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ON ENERGY AND CLUSTERS IN STOCHASTIC SYSTEMS OF STICKY GRAVITATING PARTICLES

VLADISLAV V. VYSOTSKY

Abstract. We consider one-dimensional model of a gravitational gas. The gas consists of n particles whose initial positions and speeds are random. At collisions particles stick together forming "clusters". Our main goal is to study the properties of the gas as $n \to \infty$. We separately consider "cold gas" (each particle has zero initial speed) and "warm gas" (each particle has non-zero initial speeds). For the cold gas, the asymptotics of the number of clusters $K_n(t)$ is studied. We also explore the kinetic energy $E_n(t)$. It is proved that the warm gas instantly "cools", i. e., $E_n(+0) \to 0$ as $n \to \infty$.

Keywords: gravitational gas, sticky particles, non-elastic collisions, system of particles, number of clusters, energy.

1. Introduction

In this paper we study a one-dimensional model of a gravitational gas. The gas initially consists of n particles located at the real axis. Each particle is characterized by its mass, initial position, and initial speed. Particles begin to move under the forces of mutual attraction. When a collision happens, the particles *stick* together forming new particles ("clusters"). The cluster's characteristics are defined by laws of mass and momentum conservation. Between collisions particles move according to the laws of classical mechanics.

The aggregation of particles into a big cluster reminds us the formation of a star from space dust. For more information on applications in astrophysics, see [1].

We suppose that initial positions and initial speeds of the particles are *random* variables. Our goal is to study the gas' properties as the number of initial particles n tends to infinity. Various limit properties of the considered particle system were analyzed in [2]–[6]. We explore the *number of clusters*, which is denoted by $K_n(t)$; apparently it was never considered before. We are also interested in the system's *energy* $E_n(t)$, which is a sum of clusters' kinetic energies.

It turns out that the results for models with zero (*"cold gas"*) and non-zero (*"warm gas"*) initial speeds are absolutely different. We prove that for the cold gas for any $t \geq 0$

$$\frac{K_n(t)}{n} \xrightarrow{\mathbb{P}} f(t), \qquad n \to \infty,$$

where $f(t)$ is a deterministic function such that $f(t) = 0$ for $t \geq 1$. Thus for the cold gas the average number of particles in a cluster is constant at any time $t \in [0, 1)$.

In the warm gas, conversely, clusters of unlimited size appear immediately. That is why the gas " cools", i. e., its energy instantly decreases to zero. This phenomena was empirically discovered by the authors of [5]. Here we rigorously prove it. We also show that in both cold and warm gases, at any instant $t \in (0,1)$

$$E_n(t) \xrightarrow{\mathbb{P}} \frac{t^2}{6}, \qquad n \to \infty.$$

2. Model description

2.1. **Deterministic model.**

At time $t = 0$ the gas consists of n particles located on the real axis at points $x_1(0) \le \cdots \le x_n(0)$. Each particle has initial speed $v_i(0)$. We suppose that the particles are material points characterized by their masses m_i, speeds $v_i(t)$, and coordinates $x_i(t)$. Particles move under the gravitation. The force of attraction is proportional to the product of masses, i. e., a particle with a mass $\widetilde{m}_1$, located at a point $\widetilde{x}_1$, acts at a particle with a mass $\widetilde{m}_2$, located at an $\widetilde{x}_2$, with the force

$$F = \gamma \widetilde{m}_1 \widetilde{m}_2 \, sign(\widetilde{x}_1 - \widetilde{x}_2),$$

where $\gamma > 0$ is the gravitational constant. We emphasize that this force does not depend on distance; this is typical for one-dimensional models. It is clear that the acceleration of a particle is proportional to the difference of masses to the right and to the left of the particle.

Thus the system is completely defined by the number of particles n, particles' masses, initial coordinates, initial speeds, and the gravitational constant γ. We say that the gas is *cold* if all of initial speeds are zero and *warm* otherwise. To simplify it, we suppose that $x_i(0)$ are in ascending order.

At a collision, particles *stick* together forming the new particle called *cluster*. Its mass and speed are defined by the laws of mass and momentum conservation. The well known fact is that at such *non-elastic* collisions the energy decreases. In what follows, the term "cluster" will be used to signify not only a result of merging, but also an initial particle that has not yet experienced any collisions. Thus at any instant $t > 0$ the gas is the set of clusters and each cluster contains one or more particles.

It is obvious that there exists a time moment in which all the particles merge into one big cluster. This time moment is called the *instant of collapse* and is denoted by t_n^{coll}.

We suppose that all (initial) particles have equal masses, and the total mass is 1, i. e., $m_i = n^{-1}$, where $1 \le i \le n$. Without loss of generality, assume that $\gamma = 1$. Finally, we denote $x_i := x_i(0)$ and $v_i := v_i(0)$.

2.2. **Stochastic model.**

We now suppose that x_i and v_i are random variables; all properties and laws of dynamics for the system remain the same as in the previous subsection. Hence every model of a gravitational gas is completely defined by the choice of random variables mentioned above. This brings up the of describing the limit properties of the system as $n \to \infty$. Thus we study a "typical" system consisting of big number of particles.

Initial positions of particles differ in the three models mentioned below. It turns out that these models have many identical limit properties.

In the *Poisson model*, x_i is the moment of the ith jump of a Poisson process with intensity n. This model is very useful because for any $i \ge 0$ the differences $x_{i+1} - x_i$ are i.i.d. exponential random variables, with mean n^{-1}; we formally put $x_0 := 0$. The initial positions of the particles in the Poisson model are denoted by X_i.

In the *i.i.d. model*, x_i is the ith order statistic of a sample of n i.i.d. uniformly distributed on $[0,1]$ random variables. The initial positions of the particles in the i.i.d. model are denoted by X'_i.

In the *lattice model*, the initial positions are $x_i = i/n$.

The following well-known Lemma clears the relations between Poisson and i.i.d. models.

Lemma 1. *Let X_i be the moment of the ith jump of a Poisson process (with arbitrary intensity). Then the joint distribution of $X'_i = X_i/X_{n+1}$, where $1 \le i \le n$, coincides with the joint distribution of order statistics for a sample of n i.i.d. random variables uniformly distributed on* $[0,1]$.

Initial speeds v_i are assumed to be *mutually independent* of x_i. Moreover, v_i are supposed to be *i.i.d.* Recall that in the cold gas $v_i = 0$. In the warm gas, we require only $\mathbb{E}v_1^2 := \sigma^2 < \infty$.

Suppose $\mathbb{E}v_1 = 0$; the general case could be easily obtained by adding the drift with constant speed $\mathbb{E}v_1$. In doing this, *all* the main characteristics of the system, except the energy, remain the same.

3. Dynamics and kinematics of the system

Proposition 1, which is stated below, plays important role in the investigation of the particle system. It was used in [2]–[7] and is widely known. For the sake of generality, all the propositions of this section are formulated for systems with arbitrary initial masses. Here the number of initial particles n is fixed.

3.1. **Barycenters and blocks of particles.** A set of particles with consecutive numbers is called the *block*. The notation $J = (i, i+k]$ means the block J consisting of particles with numbers $i+1, \dots, i+k$. Thus a block is related to a subset of the natural numbers interval $[1,n]$. The size of a block is the number of its elements. We note that there can be such a moment, when some of a block's particles are contained in a cluster that includes particles from outside of the block. A block is *free* up to a time t if any of its particles have not yet collided with particles from out of the block. Let us generalize this definition: a block is *free to the right (left)* up to a time t if any of its particles has not yet collided with particles initially located to the right (left) of the block. At last, $K(i, i+k] = KJ$ denotes the cluster consisting of the particles from the block $J = (i, i+k]$ (and only of them).

It is very suitable to suppose that the initial particles do not vanish at collisions but continue to exist in generated clusters. In doing this, we consider their coordinates $x_i(t)$ and speeds $v_i(t)$ equal to the appropriate attributes of the cluster containing them. Hence $x_i(t)$ and $v_i(t)$ are defined for all $t \ge 0$. Clearly, $v_i(t)$ makes discontinuous jumps at times of collisions.

