}=\{(i_1,\dots,i_{\nu_1})\,:\,(i_1,\dots,i_{\nu_1},1,\dots,1)\in B_{1,1}\}\,,$$
$$J_{1,1}=\bigcup_{(i_1,\dots,i_\nu)\in B_{1,1}} J_{i_1,\dots,i_\nu}\,,$$
$$B_{1,2}=\{(1,\dots,1,i_{\nu_1+1},\dots,i_\nu)\in B_1\,:\,2\leq i_\alpha\leq b_\alpha+1\,\forall\,\nu_1+1\leq\alpha\leq\nu\}\,,$$
$$\mathcal{B}_{1,2}=\{(i_{\nu_1+1},\dots,i_\nu)\,:\,(1,\dots,1,i_{\nu_1+1},\dots,i_\nu)\in B_{1,2}\}\,,$$

$$J_{1,2} = \bigcup_{(i_1,\dots,i_\nu)\in B_{1,2}} J_{i_1,\dots,i_\nu}\,,$$
$$B_{2,1} = \{(\alpha,i)\in B_2\,:\,1\le\alpha\le\nu_1\},$$
$$J_{2,1} = \bigcup_{(\alpha,i)\in B_{2,1}} J_{\alpha;i}\,,$$
$$B_{2,2} = \{(\alpha,i)\in B_2\,:\,\nu_1+1\le\alpha\le\nu\},$$
$$J_{2,2} = \bigcup_{(\alpha,i)\in B_{2,2}} J_{\alpha;i}.$$

Below we use the definition of a component which maximal with respect to a set of indices. This definition is given in Section 1. Applying Theorem 1 [10] we construct the family Ξ_j, $j\in I_{1,1}$ (respectively $j\in I_{1,2}$, $j\in I_{2,1}$, $j\in I_{2,2}$), such that

- for every $j\in I_{1,1}$ (respectively $j\in I_{1,2}$, $j\in I_{2,1}$, $j\in I_{2,2}$) the set $\Xi_j=\Xi_E$ for a defined over k and irreducible over k component E of $W^{(j')}\cap\overline{\Delta}$ with $j'\in J_{1,1}$ (respectively $j'\in J_{1,2}$, $j'\in J_{2,1}$, $j'\in J_{2,2}$) such that $E\not\subset\mathcal{Z}(X_{0,0})$ and E is maximal with respect to $J_{1,1}\cup J_{1,2}$ (respectively with respect to $J_{1,2}$, $J_{2,1}\cup J_{2,2}$, $J_{2,2}$).
- conversely, for every $j'\in J_{1,1}$ (respectively $j'\in J_{1,2}$, $j'\in J_{2,1}$, $j'\in J_{2,2}$) for every defined over k and irreducible over k component E of $W^{(j')}\cap\overline{\Delta}$ such that $E\not\subset\mathcal{Z}(X_{0,0})$ and E is maximal with respect to $J_{1,1}\cup J_{1,2}$ (respectively with respect to $J_{1,2}$, $J_{2,1}\cup J_{2,2}$, $J_{2,2}$) there is $j\in I_{1,1}$ (respectively $j\in I_{1,2}$, $j\in I_{2,1}$, $j\in I_{2,2}$) with $\Xi_j=\Xi_E$,
- for all $j_1,j_2\in I_{1,1}$ (respectively $j_1,j_2\in I_{1,2}$, $j_1,j_2\in I_{2,1}$, $j_1,j_2\in I_{2,2}$) if $j_1\ne j_2$ then $\Xi_{j_1}\ne\Xi_{j_2}$, and hence $E_{j_1}\ne E_{j_2}$ by Lemma 2.

For every $j\in I_{1,1}\cup I_{1,2}\cup I_{2,1}\cup I_{2,2}$ put $\overline{E}_j=\overline{E}$, where $\Xi_j=\Xi_E$ and $\overline{E}$ is the the closure of $E\setminus\mathcal{Z}(X_{0,0})$ in $\mathbb{P}^n(\overline{k})$ with respect to the Zariski topology by inclusion (27). Now $\overline{E}_j$, $j\in I_{1,1}$ (respectively $j\in I_{1,2}$, $j\in I_{2,1}$, $j\in I_{2,2}$) is the family of all the defined over k and irreducible over k components of $\bigcup_{(i_1,\dots,i_{\nu_1})\in\mathcal{B}_{1,1}}\bigcap_{1\le\alpha\le\nu_1}W_\alpha^{(i_\alpha)}$ (respectively $\bigcup_{1\le\alpha\le\nu_1}\ \bigcup_{b_\alpha+1\le i\le a_\alpha}W_\alpha^{(i)}$, $\bigcup_{(i_{\nu_1+1},\dots,i_\nu)\in\mathcal{B}_{1,2}}\bigcap_{\nu_1+1\le\alpha\le\nu}W_\alpha^{(i_\alpha)}$, $\bigcup_{\nu_1+1\le\alpha\le\nu}\ \bigcup_{b_\alpha+1\le i\le a_\alpha}W_\alpha^{(i)}$) by Lemma 2, by property (a) from Section 1 and since $H\in\mathcal{H}_1\subset\mathcal{H}_0$.

