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Properties $\mathrm{AP}(l_p)$ and
l_p-factorizations of operators[1]

Is it true that if the adjoint to T operator T^* compactly factors through the space l_p, then the operator T itself can be (strictly) factored through the space $l_{p'}$?

The answer to this question is negative if $p = 1$ or $p = \infty$ (see [2], [3], [4]). The answer is still unknown for the other values of parameter p, i.e. in the case where $1 < p \neq 2 < \infty$.

Let us stop for a while and look at the situation somewhat clearing the difficulties in solving this problem. Definitions and details connecting with the considerations to be given now can be found below.

Let us consider the Banach tensor product $X^*\tilde{\tilde{\otimes}}_p Y$ (the completion in the norm (1) of $X^*\otimes Y$), associated with the ideal K_p of the compactly l_p-factored operators in Banach spaces (in particular, every tensor in the considering situation generates an operator which can be factored through the space l_p compactly). Each tensor $z \in X^*\tilde{\tilde{\otimes}}_p Y$, as well as every operator $S \in \mathrm{K}_p(X, Y)$, admits a representation in the form (3), and, moreover, for a given number $\varepsilon > 0$, this representation for z can be taken in such a way that the relation (4) if fulfilled, where $k_p^0(z)$ is the norm of the element z in its tensor product. Every operator S, as was said just before, also admit an expansion of such a kind into a series of 1-dimensional operators, but with the estimation at the right by its K_p-norm: $\ldots \leq k_p(S) + \varepsilon$. Besides, the norms k_p^0 and k_p are defined for the finite rank operators of the kind (2) very similarly (see below).

At the first sight, it is not seen any difference between the tensor $z \in X^*\tilde{\tilde{\otimes}}_p Y$ and the corresponding to it operator $S = \tilde{z}$, especially, if to look at their expansions into the series of the kind of (3). However, an essential difference in their definitions can be seen if we consider the finite dimensional operators (or tensors): for the finite dimensional operator S from (2) (let z be a tensor which corresponds to it), its norm in the space $\mathrm{K}_p(X, Y)$ is

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computed as the infimum of the left part of the relation (4) *over all* corresponding representations of the operator S in the form (3) , while the norm of the associated tensor element z — as the infimum of the left part of the relation (4) over all expansions of the operator S *into a finite sum of one dimensional mappings.* This difference appears to be very essential, and we know that at least for the cases where $p = 1, p = \infty$, the considered norms, generally speaking, are not equal each to other even on the finite dimensional operators (about the case $p = \infty$ see [2, Corollary 3.4 and the proofs before the corollary] and also [4]; the case $p = 1$ was considered in [3], but the corresponding assertion was appeared, without any proof, already in the paper [2, Corollary 4.2]; cf. also [4]).

If $1 \leq p \leq \infty$ and S is a finite dimensional operator from X to Y, then its K_p^0-norm is defines as the quantity

$$k_p^0(S) := \inf \left\{ (\sup_{1\leq i \leq} |a_i|) \, \varepsilon_p(x'_j) \, \varepsilon_{p'}(y_j) \right\}, \tag{1}$$

where the sup is taken over all *finite* representations of the operator S in the form

$$S = \sum_{j=1}^{n} a_j \, x'_j \otimes y_j. \tag{2}$$

The completion of the algebraic tensor product $X^* \otimes Y$ (which we consider as a linear space of all finite dimensional operators) in this norm k_p^0 will be denoted by $X^* \tilde{\tilde{\otimes}}_p Y$, and for the norm in this completion we take the same notation k_p^0. Every tensor element $z \in X^* \tilde{\tilde{\otimes}}_p Y$ can be represented in the form of a convergent series

$$\sum_{j=1}^{\infty} a_j \, x'_j \otimes y_j, \qquad \alpha_j \to 0, \tag{3}$$

and, for a given number $\varepsilon > 0$, this representation can be chosen in such a way that

$$(\sup_{1\leq i<\infty} |a_i|) \, \varepsilon_p(x'_j) \, \varepsilon_{p'}(y_j) \leq k_p^0(z) + \varepsilon \tag{4}$$