Thus for any $t \ge 0$ for a block J the sum

$$x_J(t) := M_J^{-1} \sum_{j \in J} m_j x_j(t), \qquad \text{where } M_J := \sum_{j \in J} m_j.$$

is well defined. We say that $x_J(t)$ is *barycenter* of J. In the same way, put

$$v_J(t) := M_J^{-1} \sum_{j \in J} m_j v_j(t).$$

Proposition 1. *Suppose that a block $J = (i, i+k]$ is free up to a time t; then its barycenter moves as if it were a single particle with summarized mass and impulse, i. e.,*

$$x_J(s) = x_J(0) + v_J(0)\, s + \frac{(M_J^{(R)} - M_J^{(L)})\, s^2}{2}, \qquad 0 \le s \le t,$$

where $M_J^{(R)} := \sum_{j > i+k} m_j$ and $M_J^{(L)} := \sum_{j \le i} m_j$ are total masses of the particles initially located to the right and to the left of J, respectively.

Corollary. *In the cold gas for any initial positions of the particles, one has $t_n^{\mathrm{coll}} \le \sqrt{2}$.*

Proof. There exists a $k \in [1, n]$ such that the blocks $[1, k]$ and $(k, n]$ are free up to the instant t_n^{coll}. As the gas is cold, i. e., $v_i = 0$, we get $x_{(k,n]}(s) - x_{[1,k]}(s) = x_{(k,n]}(0) - x_{[1,k]}(0) - s^2/2$ for $s \le t_n^{\mathrm{coll}}$. But at the moment of final collision $x_{[1,k]}(t_n^{\mathrm{coll}}) = x_{(k,n]}(t_n^{\mathrm{coll}})$ and therefore $(t_n^{\mathrm{coll}})^2/2 = x_{(k,n]}(0) - x_{[1,k]}(0) \le 1$. □

We now generalize the situation allowing collisions with particles to the right (left) of a block.

Proposition 2. *Let a block $J = (i, i+k]$ be free to the right (left) up to a time t; then*

$$x_J(s) \ge (\le)\; x_J(0) + v_J(0)\, s + \frac{(M_J^{(R)} - M_J^{(L)})\, s^2}{2}, \qquad 0 \le s \le t.$$

Finally, we introduce

$$x_J^*(s) := x_J(0) + v_J(0)\, s + \frac{(M_J^{(R)} - M_J^{(L)})\, s^2}{2}, \qquad s \ge 0,$$

which describes the imaginary path of a block's J barycenter as if J had not collided with outer particles.

3.2. **Clusterization times.** We start with the next definitions. For a block J, the first moment, when all of its particles are included in a common cluster, is called the *clusterization time* and is denoted by t_J^{cl}. For the jth particle, the moment, when it sticks with the $(j+1)$th one, is denoted by t_j^{cl}. It is obvious that $t_j^{\mathrm{cl}} = t_{[j,j+1]}^{\mathrm{cl}}$. Clearly, for any block $J = (i, i+k]$ it is true that

$$t_J^{\mathrm{cl}} = \max_{i < j < i+k} t_j^{\mathrm{cl}}. \tag{1}$$

Thus we would like to find the way of calculating t_j^{cl}. It is given by

Proposition 3. *For any particle $1 \le j < n$, its clusterization time is*

$$t_j^{\mathrm{cl}} = \inf \left\{ s \ge 0 : \min_{\substack{0 \le l < j \\ j < k \le n}} \left(x_{(j,k]}^*(s) - x_{(l,j]}^*(s) \right) = 0 \right\}. \tag{2}$$

The proof is easy and follows from Proposition 2; one can find it in [4].

Remark. *For any j,k,l the equation $x^*_{(j,k]}(s) - x^*_{(l,j]}(s) = 0$ has exactly one non-negative root.*

Proof. From the definition of $x^*_J(s)$,

$$x^*_{(j,k]}(s) - x^*_{(l,j]}(s) = x_{(j,k]}(0) - x_{(l,j]}(0) + \left(v_{(j,k]}(0) - v_{(l,j]}(0)\right) s - \frac{M_{(l,k]} s^2}{2}. \tag{3}$$

The higher coefficient is negative and the lower one is non-negative. □

Now we are ready to study stochastic systems of particles.

4. The limit of clusterization times

In this section we study asymptotics of a particle's clusterization time as $n \to \infty$. As we vary the number of particles n, the old notation t^{cl}_j is replaced with $t^{cl}_{j,n}$.

Since n changes, particles' numbers should also change. For instance, the "middle" particle is the $(n/2)$th one. We say that the sequence $r(n)$ is *typical* if $\lim r(n) = \infty$ as $n \to \infty$, and $r(n) \in [1, n/2]$ for all n.

Lemma 2. *In the cold gas for both the Poisson and i.i.d. models for any typical sequence $r(n)$ the random variables $t^{\mathrm{cl}}_{r(n),n}$ converge in law to a limit, which does not depend on $r(n)$; denote*

$$f(t) := \lim_{n\to\infty} \mathbb{P}\{t^{\mathrm{cl}}_{r(n),n} > t\}, \qquad t \geq 0.$$

For $f(t)$ there exists the explicit expression (8). This function is continuous, decreasing, and equals zero for $t \geq 1$.

Moreover, for $t \geq 0$

$$\max_{j\in[r(n),n-r(n)]} \left|\mathbb{P}\{t^{\mathrm{cl}}_{j,n} > t\} - f(t)\right| \to 0, \qquad n \to \infty. \tag{4}$$

Remark. *Unfortunately, the author was unable to compute $f(t)$. Numerical simulations show that $f(t) = 1 - t^2$ for $t \in [0, 1]$.*

The function $f(t)$ is called the *clusterization function* of the cold gas. Theorem 1 from the next section makes this title clear.

Proof. *First part.* The Poisson model of initial positions.

Let n and $j \in [r(n), n - r(n)]$ be fixed. We use (2) to calculate $t^{\mathrm{cl}}_{j,n}$. Since $m_i = n^{-1}$ and the gas is cold, (3) transforms to

$$x^*_{(j,k]}(s) - x^*_{(l,j]}(s) = \frac{1}{k-j} \sum_{i=j+1}^{k} X_i - \frac{1}{j-l} \sum_{i=l+1}^{j} X_i - \frac{k-l}{2n}\, s^2.$$

This expression decreases for $s \geq 0$. Therefore it follows from (2) that for every $t \geq 0$

$$\{t^{\mathrm{cl}}_{j,n} > t\} \Longleftrightarrow \left\{ \min_{\substack{0\le l<j\\ j<k\le n}} \left(\frac{1}{k-j} \sum_{i=j+1}^{k} X_i - \frac{1}{j-l} \sum_{i=l+1}^{j} X_i - \frac{k-l}{2n}\, t^2 \right) > 0 \right\}. \tag{5}$$

Transforming the expression in the parentheses,

$$\begin{aligned}\frac{1}{k-j}\sum_{i=j+1}^{k} X_i - \frac{1}{j-l}\sum_{i=l+1}^{j} X_i &= \frac{1}{k-j}\sum_{i=j+1}^{k}(X_i - X_j) + X_j - \frac{1}{j-l}\sum_{i=l+1}^{j} X_i \\ &= \frac{1}{k-j}\sum_{i=j+1}^{k}(X_i - X_j) + \frac{1}{j-l}\sum_{i=l+1}^{j}(X_j - X_i). \qquad (6)\end{aligned}$$

By independence of increments and stationarity of a Poisson process, this random variable has the same distribution as

$$\frac{1}{k-j}\sum_{i=1}^{k-j} Z_i + \frac{1}{j-l}\sum_{i=1}^{j-l-1} W_i,$$

where Z_i and W_i are moments of jumps of two independent Poisson processes with intensity n. Thus, taking into account (5),

$$\mathbb{P}\{t^{\mathrm{cl}}_{j,n} > t\} = \mathbb{P}\Bigg\{ \min_{\substack{1\le q\le j \\ 1\le p\le n-j}} \left(\frac{1}{p}\sum_{i=1}^{p} Z_i + \frac{1}{q}\sum_{i=1}^{q-1} W_i - \frac{p+q}{2n} t^2 \right) > 0 \Bigg\}.$$

We finish by passing to moments of jumps P_i and Q_i of standard independent Poisson processes:

$$\mathbb{P}\{t^{\mathrm{cl}}_{j,n} > t\} = \mathbb{P}\Bigg\{ \min_{\substack{1\le q\le j \\ 1\le p\le n-j}} \left(\frac{1}{p}\sum_{i=1}^{p} P_i + \frac{1}{q}\sum_{i=1}^{q-1} Q_i - \frac{p+q}{2} t^2 \right) > 0 \Bigg\}.$$

Since the sequence $r(n)$ is typical, there exists the limit $\lim \mathbb{P}\{t^{\mathrm{cl}}_{r(n),n} > t\} =: f(t)$ as $n \to \infty$. As $j \in [r(n), n-r(n)]$, we have

$$f(t) < \mathbb{P}\{t^{\mathrm{cl}}_{j,n} > t\} < \mathbb{P}\Bigg\{ \min_{\substack{1\le q\le r(n) \\ 1\le p\le r(n)}} \left(\frac{1}{p}\sum_{i=1}^{p} P_i + \frac{1}{q}\sum_{i=1}^{q-1} Q_i - \frac{p+q}{2} t^2 \right) > 0 \Bigg\}, \qquad (7)$$

thus the uniform estimate (4) holds.