Let $j_1\in I_{i,1}$, $j_2\in I_{i,2}$ for some $i=1,2$. Let $\mathbb{P}^{n\nu}(\overline{k})\supset E_\gamma\setminus\mathcal{Z}(X_{0,0})=E_{j_\gamma}\setminus\mathcal{Z}(H)$, $\gamma=1,2$, according to (27). Again applying Theorem 1 [10] we decide for every $i=1,2$ for every $j_1\in I_{i,1}$ for every $j_2\in I_{i,2}$ whether $\Xi_{E_1}\subset E_2$. It is equivalent to $\Xi_{j_1}\subset\overline{E}_{j_2}$ and also to $\overline{E}_{j_1}\subset\overline{E}_{j_2}$ by Lemma 2. Thus, we construct for $i=1,2$ the subset

$$J^{(i)}=\{j\in I_{i,1}\,:\,\Xi_j\not\subset\overline{E}_{j_2}\quad\text{for all}\quad j_2\in I_{i,2}\}.$$

Hence by Lemma 2 for every $j\in J^{(i)}$ for every $j_2\in J_{i,2}$ the set $\Xi_j\not\subset W^{(j_2)}\cap\overline{\Delta}$ for $i=1,2$. Recall that the sets $A^{(1)}=\{1,\dots,\nu_1\}$, $A^{(2)}=\{\nu_1+1,\dots,\nu\}$, see the statement of Theorem 2. For every $j\in J^{(i)}$ denote

$$E_j=\overline{E}\setminus\Big(\bigcup_{\alpha\in A^{(i)}}\ \bigcup_{b_\alpha+1\le i\le a_\alpha}W_\alpha^{(i)}\Big),$$

where E is a defined over k and irreducible over k component of $W^{(j_1)} \cap \overline{\Delta}$ for some $j_1 \in J_{i,1}$ such that the set $\Xi_j = \Xi_E$, and $\overline{E}$ is defined above. Therefore, E_j, $j \in J^{(i)}$ is the family of all the defined over k and irreducible over k components of $\bigcap_{\alpha \in A^{(i)}} W_\alpha$ by property (a) from Section 1 and since $H \in \mathcal{H}_1 \subset \mathcal{H}_0$.

Now we slightly change the notations. *In what follows in the general case to take into account the dependence on H for every $i = 1, 2$ we shall denote $J^{(i)}(H)$ in place of $J^{(i)}$, and $\Xi_j(H)$ in place of Ξ_j for every $j \in J^{(i)}(H)$. Let us choose $H_0 \in \mathcal{H}_1$, cf. the construction from the proof of Theorem 1. Put $J^{(i)} = J^{(i)}(H_0)$, $\Xi_j = \Xi_j(H_0)$ for every $j \in J^{(i)}(H_0)$, and $L'_0, L'_{\sigma+1}, \ldots, L'_n$, $L_0, \ldots, L_{n+1}$ to be the constructed linear forms corresponding to H_0.* Now assertion (a) of Theorem 2 holds. Assertion (a) is proved.

For every $i = 1, 2$ for every $H \in \mathcal{H}_1$ for every $j \in J^{(i)}(H)$ for every $j_0 \in J^{(i)}$ applying assertion (c) of Theorem 1 one can decide whether $\Xi_j \subset E_{j_0}$. By Lemma 2 the set $\Xi_j \subset E_{j_0}$ if and only if $E_j = E_{j_0}$. Thus, we are able to construct the bijection $\gamma_{i,H} : J^{(i)}(H) \to J^{(i)}$ such that for every $j \in J^{(i)}(H)$ the irreducible component $E_{\gamma_{i,H}(j)} = E_j$. In what follows replacing $J^{(i)}(H)$ by $J^{(i)}$ using $\gamma_{i,H}$ we shall suppose without loss of generality that $J^{(i)}(H) = J^{(i)}$ and $\gamma_{i,H}$ is the identity mapping for every $H \in \mathcal{H}_1$ for every $i = 1, 2$.

