SYMMETRIES OF WEAK-CONTROLLABLE SYSTEMS

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1 Introduction

Hereafter all objects belong to the C^∞-category. The completeness of the vector field means that every trajectory is defined on whole $\mathbb{R}$. In other words, the flow forms one-parameter group of transformations.

After the basic facts about extendability of trajectories in O.D.E. theory, it turns out that in the others theories dealing with vector fields there are no difficult theorems about completeness. The notion of completeness stands in the row of properties which either have place obviously or one can say nothing of important about it. The Palais example shows that the set of complete fields is not even the linear space. Because of this the theory of infinite dimensional Lie algebras and groups are different ones.

From the other side in the affine and semi-Riemannian geometry the incompleteness is unavoidable phenomenon. An example of incomplete spray of affine connection on closed manifold (nonmetrizable) one can find even on one-dimensional manifold — on the circle. Moreover, many physically justified models with incomplete geodesic flows appeared.

Such situation might be explained by the absence of "an uniform reason" for completeness. There are several direct arguments of that sort: explicit integration; boundedness in infinity of the vector length (in particular, the compactness of support); the existence of transitive group of transformation. We used in the article the more effective indirect methods which ascertain the completeness of field in the presence of another complete field. Namely, when one considers the symmetries. The typical examples of such sort theorems are the Palais and Kobayashi ones. We will show the tight relation between the completeness of symmetries and different notions of controllability of Nonlinear Control Systems. The main results are the theorems 2 and 5

2 Controllable Systems

Let φ_t^X be the flow of the field X. Consider the set of vector fields $\{Y_\alpha\}$.

Definition 1. The point z' is *accessible* from the point z, $z' \sim z$, if there exist the set of numbers $\{t_1, \ldots, t_N\}$ and set of indices $\{\alpha_1, \ldots, \alpha_N\}$ such, that

$$z' = \varphi_{t_1}^{Y_{\alpha_1}} \circ \cdots \circ \varphi_{t_N}^{Y_{\alpha_N}}(z). \tag{1}$$

Geometrically the accessibility means that one can reach the point z by the arcs of trajectories of fields $\{Y_\alpha\}$. Accessibility is the equivalence on the manifold and each point z define the class of equivalence $\mathfrak{O}(z)$. Due to the Sussmann-Nagano theorem [1] the orbit $\mathfrak{O}(z)$ is connected immersion submanifold of M. The set $\{Y_\alpha\}$ is called *full controllable* if the class of equivalence equals M. Consider the system $\{Y_\alpha\}$ of complete fields. If these fields generate the finite dimensional Lie algebra $\mathfrak{g}$, then according the Palais theorem this algebra consists of complete fields and their flows possess the structure of connected Lie group G acting on manifold. For the system of complete fields $\{Y_\alpha\}$ generating the finite dimensional Lie algebra the notions of full-controllability and transitive action of group G are equivalent. In general situation the system of complete fields does not generate the finite dimensional Lie algebra and the linear combination of complete fields can be incomplete, however the conclusion of Sussmann theorem is true.

It is useful to introduce the notion of controllability *on the set*. The system is full-controllable on the set A if there exists the point $x_0 \in M$ such that $A \subset \mathfrak{O}(x_0)$.

Proposition 1. *If the system of complete fields $\{Y_\alpha\}$ generates the finite dimensional Lie algebra, then it is full-controllable on each orbit of action of corresponding Lie group.*

In the paper [3] the sufficient conditions for full controllability are found. These conditions are applicable for linear systems and systems satisfying the Palais theorem conditions. The further extension of the notion of controllability leads to criteria of topological nature.

Definition 2. The system $\{Y_\alpha\}$ is *weak-controllable* from the point x_0 on the set A, if

$$\forall U_{x_0}, \forall z \in A => \exists z' \in U_{x_0} : z' \sim z, \tag{2}$$

where U_{x_0} is the neighborhood of x_0.

Analogously we denote the set of accessibility $\mathfrak{O}(U) = \{z \in M | \exists z' \in U : z \sim z'\}$. Now we can rewrite the definition 2 in the form

$$A \subset \bigcap_{U_{x_0}} \mathfrak{O}(U_{x_0}) \tag{3}$$

It is clear that weak-controllability follows from the full controllability. The inverse is not true as the next example shows. Let us consider the system consisting of one field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on the plane (x, y). This system is weak-controllable from the origin on the whole plane, but it is not full-controllable.

The Ω-limit of the vector field Y flow on the set A is the set

$$\Omega^Y(A) = \bigcap_{z \in A} Cl\, \gamma(z), \tag{4}$$

where $\gamma(z)$ is the trajectory passing through the point z. Consider the system $\{Y\}$ consisting of one vector field.