It remains to check that $1 - f(t)$ is a distribution function. This immediately follows from Prokhorov's Theorem. Indeed, as $0 \le t^{\mathrm{cl}}_{r(n),n} \le t^{\mathrm{coll}}_n$, by the Corollary of Proposition 1, distributions of the random variables $t^{\mathrm{cl}}_{r(n),n}$ are concentrated on $[0, \sqrt{2}]$.

Let us prove the stated properties of $f(t)$. As $1 - f(t)$ is a distribution function, $f(t)$ is right continuous. To show that it is left continuous, put $r(n) = n/3$. For each t the right-hand side of (7) decreases monotonically to $f(t)$ as $n \to \infty$. We see that the converging functions are continuous, the limit function $f(t)$ is decreasing, thus we have the desired continuity.

The statement that $f(t) = 0$ for $t > 1$ immediately follows from the next well known fact (for instance, see [3]): in the cold gas (for any of three models of initial positions)

$$t^{\mathrm{coll}}_n \xrightarrow{\mathbb{P}} 1, \qquad n \to \infty.$$

Finally, to get more suitable expression for $f(t)$, we proceed to the limit in (7) as $n \to \infty$. By continuity of a probability measure,

$$f(t) = \mathbb{P}\left\{ \min_{p\in\mathbb{N}} \left(\frac{1}{p} \sum_{i=1}^{p} P_i - \frac{p+1}{2}\, t^2 \right) + \min_{q\in\mathbb{N}} \left(\frac{1}{q} \sum_{i=1}^{q-1} Q_i - \frac{q-1}{2}\, t^2 \right) > 0 \right\}. \tag{8}$$

We recall that P_i and Q_i are moments of jumps of standard independent Poisson processes. Also, it can be shown that

$$f(t) = e^{t^2} \mathbb{P}\left\{ \min_{k\in\mathbb{N}} \sum_{i=1}^{k} (P_i - it^2) > 0 \right\}^2 .$$

Second part. I.i.d. model. We should only check that (4) is true.

Let us rearrange (5) to separate t. Denoting

$$G_j[y_1, \dots, y_n] := \min_{\substack{1\le l<j \\ j<k\le n}} \frac{2n}{k-l} \left(\frac{1}{k-j} \sum_{i=j+1}^{k} y_i - \frac{1}{j-l} \sum_{i=l+1}^{j} y_i \right),$$

we rewrite (5) as

$$\{ t^{\mathrm{cl}}_{j,n} > t \} \iff \{ G_j[X_1, \dots, X_n] > t^2 \}$$

(clearly, $t^{\mathrm{cl}}_{j,n}$ is for the Poisson model). The expression for the clusterization time in the i.i.d. model is the same, one should only replace X_i with X'_i. Then we pass from the i.i.d. model to the Poisson one via Lemma 1:

$$\mathbb{P}\{ G_j[X'_1, \dots, X'_n] > t^2 \} = \mathbb{P}\left\{ G_j\left[\frac{X_1}{X_{n+1}}, \dots, \frac{X_n}{X_{n+1}} \right] > t^2 \right\}.$$

It easily comes from the definition of G_j that

$$G_j\left[\frac{X_1}{X_{n+1}}, \dots, \frac{X_n}{X_{n+1}} \right] = \frac{G_j[X_1, \dots, X_n]}{X_{n+1}}.$$

Hence our goal is to prove

$$\lim_{n\to\infty} \max_{j\in[r(n), n-r(n)]} \left| \mathbb{P}\{ G_j[X_1, \dots, X_n] > X_{n+1}\, t^2 \} - f(t) \right| = 0. \tag{9}$$

At first notice that

$$X_{n+1} \xrightarrow{\mathbb{P}} 1. \tag{10}$$

In fact, switching to the moments of jumps P_i of a standard Poisson process, we get $X_{n+1} \stackrel{d}{=} P_{n+1}/n$. Since P_{n+1} is the sum of $n+1$ independent exponential random variables with mean 1, by the law of large numbers (10) holds.

Then, combining (10) with continuity of $f(t)$, we obtain (9). □

Let us turn to the warm gas. In this case it is typical that a particle merges instantly with its neighbors. Besides, clusters of unlimited size appear immediately.

Lemma 3. *In the warm gas (for any of three models of initial positions) for a typical sequence $r(n)$*

$$t^{\mathrm{cl}}_{r(n),n} \xrightarrow{\mathbb{P}} 0, \qquad n \to \infty.$$

Moreover, for any $\tau > 0$

$$\min_{j \in [r(n), n-r(n)]} \mathbb{P}\{t^{cl}_{j,n} < \tau\} \to 1, \qquad n \to \infty.$$

Proof. Fix a $\tau > 0$ and $j \in [r(n), n - r(n)]$. We reduce (3) to

$$x^*_{(j,k]}(s) - x^*_{(l,j]}(s) = x_{(j,k]}(0) - x_{(l,j]}(0) + (v_{(j,k]}(0) - v_{(l,j]}(0))\, s - \frac{k-l}{2n}\, s^2.$$

and denote by $a_{j-l,k-j}$, $b_{j-l,k-j}$ and $c_{j-l,k-j}$ the coefficients of this polynomial, from the higher degree. By $h_{j-l,k-j}(s)$ denote the polynomial itself. Then (2) may be written as

$$t^{\mathrm{cl}}_{j,n} = \inf\Big\{ s \geq 0 : \min_{\substack{1 \leq l \leq j \\ 1 \leq k \leq n-j}} h_{l,k}(s) = 0 \Big\}.$$

Consider the equations $h_{m,m}(s) = a_{m,m}s^2 + b_{m,m}s + c_{m,m} = 0$. It is true that $h_{m,m}(0) = c_{m,m} \geq 0$ and $h_{m,m}(-c_{m,m}/b_{m,m}) = a_{m,m} \cdot (c_{m,m}/b_{m,m})^2 < 0$. Hence, if there is an m such that $b_{m,m} < 0$, then a unique non-negative root of $h_{m,m}(s)$ is less then $-c_{m,m}/b_{m,m}$. Therefore

$$\{b_{m,m} < 0\} \Longrightarrow \Big\{ t^{\mathrm{cl}}_{j,n} < -\frac{c_{m,m}}{b_{m,m}} \Big\}. \tag{11}$$

Now we estimate the probability to find an m such that $b_{m,m} < 0$. Put $\tilde{v}_i = v_1 - v_2$ and consider a new sequence of i.i.d. random variables $\{\tilde{v}_i\}_{i=1}^{\infty}$. Then

$$m b_{m,m} = \sum_{i=j+1}^{j+m} v_i - \sum_{i=j-m+1}^{j} v_i \stackrel{d}{=} \sum_{i=1}^{m} \tilde{v}_i. \tag{12}$$

For any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\mathbb{P}\Big\{ \min_{1 \leq m \leq N} \sum_{i=1}^{m} \tilde{v}_i < -1 \Big\} > 1 - \frac{\varepsilon}{2}. \tag{13}$$

This assertion follows from the next well known fact (see [8, Chap. 2]):

$$\mathbb{P}\Big\{ \frac{1}{\sqrt{n \mathbb{D} u_1}} \max_{1 \leq m \leq n} \sum_{i=1}^{m} u_i > \alpha \Big\} \to \frac{2}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-x^2/2} dx, \qquad n \to \infty,$$

where $\alpha \geq 0$ and u_i are i.i.d. random variables such that $\mathbb{D}u_1 < \infty$ and $\mathbb{E}u_1 = 0$. Recall that here $\mathbb{D}\tilde{v}_1 = 2\sigma^2 < \infty$ and $\mathbb{E}\tilde{v}_1 = 0$.