Let us prove (b). Let $j_1 \in J^{(1)}$, $j_2 \in J^{(2)}$ be arbitrary indices. Let $H \in \mathcal{H}_1$ be an arbitrary linear form. Recall that there are $j'_i \in J_{1,i}$, $i = 1, 2$, and a defined over k and irreducible over k component E_i of $W^{(j'_i)} \cap \overline{\Delta}$ such that $\Xi_{j_i} = \Xi_{E_i}$. We choose such j'_i and E_i for every $H \in \mathcal{H}_1$ and every $i = 1, 2$. Then by Lemma 2 and the described construction the inclusion $E_{j_1} \subset \overline{E}_{j_2}$ holds if and only if $\Xi_{E_1} \subset E_2$ for $H = H_0$ (or any other fixed $H \in \mathcal{H}_1$). We decide whether $\Xi_{E_1} \subset E_2$ using assertion (c) of Theorem 1. Assertion (b) is proved.

Let us prove (c). Let $j_1 \in J^{(1)}$, $j_2 \in J^{(2)}$ be arbitrary indices. Recall that at present $\mathcal{H}_1$ is from the construction from the proof of Theorem 1 with $\mathcal{V}_\alpha$, $\mathcal{W}_\alpha$ in place of V_α, W_α, see above, and $\bigcap_{H \in \mathcal{H}_1} \mathcal{Z}(H) = \emptyset$ in $\mathbb{P}^n(\overline{k})$ by property (b) from Section 1. Now $E_{j_1} \subset E_{j_2}$ if and only if $E_{j_1} \subset \overline{E}_{j_2}$ and for every $H \in \mathcal{H}_1$

$$(E_{j_1} \setminus \mathcal{Z}(H)) \cap \Big(\bigcup_{\nu_1+1 \le \alpha \le \nu} \bigcup_{b_\alpha+1 \le j \le a_\alpha} W_\alpha^{(j)} \Big) \subset \bigcup_{1 \le \alpha \le \nu_1} \bigcup_{b_\alpha+1 \le j \le a_\alpha} W_\alpha^{(j)}. \quad (28)$$

Thus, it remains to decide for every $H \in \mathcal{H}_1$ whether (28) holds.

We enumerate $H \in \mathcal{H}_1$. Let $j'_i \in J_{1,i}$, $i = 1, 2$, and the irreducible components E_i of $W^{(j'_i)} \cap \overline{\Delta}$ be as above. Recall that $j'_1 \in J_{i_1,\ldots,i_\nu}$, where $i_1, \ldots, i_\nu$ are known according to our construction (and $i_1, \ldots, i_{\nu_1} \ge 2$, $i_{\nu_1+1} = \ldots = i_\nu = 1$). We identify $E_{j_1} \setminus \mathcal{Z}(H)$ with $E_1 \setminus \mathcal{Z}(X_{0,0})$. Recall that in the proof of Theorem 1 the algebraic varieties $V_{i_1,\ldots,i_\nu}$, $\overline{V}_{i_1,\ldots,i_\nu}$ are defined. Now $V_{i_1,\ldots,i_\nu}$, $\overline{V}_{i_1,\ldots,i_\nu}$ correspond to $\mathcal{V}_\alpha$, $\mathcal{W}_\alpha$ in place of V_α, W_α, see above. Similarly for every $1 \le \alpha \le \nu$ for every $b_\alpha + 2 \le j \le a_\alpha + 1$ the algebraic varieties $V_{\alpha;j}$, $W_{\alpha;j}$, $\overline{V}_{\alpha;j}$, $\overline{W}_{\alpha;j}$ are defined. They also correspond to $\mathcal{V}_\alpha$, $\mathcal{W}_\alpha$ in place of V_α, W_α.

Further, we identify $E_1 \setminus \mathcal{Z}(X_{0,0})$ with an irreducible component of $V_{i_1,\ldots,i_\nu} \cap \Delta$, and E_1 with an irreducible component of $\overline{V}_{i_1,\ldots,i_\nu} \cap \overline{\Delta}$. Hence for every $\nu_1 + 1 \le \alpha \le \nu$ for every $b_\alpha + 2 \le j \le a_\alpha + 1$ the algebraic variety $(E_{j_1} \setminus \mathcal{Z}(H)) \cap W_\alpha^{(j)}$ is identified with $(E_1 \setminus \mathcal{Z}(X_{0,0})) \cap W_{\alpha;j} = E_1 \cap \overline{W}_{\alpha;j} \setminus \mathcal{Z}(X_{0,0})$. Now (28) is equivalent to

$$E_1 \cap \Big(\bigcup_{\nu_1+1\le\alpha\le\nu} \bigcup_{b_\alpha+2\le j\le a_\alpha+1} \overline{W}_{\alpha,j} \Big) \subset \bigcup_{1\le\alpha\le\nu_1} \bigcup_{b_\alpha+2\le j\le a_\alpha+1} (\overline{W}_{\alpha,j} \cup \mathcal{Z}(X_{0,0})). \tag{29}$$