Theorem 1. *The system $\{Y\}$ is weak-controllable if and only if $\Omega^Y(A) \neq \varnothing$*

Proof . Weak-controllability of the system from the point x_0 means that this point is the limit point of each trajectory intersecting the set A, i.e. $x_0 \in \Omega^Y(A)$. $\square$

Corollary 1. *The system $\{Y\}$ is weak-controllable from any point on any subset of everywhere dense trajectories.*

In particular, if all trajectories are dense on manifold, the system is weak-controllable from any point of M. Let us consider now the system $\{Y_\alpha\}$ of complete fields generating the finite dimensional Lie algebra $\mathfrak{g}$ and corresponding Lie group G. In view of proposition 1 the set $\mathfrak{O}(z)$ coincides with the G-orbit $O(z)$ for any point z. In this case the weak-controllability on A means that there exists the point x_0 which belongs to the closure of each orbit intersecting the set A, $x_0 \in \Omega(A) = \bigcap_{z \in A} Cl\, O(z)$. It is clear that nonemptiness of the set $\Omega(A)$ is the criterion of weak-controllability.

3 Controllability and completeness of symmetries

In this section we bind the notions introduced in section 2 with completeness of symmetries. The first example of such connection could be the following sufficient condition of completeness. We will distinguish the positive and negative Ω-limits of flow on the set A

$$\Omega^{\pm}_X(A) = \bigcap_{z \in A} Cl\, \gamma^{\pm}(z). \tag{5}$$

If for every trajectory γ on manifold holds $\Omega^+_X(\gamma) \neq \varnothing$ and $\Omega^-_X(\gamma) \neq \varnothing$, the field X is complete. Indeed, let us consider the point $x_0 \in \Omega^+_X(\gamma)$. This point has a neighborhood U^ε, in every point of which the trajectory is extendable on interval $[0, \varepsilon)$. The point x_0 is the limit point for the positive semi-trajectory of any point $z \in \gamma$. Therefore, there exists such point $T > 0$, that $\varphi^X_T(z) \in U^\varepsilon$. Hence, the point $\varphi^X_{T+\varepsilon/2}(z)$ is defined. It means that for every point of this trajectory it is extendable on $\varepsilon/2$ in positive direction. Analogously one can prove the extendability in negative direction. For instance, any flow on closed manifold satisfies above condition. Let us formulate now the main result of this section.

Theorem 2. *The system of complete fields $\{Y_\alpha\}$ is weak-controllable. If the field X commutes with all fields of the system, it is complete.*

Proof . Let's say the point z has ε-property with respect to field X, if the trajectory $\gamma(z)$ is extendable on interval $(-\varepsilon, \varepsilon)$. Let Y be complete and the field X commute with it, $[Y, X] = 0$. If the point z has ε-property with respect to the field X, then all points $\varphi^Y_t(z)$ have ε-property. Indeed, let us fix the number T and consider the curve $\varphi^X_s\left(\varphi^Y_T(z)\right) = \varphi^Y_T\left(\varphi^X_s(z)\right)$. The equality holds by virtue of commutativity and the right part of equality is defined for all $s \in (-\varepsilon, \varepsilon)$. Hence, the left part is defined too. Let now all points of the set U^ε_X have ε-property with respect to X. Under this condition all points of the accessibility set $\mathfrak{O}(U^\varepsilon_X)$ of the system $\{Y_\alpha\}$ of complete fields possess this property. The system is weak-controllable on manifold, therefore there exists the point x_0 form every neighborhood of which all points of manifold are accessible. The point x_0 in turn has the neighborhood U^ε_X possessing ε-property and, hence, $\mathfrak{O}(U^\varepsilon_X) = M$. Finally, two mutually inverse map $\varphi^X_{\pm\varepsilon/2}$ are defined globally on manifold. It means that the field X is complete. □

4 The Kobayashi Theorem Extension

Consider the manifold M provided with symmetrical affine connection. It means that the spray S is given on the tangent bundle TM. The field Y on TM is called the polynomial symmetry of spray if $[S, Y] = 0$ and $[Y, L] = kL$, $k \geqslant 0$, where L is the Liouville field. The number k is the rank of the symmetry. All polynomial spray symmetries are complete lifts [4] on TM of symmetrical tensor fields of type $(1, k)$ on M. The symmetry is covariant constant if it is the complete lift of covariant constant tensor field. The two known results about spray symmetries are the following theorems.

Theorem 3. (Kobayashi [2]) *If the spray is complete then all symmetries of rank 0 are complete.*

Theorem 4. ([5]) *If the spray is complete then all covariant constant symmetries of rank 1 are complete.*

The space of all polynomial spray symmetries is, in general, infinite dimensional. It is however known [5] that for each rank the space is finite dimensional. The space of all covariant constant symmetries is commutative Lie algebra. By virtue of the above two theorems we can consider the system $\{Y_\alpha\}$ of covariant constant symmetries of ranks 0 and 1 which are complete if the spray is.

Theorem 5. *If the system $\{Y_\alpha\}$ is weak-controllable on $TM\backslash M$ and the spray is complete, then all covariant constant spray symmetries are complete.*

Proof . In view of theorem 2 all symmetries are complete on $TM\backslash M$ and the null section M of tangent bundle TM consists of null points of complete lifts. □

Note 1. In the article of the author [6] the wide class of covariant constant symmetries of all ranks is described which are complete if the spray is. Hence, one can weaken the condition of the theorem 5.

The described approach allows to inverse the statement about the spray completeness.

Theorem 6. *If the set of complete spray symmetries (even not polynomial) is weak-controllable on $TM\backslash M$, then the spray is complete.*

Acknowledgements

This work is supported by VNP of Ministry of Education of RF 3.1N4733

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