Take a large n such that $N \leq r(n)$ and $r(n) + N \leq n$; then from

$$\Big\{ \min_{1 \leq m \leq N} m\, b_{m,m} < -1 \Big\} \Longrightarrow \Big\{ \min_{1 \leq m \leq N} b_{m,m} < -\frac{1}{N} \Big\},$$

(12) and (13) we deduce

$$\mathbb{P}\left\{ \min_{1\le m\le N} b_{m,m} < -\frac{1}{N} \right\} > 1-\frac{\varepsilon}{2}. \tag{14}$$

Notice that N does not depend on j and (14) is true for every $j \in [r(n), n-r(n)]$.

Further, consider the coefficients $c_{m,m}$. Taking into account that $x_{(j,k]}(0) \le x_k$ and $x_{l+1} \le x_{(l,j]}(0)$, we get

$$\max_{1\le m\le N} c_{m,m} = c_{N,N} \le x_{j+N} - x_{j-N+1}. \tag{15}$$

Let us show that for any of three models of initial positions

$$\max_{j\in[r(n),n-r(n)]} (x_{j+N} - x_{j-N+1}) \xrightarrow{\mathbb{P}} 0, \qquad n\to\infty. \tag{16}$$

For the lattice model, there is nothing to prove. By Glivenko-Cantelli Theorem,

$$\max_{1\le i\le n} \left| X_i' - \frac{i}{n} \right| \xrightarrow{\mathbb{P}} 0, \qquad n\to\infty. \tag{17}$$

whence (16) holds for the i.i.d. model. For the Poisson model, we should only switch from the i.i.d. to the Poisson (via Lemma 1) and then use (10).

For all n large enough, from (15) and (16) it follows that

$$\mathbb{P}\left\{ \max_{1\le m\le N} c_{m,m} < \frac{\tau}{N} \right\} > 1-\frac{\varepsilon}{2}. \tag{18}$$

Combining (14) and (18),

$$\mathbb{P}\left\{ \exists\, m \in [1,\, N] : 0 < -\frac{c_{m,m}}{b_{m,m}} < \tau \right\} > 1-\varepsilon.$$

Finally, taking into account (11), from the last inequality we have $\mathbb{P}\{t^{\mathrm{cl}}_{j,n} < \tau\} > 1-\varepsilon$ for all large n. This estimate is uniform over $j \in [r(n), n-r(n)]$ because (18) is uniform (as (16) is uniform) and (14) is independent of j. □

Corollary. *For any positive integer M,*

$$\min_{j\in[r(n),n-r(n)]} \mathbb{P}\{t^{\mathrm{cl}}_{[j,j+M],n} < \tau\} \to 1, \qquad n\to\infty.$$

Proof. Easily follows from (1). □

5. The limit of the number of clusters

In this section we prove one of the main results, which is the formula for the number of clusters in the cold gas.

Theorem 1. *In the cold gas for both the Poisson and i.i.d. models for any $t \ge 0$*

$$\frac{K_n(t)}{n} \xrightarrow{\mathbb{P}} f(t), \qquad n\to\infty. \tag{19}$$

Proof. We prove (19) only for the Poisson model. For the i.i.d. model the proof is omitted, since the method of switching from the Poisson to the i.i.d. was already shown.

First, we note that it is sufficient to provide the proof only for $0 \le t < 1$. Indeed, if (19) holds for $t < 1$, then it is true for $t = 1$ by continuity of $f(t)$, the equality $f(1) = 0$, and decrease of $K_n(t)$ in t. Then, by the same argument, (19) holds for $t > 1$.

Consider a $t < 1$. When a particle merges with its right hand neighbor, the number of clusters decreases by one. Therefore $K_n(t)$ is the difference between n and the number of particles that have merged with their right hand neighbors up to the time t. In other words, $K_n(t)$ equals the number of particles that have *not* yet merged with their right neighbors, i. e.,

$$K_n(t) = \#\Big\{ j \in [1, n-1] : t^{\mathrm{cl}}_{j,n} > t \Big\}.$$

Consider the sequence $r(n) := \delta n$, for a $\delta \in (0, 1/6)$. Obviously, this sequence is the typical one. The main idea of the current proof is that by Lemma 2 we know, roughly speaking, the clusterization time of any particle $j \in [\delta n, n - \delta n]$.

Since

$$\#\Big\{ j \in [\delta n, n-\delta n] : t^{\mathrm{cl}}_{j,n} > t \Big\} \le K_n(t) \le \#\Big\{ j \in [\delta n, n-\delta n] : t^{\mathrm{cl}}_{j,n} > t \Big\} + 2\delta n,$$

and δ is such that $(1-2\delta)^{-1} < 1 + 3\delta$, we have

$$\limsup_{n\to\infty} \left| \frac{K_n(t)}{n} - \frac{\#\{ j \in [\delta n, n-\delta n] : t^{\mathrm{cl}}_{j,n} > t \}}{(1-2\delta)n} \right| \le 5\delta. \tag{20}$$

For each particle with number $j \in [\delta n, n - \delta n]$ we define the random variable $\xi^{(n)}_j := \mathbb{I}_{(t,+\infty)}(t^{\mathrm{cl}}_{j,n})$. Then

$$\mathbb{P}\{\xi^{(n)}_j = 0\} = \mathbb{P}\{t^{\mathrm{cl}}_{j,n} \le t\}, \qquad \mathbb{P}\{\xi^{(n)}_j = 1\} = \mathbb{P}\{t^{\mathrm{cl}}_{j,n} > t\}, \tag{21}$$

and in the terms of the introduced variables

$$\#\Big\{ j \in [\delta n, n-\delta n] : t^{\mathrm{cl}}_{j,n} > t \Big\} = \sum_{j=\delta n}^{n-\delta n} \xi^{(n)}_j.$$

We claim that

$$\limsup_{n\to\infty} \mathbb{P}\left\{ \left| \frac{1}{(1-2\delta)n} \left(\sum_{j=\delta n}^{n-\delta n} \xi^{(n)}_j - \sum_{j=\delta n}^{n-\delta n} \mathbb{P}\{t^{\mathrm{cl}}_{j,n} > t\} \right) \right| > \delta^{1/3} \right\} < 10\delta^{1/3}. \tag{22}$$

Then from (22) and (4)

$$\limsup_{n\to\infty} \mathbb{P}\left\{ \left| \frac{1}{(1-2\delta)\, n} \sum_{j=\delta n}^{n-\delta n} \xi^{(n)}_j - f(t) \right| > 2\delta^{1/3} \right\} < 10\,\delta^{1/3}.$$

Combining this with (20), we obtain (19), since $\delta \in (0, 1/6)$ is arbitrary.

The problem is to prove (22). From

$$\operatorname{cov}(\xi^{(n)}_i, \xi^{(n)}_j) = \mathbb{P}\big\{\xi^{(n)}_i = 1,\, \xi^{(n)}_j = 1\big\} - \mathbb{P}\big\{\xi^{(n)}_i = 1\big\}\, \mathbb{P}\big\{\xi^{(n)}_j = 1\big\}$$

it follows that $\operatorname{cov}(\xi_i^{(n)},\xi_j^{(n)})<1$. Then, by tedious calculations,

$$\begin{aligned}
\mathbb{D}\left(\sum_{j=\delta n}^{n-\delta n}\xi_j^{(n)}\right) &= \sum_{i,j=\delta n}^{n-\delta n}\operatorname{cov}(\xi_i^{(n)},\xi_j^{(n)}) = \sum_{k=0}^{n-2\delta n}\sum_{\substack{|i-j|=k\\ i,j\in[\delta n,n-\delta n]}}\operatorname{cov}(\xi_i^{(n)},\xi_j^{(n)})\\
&= \sum_{k=0}^{2\delta n}\sum_{\substack{|i-j|=k\\ i,j\in[\delta n,n-\delta n]}}\operatorname{cov}(\xi_i^{(n)},\xi_j^{(n)}) + \sum_{k=2\delta n+1}^{n-2\delta n}\sum_{\substack{|i-j|=k\\ i,j\in[\delta n,n-\delta n]}}\operatorname{cov}(\xi_i^{(n)},\xi_j^{(n)})\\
&< \sum_{k=0}^{2\delta n}2n + \sum_{k=2\delta n+1}^{n-2\delta n}2n\max_{\substack{2\delta n<|i-j|\\ i,j\in[\delta n,n-\delta n]}}\operatorname{cov}(\xi_i^{(n)},\xi_j^{(n)})\\
&< 4\delta n^2 + 2n^2\max_{\substack{2\delta n<|i-j|\\ i,j\in[\delta n,n-\delta n]}}\operatorname{cov}(\xi_i^{(n)},\xi_j^{(n)}).
\end{aligned}$$