The tensor product $X^* \tilde{\tilde{\otimes}}_p Y$ generates naturally a linear subspace of operators $\mathrm{K}_p(X, Y)$ in $\mathrm{L}(X, Y)$; this is a factor space of the considered tensor product over the kernel of the natural map $X^* \tilde{\tilde{\otimes}}_p Y \to \mathrm{L}(X, Y)$. The corresponding norm on this space of operators is denoted by k_p. $[\mathrm{K}_p, k_p]$ is a

Banach operator ideal and it is a particular case of the so-called ideals of (r,p,q)-nuclear operators — $\left[\mathrm{N}_{(r,p,q)}, \nu_{(r,p,q)}\right]$: $[\mathrm{K}_p, k_p] = \left[\mathrm{N}_{(\infty,p,p')}, \nu_{(\infty,p,p')}\right]$ (details can be found in the book [1]). Every operator $S \in \mathrm{K}_p(X,Y)$ can be represented in the form of a convergent in the space $\mathrm{K}_p(X,Y)$ series (3), and, for a given number $\varepsilon > 0$, this representation can be chosen in such a way that

$$(\sup_{1\le i<\infty} |a_i|)\, \varepsilon_p(x'_j)\, \varepsilon_{p'}(y_j) \le k_p(S) + \varepsilon. \tag{5}$$

Now, let us look at the discussed at the moment operator ideal from another side. We shall say that an operator $S: X \to Y$ *compactly factors through the space* l_p, (in the terminology of A. Pietsch [1], is *p-compact*), if there exist two compact operators $A: X \to l_p$ and $B: l_p \to Y$, for which $S = BA$. As a norm of such compactly l_p-factored operator we take a quantity $\inf \|A\|\,\|B\|$, where the infimum is taken over all possible factorizations of the operator S of the mentioned kind.

One of the main theorems in the theory of compactly l_p-factored operator is formulated as follows:

$$k_p(S) = \inf\{\|A\|\,\|B\|:\ S = BA: X \to l_p \to Y;\ A, B \text{ are compact}\}$$

and *the ideal* K_p *coincides with the ideal of all operators which are compactly factored through* l_p; in the case $p = \infty$ the compact factorization through l_∞ is just the same as the compact factorization through c_0 (see [1], Theorems 18.3.2, 18.1.3).

A little bellow we will consider some of the facts, obtained by us and connected with the presence or the absence of the approximation properties with respect to the tensor norms k_p^0 in Banach spaces ($1 \le p \le \infty$). Let us denote shortly by $\mathrm{AP}(l_p)$ the approximation property with respect to k_p^0 We write $X \in \mathrm{AP}(l_p)$, if the space X has the property $\mathrm{AP}(l_p.)$ It is worthwhile to recall that $X \in \mathrm{AP}(l_p)$ *if for every Banach space* Y *the canonical mapping from* $Y^*\widetilde{\widetilde{\otimes}}_p X$ *to* $\mathrm{L}(Y,X)$ *is one-to-one.* This is equivalent to the fact that for every Y we have a canonical isometry $Y^*\widetilde{\widetilde{\otimes}}_p X = \mathrm{K}_p(Y,X)$.

It is for a long time not known *if there exist the spaces without the property* $\mathrm{AP}(l_p)$ *for any* p *not equal to one, two or infinity.* It is not hard to show that every Banach space has the property $\mathrm{AP}(l_2)$. On the other hand, it was shown by the author that *there exist separable Banach spaces without the properties* $\mathrm{AP}(l_1)$ *and* $\mathrm{AP}(l_\infty)$ (cf. [2], [3], [4]).

Recall that a Banach space is said to be *completely separable* if it and all of its duals are separable.