The algebraic variety $\overline{V}_{i_1,\ldots,i_\nu} \cap \overline{\Delta}$ is given explicitly by a system of homogeneous polynomial equations with respect to $X_{0,0}$, $X_{i,j}$, $1 \le i \le \nu$, $1 \le j \le n$. According to the described construction one can find the representation ρ_1 of E_1 such that $n(\rho_1) = n\nu$, $a(\rho_1) = b(\rho_1) = 1$, $q_1(1,\rho_1) = (\overline{V}_{i_1,\ldots,i_\nu} \cap \overline{\Delta}) \cup \mathcal{Z}(X_{0,0})$, $q_2(1,\rho_1) = E_1$.

Let us construct some bijections

$$\begin{aligned}
&\tau_1 : \Big\{ 1, \ldots, \sum_{1\le\alpha\le\nu_1} (a_\alpha - b_\alpha) \Big\} \to \{(\alpha, j) : 1 \le \alpha \le \nu_1 \,\&\\
&b_\alpha + 2 \le j \le a_\alpha + 1\},\\
&\tau_2 : \Big\{ 1, \ldots, \sum_{\nu_1+1\le\alpha\le\nu} (a_\alpha - b_\alpha) \Big\} \to \{(\alpha, j) : \nu_1 + 1 \le \alpha \le \nu \,\&\\
&b_\alpha + 2 \le j \le a_\alpha + 1\}.
\end{aligned}$$

The algebraic variety $\overline{V}_{\alpha;j}$ is given explicitly by a system of homogeneous polynomial equations with respect to $X_{0,0}$, $X_{i,w}$, $1 \le i \le \nu$, $1 \le w \le n$. According to the described construction one can find the representation ρ' (respectively ρ'') of $\bigcup_{1\le\alpha\le\nu_1} \bigcup_{b_\alpha+2\le j\le a_\alpha+1} (\overline{W}_{\alpha,j} \cup \mathcal{Z}(X_{0,0}))$ (respectively $\bigcup_{\nu_1+1\le\alpha\le\nu} \bigcup_{b_\alpha+2\le j\le a_\alpha+1} \overline{W}_{\alpha,j}$) such that $n(\rho') = n\nu$, $a(\rho') = b(\rho') = \sum_{1\le\alpha\le\nu_1} (a_\alpha - b_\alpha)$, $q_1(i,\rho') = \overline{V}_{\alpha;j} \cup \mathcal{Z}(X_{0,0})$, $q_2(i,\rho') = \overline{W}_{\alpha;j} \cup \mathcal{Z}(X_{0,0})$ (respectively $n(\rho'') = n\nu$, $a(\rho'') = b(\rho'') = \sum_{\nu_1+1\le\alpha\le\nu} (a_\alpha - b_\alpha)$, $q_1(i,\rho'') = \overline{V}_{\alpha;j} \cup \mathcal{Z}(X_{0,0})$, $q_2(i,\rho'') = \overline{W}_{\alpha;j}$) where $(\alpha, j) = \tau_1(i)$ (respectively $(\alpha, j) = \tau_2(i)$).

Finally, applying assertion (b) of Theorem 2 with $(3, 2, n\nu)$ in place of (ν, ν_1, n), and with

$$E_1, \quad \bigcup_{\nu_1+1\le\alpha\le\nu} \bigcup_{b_\alpha+2\le j\le a_\alpha+1} \overline{W}_{\alpha,j}\,, \quad \bigcup_{1\le\alpha\le\nu_1} \bigcup_{b_\alpha+2\le j\le a_\alpha+1} (\overline{W}_{\alpha,j} \cup \mathcal{Z}(X_{0,0}))$$

given by ρ_1, ρ'', ρ' (in place of $W_1, \ldots, W_\nu$ given by $\rho_1, \ldots, \rho_\nu$, see the notations before the statement of Theorem 1) we decide whether (29) holds. As we have seen $E_{j_1} \subset E_{j_2}$ if and only if $E_{j_1} \subset \overline{E}_{j_2}$ and (29) is satisfied for every $H \in \mathcal{H}$. Thus, assertion (c) is proved.

The required bounds for the working time of the algorithms from Theorem 2 follow immediately from the ones for the applied algorithms. The theorem is proved.

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