If we show that

$$\limsup_{n\to\infty}\max_{\substack{2\delta n<|i-j|\\ i,j\in[\delta n,n-\delta n]}}\operatorname{cov}(\xi_i^{(n)},\xi_j^{(n)})\le 0, \tag{23}$$

then, via Chebyshev inequality,

$$\begin{aligned}
&\limsup_{n\to\infty}\mathbb{P}\left\{\left|\frac{1}{(1-2\delta)n}\left(\sum_{j=\delta n}^{n-\delta n}\xi_j^{(n)} - \sum_{j=\delta n}^{n-\delta n}\mathbb{P}\{t_j^{\mathrm{cl}}>t\}\right)\right| > \delta^{1/3}\right\}\\
\le\ &\limsup_{n\to\infty}\frac{1}{(1-2\delta)^2n^2\delta^{2/3}}\mathbb{D}\left(\sum_{j=\delta n}^{n-\delta n}\xi_j^{(n)}\right)\le\frac{4\delta^{1/3}}{(1-2\delta)^2}
\end{aligned}$$

which implies (22).

Thus it remains to check (23) to finish the proof. Let us verify that for any $\varepsilon\in(0,1)$ there exists $N(\varepsilon)$ such that for all $n>N(\varepsilon)$ for all $i,j\in[\delta n,n-\delta n]$ it is true that

$$2\delta n<|i-j|\Longrightarrow\operatorname{cov}(\xi_i^{(n)},\xi_j^{(n)})<\varepsilon. \tag{24}$$

Denote

$$C_m[l,k,y_1,\dots,y_n] := \frac{1}{k-m}\sum_{i=m+1}^{k}y_i - \frac{1}{m-l}\sum_{i=l+1}^{m}y_i - \frac{k-l}{2n}t^2.$$

It was shown in (5) that in the cold gas for the Poisson model

$$\mathbb{P}\{\xi_m^{(n)}=1\} = \mathbb{P}\{t_{m,n}^{\mathrm{cl}}>t\} = \mathbb{P}\left\{\min_{\substack{0\le l<m\\ m<k\le n}}C_m[l,k,X_1,\dots,X_n]>0\right\}.$$

First let us prove that for any $\varepsilon\in(0,1)$ there exists $N(\varepsilon)$ such that for all $n>N(\varepsilon)$

$$m\in[\delta n,n-\delta n]\Longrightarrow\mathbb{P}\left\{\min_{\substack{\delta n<k-l\\ 0\le l<m<k\le n}}C_m[l,k,X_1,\dots,X_n]>0\right\}>1-\frac{\varepsilon}{3}, \tag{25}$$

and, consequently,

$$m \in [\delta n, n-\delta n] \Longrightarrow \mathbb{P}\Big\{ \min_{\substack{k-l \le \delta n \\ 0 \le l < m < k \le n}} C_m[l,k,X_1,\dots,X_n] > 0 \Big\} < \mathbb{P}\big\{\xi_m^{(n)} = 1\big\} + \frac{\varepsilon}{3}. \tag{26}$$

Indeed, it follows from (17) and (10) that

$$\max_{1 \le i \le n} \left| X_i - \frac{i}{n} \right| \xrightarrow{\mathbb{P}} 0, \qquad n \to \infty.$$

Hence for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that

$$n > N(\varepsilon) \Longrightarrow \mathbb{P}\Big\{ \max_{1 \le i \le n} \left| X_i - \frac{i}{n} \right| < \delta \, \frac{1-t^2}{4} \Big\} > 1 - \frac{\varepsilon}{3}. \tag{27}$$

This $N(\varepsilon)$ is the one we need. In fact,

$$\begin{aligned} &\max_{1 \le i \le n} \left| X_i - \frac{i}{n} \right| < \delta \, \frac{1-t^2}{4} \\ \Longrightarrow \quad &\forall\, l,k,m \in [1,\, n] \quad \left| \frac{1}{k-m} \sum_{i=m+1}^{k} X_i - \frac{1}{m-l} \sum_{i=l+1}^{m} X_i - \frac{k-l}{2n} \right| < \delta \, \frac{1-t^2}{2} \end{aligned}$$

implies

$$\begin{aligned} &\max_{1 \le i \le n} \left| X_i - \frac{i}{n} \right| < \delta \, \frac{1-t^2}{4} \\ \Longrightarrow \quad &\forall\, l,k,m \in [1,\, n] \quad C_m[l,k,X_1,\dots,X_n] > \Big(\frac{k-l}{2n} - \frac{\delta}{2} \Big)(1-t^2). \end{aligned}$$

The last implication and (27) yield (25).

As (25) holds, we can prove (23), or, to be precise, its equivalent form (24). Take $N(\varepsilon)$ from (25) as the required one. If $n > N(\varepsilon)$ and $i, j \in [\delta n, n - \delta n]$ are such that $2\delta n < |i-j|$, then, assuming that $i < j$,

$$\begin{aligned} &\mathbb{P}\big\{\xi_i^{(n)} = 1, \xi_j^{(n)} = 1\big\} \\ = \ &\mathbb{P}\Big\{ \min_{\substack{0 \le l < i \\ i < k \le n}} C_i[l,k,X_1,\dots,X_n] > 0, \ \min_{\substack{0 \le p < j \\ j < q \le n}} C_j[\,p,q,X_1,\dots,X_n] > 0 \Big\} \\ < \ &\mathbb{P}\Big\{ \min_{\substack{k-l \le \delta n \\ 0 \le l < i < k \le n}} C_i[l,k,X_1,\dots,X_n] > 0, \ \min_{\substack{q-p \le \delta n \\ 0 \le p < j < q \le n}} C_j[\,p,q,X_1,\dots,X_n] > 0 \Big\} \\ = \ &\mathbb{P}\Big\{ \min_{\substack{k-l \le \delta n \\ 0 \le l < i < k \le n}} C_i[l,k,X_1,\dots,X_n] > 0 \Big\} \cdot \mathbb{P}\Big\{ \min_{\substack{q-p \le \delta n \\ 0 \le p < j < q \le n}} C_j[\,p,q,X_1,\dots,X_n] > 0 \Big\}. \end{aligned}$$

The last equality should be explained in detail. The first minimum is the function of $X_1, \dots, X_{i+\delta n-1}$. On the other hand, according to (6),

$$C_j[\,p,q,X_1,\dots,X_n] = C_j[\,p,q,X_j - X_1,\dots,X_j - X_{j-1}, 0, X_{j+1} - X_j, \dots, X_n - X_j],$$

and the second minimum is the function of $X_j - X_{j-\delta n+1}, \dots, X_j - X_{j-1}, 0, X_{j+1} - X_j, \dots, X_n - X_j$. We should only recall that $2\delta n < j - i$ and apply the independence of increments of a Poisson process.

From (26) we get

$$\mathbb{P}\{\xi_i^{(n)} = 1, \xi_j^{(n)} = 1\} < \left(\mathbb{P}\{\xi_i^{(n)} = 1\} + \frac{\varepsilon}{3}\right)\left(\mathbb{P}\{\xi_j^{(n)} = 1\} + \frac{\varepsilon}{3}\right),$$

whence $\operatorname{cov}(\xi_i^{(n)}, \xi_j^{(n)}) < 2\varepsilon/3 + \varepsilon^2/9 < \varepsilon$, this is (24). □

Let us obtain the similar result for the warm gas. The following Lemma will be useful in the next section, although it does not give an accurate estimate for the number of clusters.