Theorem 1. *Let $p \in [1, \infty]$. Suppose that there exists a Banach space without the property* $\mathrm{AP}(l_p)$. *Then*

1) *there exist a completely separable Banach space X and a separable reflexive space Y, an operator $U \in \mathrm{QN}_p^{\mathrm{dual}} \circ \mathrm{QN}_{p'}(X, Y)$ such that $X \in \mathrm{AP}$, U lies in the closure of the subspace $X^* \otimes Y$ of the space $\mathrm{D}_{p'}$, dual to $X^* \otimes Y$, in the topology $\sigma(\mathrm{D}_{p'}(X, Y), Y^* \widetilde{\widetilde{\otimes}}_p X)$ (and, therefore, in the topology of compact convergence), but U is not in the closure of the ball of radius C of the subspace $X^* \otimes Y$ of the space $\mathrm{D}_{p'}(X, Y)$ in the topology of the compact convergence for any $C < \infty$;*

2) *there exist a completely separable Banach space X and a separable reflexive space Y, an operator $T \in \mathrm{L}(Y, X)$ such that $X \in \mathrm{AP}$, $T \in \mathrm{K}_p(Y, X^{**})$ and $T \notin \mathrm{K}_p(Y, X)$; moreover, there exists a sequence of operators $\{T_n\}$ v $Y^* \otimes X$, which is convergent to T in norm induced from $\mathrm{K}_p(Y, X^{**})$. Thus, in this case the natural injection $\mathrm{K}_p(Y, X) \subset \mathrm{K}_p(Y, X^{**})$ is not isometric.*

Theorem 2. *If there exists a Banach space which does not possess the property $\mathrm{AP}(l_p)$ then there exist a separable reflexive Banach space X, a completely separable Banach space Z and an operator $T \in \mathrm{L}(X, Z)$ such that Z^{**} has a basis, $X \notin \mathrm{AP}(l_p)$, $T^{**} \in \mathrm{K}_p(X, Z^{**})$, but $T \notin \mathrm{K}_p(X, Z)$. Moreover, in this case the natural injection $\mathrm{K}_p(X, Z) \subset \mathrm{K}_p(X, Z^{**})$ is isometric, but not onto.*

Now, let us gather some of the facts obtained by us about the spaces with the properties $\mathrm{AP}(l_p)$ and formulate them as one very general theorem.

Theorem 3. *For every $p \in [1, \infty]$ the following statements are equivalent.*

1) *Every Banach space possesses the property* $\mathrm{AP}(l_p)$;

1′) *Every separable reflexive Banach space possesses the property* $\mathrm{AP}(l_p)$;

2) *Every Banach space possesses the property* $\mathrm{AP}(l_{p'})$;

2′) *Every separable reflexive Banach space possesses the property* $\mathrm{AP}(l_{p'})$;

3) *There does not exist an operator acting in Banach spaces which is not a K_p-operator, but whose second adjoint lies in the ideal* K_p;

3′) *There does not exist an operator acting in completely separable Banach spaces with bases which is not a K_p-operator, but whose second adjoint lies in the ideal* K_p;

4) *the ideal K_p is regular;*

5) *the ideal* $\mathrm{K}_{p'}$ *is regular;*

6) *the ideal* $\mathrm{K}_p^{\mathrm{reg}}$ *is minimal;*

7) *the ideal* $\mathrm{K}_{p'}^{\mathrm{reg}}$ *is minimal;*

8) *the ideal of the finite dimensional operators is dense in the ideal* $\mathrm{K}_p^{\mathrm{reg}}$;

9) *the ideal of the finite dimensional operators is dense in the ideal* $\mathrm{K}_{p'}^{\mathrm{reg}}$;

10) *for every separable reflexive Banach space* X *the canonical embedding* $X^*\tilde{\tilde{\otimes}}_p X \to \mathrm{L}(X,X)$ *is one-to-one;*

11) *for every Banach space* X *the canonical embedding* $X^*\tilde{\tilde{\otimes}}_p X \to \mathrm{L}(X,X)$ *is one-to-one.*

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