For a natural number M and a $\tau > 0$, define

$$\mathcal{K}_n^{M,\tau} := \left\{k \in \left[1,\ \frac{n}{M}\right] : t^{\text{cl}}_{((k-1)\,M,kM],n} < \tau\right\}. \tag{28}$$

This quantity describes the number of blocks such that they are of the form $((k-1)M, kM]$ and are clusterized before the time τ .

Lemma 4. *In the warm gas (for any of three models of initial positions) for any $M \in \mathbb{N}$ and any $\tau > 0$,*

$$\frac{\#\mathcal{K}_n^{M,\tau}}{n/M} \xrightarrow{\mathbb{P}} 1, \qquad n \to \infty.$$

Proof. The proof is identical to the proof of Theorem 1 and is not interesting by itself. There we used Lemma 2; here the Corollary of Lemma 3 will be involved.

Consider a $\delta \in (0, 1/6)$. Similarly to (21), for all $k \in [\delta n/M, (1-\delta)n/M]$ we put $\xi_k^{(n)} = \mathbb{I}_{[0,\tau)}(t^{\text{cl}}_{((k-1)\,M,kM],n})$. As it was above, it is sufficient to derive the next assertion analogous to (24): for any $\varepsilon \in (0,1)$ there exists $N(\varepsilon)$ such that for every $n > N(\varepsilon)$ for all $i, j \in [\delta n/M, (1-\delta)n/M]$

$$\frac{2\delta n}{M} < |i - j| \Longrightarrow \operatorname{cov}(\xi_i^{(n)}, \xi_j^{(n)}) < \varepsilon.$$

Unlike the previous proof, it is easy to check the required. By the Corollary of Lemma 3, there exists $N(\varepsilon)$ such that if $n > N(\varepsilon)$, then

$$j \in [\delta n, (1-\delta)\, n] \Longrightarrow \mathbb{P}\{t^{\text{cl}}_{[j,j+M],n} < \tau\} > 1 - \frac{\varepsilon}{2}.$$

Hence for $n > N(\varepsilon)$

$$k \in \left[\frac{\delta n}{M},\ (1-\delta)\,\frac{n}{M}\right] \Longrightarrow \mathbb{P}\{\xi_k^{(n)} = 1\} > 1 - \frac{\varepsilon}{2}.$$

Consequently, if $n > N(\varepsilon)$ and $i, j \in [\delta n/M, (1-\delta)n/M]$, then

$$\operatorname{cov}(\xi_i^{(n)}, \xi_j^{(n)}) < 1 - \mathbb{P}\{\xi_i^{(n)} = 1\}\,\mathbb{P}\{\xi_j^{(n)} = 1\} < 1 - \left(1 - \frac{\varepsilon}{2}\right)^2 < \varepsilon.$$

□

Considering $M = 1$, we obtain the following

Corollary. *In the warm gas for any $t > 0$*

$$\frac{K_n(t)}{n} \xrightarrow{\mathbb{P}} 0, \qquad n \to \infty.$$

6. The limit of energy

The description of the instant cooling of the warm gas is the main result of this section. Here we also give the explicit formula for the energy.

Let us explain the obtained results. In the cold gas the energy varies because of the next reasons. On the one hand, the moduli of speeds of a majority of particles increase through gravitation, thus the energy of the system increases. On the other hand, the energy vanishes at collisions. In Theorem 3, in particular, it is shown that in the cold gas one can neglect the decrease of energy at collisions. In the warm gas big clusters appear immediately, thus the initial speeds average and hence almost disappear. After this they vary in the same way as in the cold gas, i. e., one can ignore the decrease of energy at collisions.

Theorem 2. *The warm gas (for any of three models of initial positions) instantly "cools": at the time zero it holds that*

$$E_n(0) \xrightarrow{\mathbb{P}} \frac{\sigma^2}{2}, \qquad n \to \infty,$$

but for any $\rho > 0$ for any $t \in (0, \min\{\sqrt{\frac{2\rho}{3}}, \frac{\rho}{3\sigma}\})$

$$\lim_{n\to\infty} \mathbb{P}\big\{E_n(t) < \rho\big\} = 1.$$

Proof. The first assertion follows from the law of large numbers.

To estimate the energy at an instant $t \in (0, \min\{\sqrt{\frac{2\rho}{3}}, \frac{\rho}{3\sigma}\})$ we will use the representation $[1, n] = \bigcup_{j=1}^m J_j$, where J_j are the blocks appropriate to the clusters that exist at the time t. Suppose $K = KJ_j$ is an arbitrary cluster and it consists of $k = \#J_j$ initial particles. By $|a_i| < 1$, we have $|v_K(t)| < k^{-1}|\sum_{i\in J_j} v_i| + t$. Then, for the energy of the cluster K

$$E_K(t) < \frac{k}{2n}\left(k^{-1}\left|\sum_{i\in J_j} v_i\right| + t\right)^2 \le \frac{1}{2nk}\left(\sum_{i\in J_j} v_i\right)^2 + \frac{t}{n}\sum_{i\in J_j}|v_i| + \frac{kt^2}{2n}$$

and hence for the energy of the whole system

$$E_n(t) < \frac{1}{2n}\sum_{j=1}^m \#J_j^{-1}\left(\sum_{i\in J_j} v_i\right)^2 + \frac{t}{n}\sum_{i=1}^n |v_i| + \frac{t^2}{2}. \tag{29}$$

We separately consider each of this three addends. For the third one, which is not random,

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{t^2}{2} < \frac{\rho}{3}\right\} = 1, \tag{30}$$

since t is small enough. For the second addend, by the law of large numbers and $\mathbb{E}|v_1| \le \sigma$,

$$\lim_{n\to\infty} \mathbb{P}\left\{ \frac{t}{n} \sum_{i=1}^{n} |v_i| < \frac{\rho}{3} \right\} = \lim_{n\to\infty} \mathbb{P}\left\{ \frac{1}{n} \sum_{i=1}^{n} |v_i| < \frac{\rho}{3t} \right\} = 1. \tag{31}$$

It remains to estimate the first addend. Let M be a natural number and $K = KJ_j$ be a cluster. By I^M we denote the maximal set of the form of $(kM, (k+i)M]$ such that I^M is contained in J_j; I^M could be empty. Now $J_j = I^M \bigcup I$, where I is the complement of I^M with respect to J_j. Dividing I^M into disjoint parts $I_1, \dots, I_s$ of the form $((k-1)M, kM]$, we have $J_j = (\bigcup_{\ell=1}^{s} I_\ell) \cup I$. Naturally, if $I_\ell = ((k-1)M, kM]$ for some ℓ and k, then $k \in \mathcal{K} := \mathcal{K}_n^{M,t}$; recall that the set $\mathcal{K}_n^{M,t}$ is defined in (28).

Then, using Cauchy-Schwarz inequality,

$$\#J_j^{-1}\Big(\sum_{i\in J_j} v_i\Big)^2 \le \sum_{\ell=1}^{s} \frac{1}{M}\Big(\sum_{i\in I_\ell} v_i\Big)^2 + \sum_{i\in I} v_i^2,$$

whence for the first addend of the sum (29) it is true that

$$\frac{1}{2n}\sum_{j=1}^{m} \#J_j^{-1}\Big(\sum_{i\in J_j} v_i\Big)^2 \le \frac{1}{2n}\sum_{k\in\mathcal{K}} \frac{1}{M}\Bigg(\sum_{i=(k-1)M+1}^{kM} v_i\Bigg)^2 + \frac{1}{2n}\sum_{k\notin\mathcal{K}}\sum_{i=(k-1)M+1}^{kM} v_i^2. \tag{32}$$

Applying Lemma 4, for any $\delta, \varepsilon > 0$ there exists $N_1(\varepsilon)$ such that for any $n > N_1(\varepsilon)$

$$\mathbb{P}\left\{ \frac{\#\mathcal{K}}{n/M} > 1-\delta \right\} > 1 - \frac{\varepsilon}{2}.$$

Therefore, with the probability greater then $1-\varepsilon/2$, in the second addend of the sum (32) less then δn of the variables v_i^2 are summed. Thus

$$\mathbb{P}\left\{ \frac{1}{2n}\sum_{k\notin\mathcal{K}}\sum_{i=(k-1)M+1}^{kM} v_i^2 < \frac{1}{2n}\sum_{i=(1-\delta)n}^{n} (v^2)_{i:n} \right\} > 1 - \frac{\varepsilon}{2}, \tag{33}$$

where $(v^2)_{i:n}$ is the ith order statistic of the variational series of $\{v_i^2\}_{i=1}^n$.

By H_δ denote an interval such that it is symmetrical with respect to 0 and $P_v(H_\delta) = 1-\delta$, where P_v is the distribution of v_1. From the law of large numbers and Glivenko-Cantelly Theorem it follows that

$$\frac{1}{2n}\sum_{i=(1-\delta)n}^{n} (v^2)_{i:n} \xrightarrow{\mathbb{P}} \frac{1}{2}\int_{\mathbb{R}\setminus H_\delta} x^2\, dP_v(x), \qquad n\to\infty.$$

As $\mathbb{E}v_1^2 < \infty$, the right hand side tends to 0 as $\delta \to 0$. Therefore for any $\delta > 0$ that is small enough for a $\varepsilon > 0$ there exists $N_2(\varepsilon)$ such that for every $n > N_2(\varepsilon)$

$$\mathbb{P}\left\{ \frac{1}{2n}\sum_{i=(1-\delta)n}^{n} (v^2)_{i:n} < \frac{\rho}{6} \right\} > 1 - \frac{\varepsilon}{2}.$$

Combining this with (33), for all $\varepsilon > 0$ for all $n > \max\{N_1(\varepsilon), N_2(\varepsilon)\}$

$$\mathbb{P}\left\{ \frac{1}{2n} \sum_{k \notin \mathcal{K}} \sum_{i=(k-1)M+1}^{kM} v_i^2 < \frac{\rho}{6} \right\} > 1 - \varepsilon.$$

This means that

$$\lim_{n\to\infty} \mathbb{P}\left\{ \frac{1}{2n} \sum_{k \notin \mathcal{K}} \sum_{i=(k-1)M+1}^{kM} v_i^2 < \frac{\rho}{6} \right\} = 1. \tag{34}$$

Let us finish the proof estimating the first addend of the sum (32). Clearly,

$$\begin{aligned} \frac{1}{2n} \sum_{k \in \mathcal{K}} \frac{1}{M} \left(\sum_{i=(k-1)M+1}^{kM} v_i \right)^2 &\le \frac{1}{2n} \sum_{k=1}^{n/M} \frac{1}{M} \left(\sum_{i=(k-1)M+1}^{kM} v_i \right)^2 \\ &= \frac{M}{2n} \sum_{k=1}^{n/M} \left(\frac{1}{M} \sum_{i=(k-1)M+1}^{kM} v_i \right)^2. \end{aligned}$$

Consider a triangular array of the random variables

$$\eta_k^{(n)} := \left(\frac{1}{M} \sum_{i=(k-1)M+1}^{kM} v_i \right)^2, \qquad k \in \left[1,\ \frac{n}{M}\right].$$

For a fixed n, these variables are i.i.d., with mean $\sigma^2 M^{-1}$. Now take an M such that $\sigma^2 M^{-1} < \rho/6$. Then by the law of large numbers,

$$\lim_{n\to\infty} \mathbb{P}\left\{ \frac{M}{2n} \sum_{k=1}^{n/M} \eta_k < \frac{\rho}{6} \right\} \ge \lim_{n\to\infty} \mathbb{P}\left\{ \left| \frac{M}{n} \sum_{k=1}^{n/M} \eta_k - \frac{\sigma^2}{M} \right| < \frac{\rho}{6} \right\} = 1. \tag{35}$$

Using (30), (31), (34), and (35), we obtain the required estimate. □

Now we are ready to prove the formulae for the limit energy. Before this, by $L_n(t)$ denote the *size of the biggest cluster* formed up to the time t. In the cold gas, for both the Poisson and i.i.d. models for $t < 1$ it is true that $L_n(t) \sim \log n$ in probability (see [3] and [4]). Hence for any of three models

$$\frac{L_n(t)}{n} \xrightarrow{\mathbb{P}} 0, \qquad n \to \infty \tag{36}$$

(in the cold gas for the lattice model $L_n(s) = 1$ for $s < 1$).

Theorem 3. *In the cold gas (for any of three models of initial positions) for any $t \in [0, 1)$,*

$$E_n(t) \xrightarrow{\mathbb{P}} \frac{t^2}{6}, \qquad n \to \infty.$$

Proof. Consider a $t \in [0, 1)$ and a cluster $K = K(l, l+k]$ existing at the time t. At the time zero the particle with a number $i \in (l, l+k]$ experiences the acceleration $a_i(0) = \frac{n-2i+1}{n}$. Since the block $(l, l+k]$ is free up to the time t and all of its particles are contained in K at the time t, we have $x_{(l,l+k]}(t) = x_K(t) = x_i(t)$ and $v_{(l,l+k]}(t) = v_K(t) = v_i(t)$. By

Proposition 1, we obtain $v_i(t) = \frac{n-2l-k}{n}t$ and thus $|v_i(t) - a_i(0)t| < \frac{k}{n}t$. As $|a_i(0)| < 1$ and $k \le n$, then

$$|v_i(t)^2 - a_i(0)^2 t^2| < 2\,\frac{k}{n}\,t^2.$$

From (36) it follows that

$$\sup_{i\in[1,n]} \left|v_i(t)^2 - a_i(0)^2 t^2\right| \xrightarrow{\mathbb{P}} 0, \qquad n \to \infty,$$

and therefrom

$$E_n(t) - \frac{1}{2n}\sum_{i=1}^{n}\left(\frac{n-2i+1}{n}\,t\right)^2 \xrightarrow{\mathbb{P}} 0, \qquad n \to \infty.$$

The calculation of the sum finishes the proof. □

We now consider the warm gas. Proving the next Theorem, we use the same method as in the proof of Theorem 3: it is possible to estimate a particle's speed if the size of the biggest cluster is known. But the conditions on v_i should be much stronger. According to [4], if $\mathbb{E}|v_1|^7 < \infty$, then for both the Poisson and i.i.d. models, for a $t < 1$ it is true that $L_n(t) \sim n^{2/3}\log n^{1/3}$. However, we only require $L_n(t) = o(n)$. For the lattice model this result is obtained in [6], under the less restrictive condition $\mathbb{E}|v_1|^{4+\varepsilon} < \infty$.

Theorem 4. *In the warm gas (for any of three models of initial positions)*

$$E_n(0) \xrightarrow{\mathbb{P}} \frac{\sigma^2}{2}, \qquad n \to \infty.$$

If (36) *holds for a* $t > 0$, *then*

$$E_n(t) \xrightarrow{\mathbb{P}} \frac{t^2}{6}, \qquad n \to \infty. \tag{37}$$

In particular, if $\mathbb{E}|v_1|^c < \infty$, *then* (37) *is true for all* $t \in (0,1)$. *Here* $c = 7$ *for both the Poisson and i.i.d. models and* $c > 4$ *for the lattice model.*

Proof. Fix a $t \in (0,1)$ and a $\tau \in (0,t)$. We consider a cluster $K = K(l, l+k]$ existing at the time t. By Proposition 1,

$$v_i(t) = \frac{1}{k}\sum_{j=l+1}^{l+k} v_j(\tau) + \frac{n-2l-k}{n}\,(t-\tau)$$

and hence, analogous to the previous proof,

$$\left|v_i(t) - a_i(0)(t-\tau)\right| < \left|\frac{1}{k}\sum_{j=l+1}^{l+k} v_j(\tau)\right| + \frac{k}{n}\,(t-\tau) < \left|\frac{1}{k}\sum_{j=l+1}^{l+k} v_j(\tau)\right| + \frac{k}{n}\,t.$$

Since $k \le n$, we get

$$\left|v_i(t)^2 - a_i(0)^2(t-\tau)^2\right| < \left(\left|\frac{1}{k}\sum_{j=l+1}^{l+k} v_j(\tau)\right| + \frac{k}{n}\,t\right)\left(\left|\frac{1}{k}\sum_{j=l+1}^{l+k} v_j(\tau)\right| + 2t\right).$$

and, removing the parentheses,

$$\left|v_i(t)^2 - a_i(0)^2(t-\tau)^2\right| < \left(\frac{1}{k}\sum_{j=l+1}^{l+k} v_j(\tau)\right)^2 + \frac{3}{k}\,t\left|\sum_{j=l+1}^{l+k} v_j(\tau)\right| + 2\,\frac{k}{n}\,t^2.$$

Thus for the energy of the cluster $K = K(l, l+k]$

$$\begin{aligned}\left|E_K(t) - \frac{1}{2n}\sum_{i=l+1}^{l+k} a_i(0)^2(t-\tau)^2\right| \;&<\; \frac{k}{2n}\left(\frac{1}{k}\sum_{i=l+1}^{l+k} v_i(\tau)\right)^2 \\ &\quad + \frac{3}{2n}\,t\left|\sum_{i=l+1}^{l+k} v_i(\tau)\right| + \frac{k^2}{n^2}\,t^2.\end{aligned}$$

Estimating the second addend with the sum of moduli and the first one via the Cauchy-Schwarz inequality,

$$\left|E_K(t) - \frac{1}{2n}\sum_{i=l+1}^{l+k} a_i(0)^2(t-\tau)^2\right| < \frac{1}{2n}\sum_{i=l+1}^{l+k} v_i(\tau)^2 + \frac{3}{2n}\,t\sum_{i=l+1}^{l+k}\left|v_i(\tau)\right| + \frac{k^2}{n^2}\,t^2.$$

As $k^2 \le L_n(t)k$ and the total number of particles in all the clusters equals n, we obtain

$$\begin{aligned}\left|E_n(t) - \frac{1}{2n}\sum_{i=1}^{n} a_i(0)^2(t-\tau)^2\right| \;&<\; \frac{1}{2n}\sum_{i=1}^{n} v_i(\tau)^2 + \frac{3}{2n}\,t\sum_{i=1}^{n}\left|v_i(\tau)\right| + \frac{L_n(t)}{n}\,t^2 \\ &\le\; \frac{1}{2n}\sum_{i=1}^{n} v_i(\tau)^2 + \frac{3}{2}\,t\left(\frac{1}{n}\sum_{i=1}^{n} v_i(\tau)^2\right)^{1/2} + \frac{L_n(t)}{n}\,t^2 \\ &=\; E_n(\tau) + \frac{3}{2}\,t\sqrt{2E_n(\tau)} + \frac{L_n(t)}{n}\,t^2; \qquad (38)\end{aligned}$$

the second addend was estimated via Cauchy-Schwarz inequality. Let us fix a $\rho > 0$. By Theorem 2 and (36), from (38) it follows that for all τ that are small enough

$$\lim_{n\to\infty}\mathbb{P}\left\{\left|E_n(t) - \frac{1}{2n}\sum_{i=1}^{n} a_i(0)^2(t-\tau)^2\right| < \rho\right\} = 1$$

Thus the proof is finished, since for all small τ and large n it is true that

$$\mathbb{P}\left\{\left|E_n(t) - \frac{t^2}{6}\right| < 2\rho\right\} \ge \mathbb{P}\left\{\left|E_n(t) - \frac{1}{2n}\sum_{i=1}^{n} a_i(0)^2(t-\tau)^2\right| < \rho\right\}.$$

□

7. Simulations

Theorem 1, which is one of the main results, was obtained empirically. Through the simulations of the cold gravitational gas, the author discovered that for large n for $t \in [0, 1]$ the graph of the function $K_n(t)/n$ calls to mind the parabola $1 - t^2$. That is appropriate to

(19). As it was mentioned above, the author was unable to prove that $f(t) = 1 - t^2$, but the numerical simulations based on (8) show the truth of this conjecture.

Through the simulations the i.i.d. model was studied. In the warm gas the initial speeds were supposed to be Gaussian, with zero mean and unit variance.

In Fig. 1 the graphs of $K_n(t)$ for $n = 100$ and $n = 1000$ are presented. Each one was obtained through a single run of the simulation program. For $n = 5000$ the deviation from the limit function was so small that it was absolutely imperceptible with the current size of the figure.

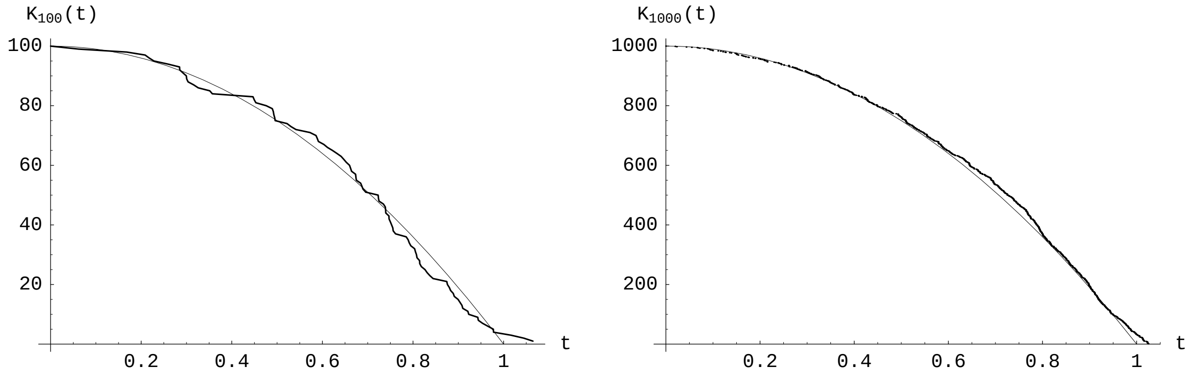

FIGURE 1. $K_{100}(t)$ and $K_{1000}(t)$ in the cold gas.

While considering the energy, it turns out that in the cold gas one gets a good approximation to the limit function (see Theorem 3) even for $n = 100$. It is confirmed by Fig. 2.

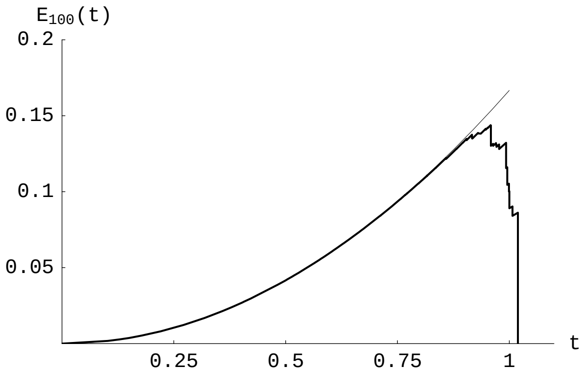

FIGURE 2. $E_{100}(t)$ in the cold gas.

In the warm gas the convergence of the energy to the limit (see Theorem 4) is much slower. In Fig. 3 the graphs of $E_n(t)$ for $n = 500$ and $n = 5000$ are shown. The deviation from the limit for $t \approx 1$ is quite significant even for $n = 10000$. However, Fig. 3 perfectly illustrates the instant cooling of the warm gas described in Theorem 2.

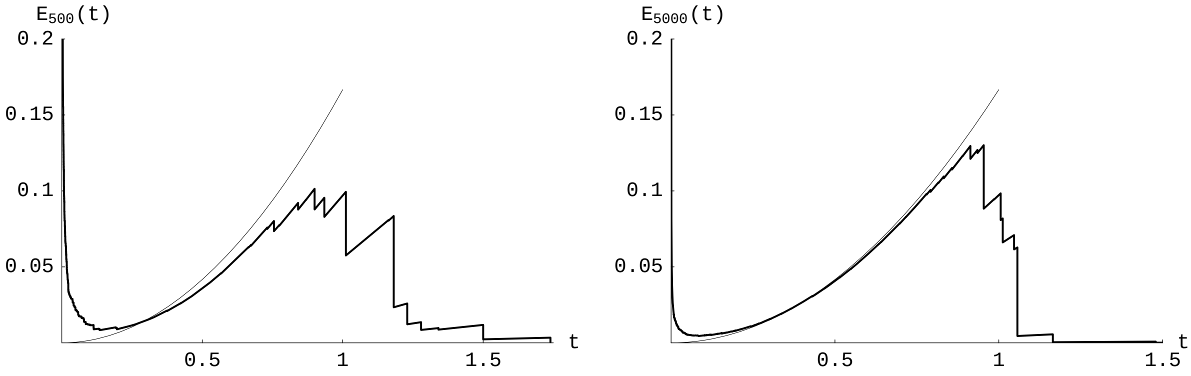

FIGURE 3. $E_{500}(t)$ and $E_{5000}(t)$ in the warm gas.

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