Pólya processes and sequences: the martingale characterization

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Abstract

It is proved that a mixed Poisson process ξ_t is either Pólya or pure Poisson process iff there exists a non-degenerate linear transformation $\xi_t \mapsto \eta_t = a(t)\xi_t + b(t)$ such that η_t is a martingale. Analogous result is demonstrated for Pólya sequences.

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1 Introduction

Let $\eta_\bullet = \{\eta_i, i \geq 1\}$ be a sequence of i.i.d. random variables such that $\mathcal{L}(\eta_i) = \mathrm{EXP}(1)$, where $\mathcal{L}(\xi)$ stands for the distribution of the random variable ξ and $\mathrm{EXP}(\lambda)$ is the exponential distribution with the parameter λ.

Consider a non-negative random variable τ with the distribution $\mathcal{P} = \mathcal{L}(\tau)$ and assume that τ is independent of $\eta_\bullet$. Then the random process $N_\bullet = \{N_t,\ t \geq 0\}$ defined by the equality

$$N_t = \max\{n : \eta_1 + \ldots + \eta_n < t\tau\} \tag{1.1}$$

is called *the mixed Poisson process with the structure distribution* $\mathcal{P}$.

Definition (1.1) means that the conditional distribution of the process $N_\bullet$ under the condition $\tau = z$ coincides with the distribution of a Poisson process with the parameter z.

Mixed Poisson processes are natural models in many fields including accidents and sickness, market research, clinical trials, etc. Numerous examples and references can be found in [3]–[5].

The most common choice of the structure distribution $\mathcal{P}$ is the Gamma distribution $\mathrm{G}(k, \mu)$ with a certain parameters $k, \mu > 0$ (k is the *shape parameter* and μ the *scale parameter*). Then $N_\bullet$ is called *the Pólya process* with the parameters k, μ.

The natural problem is to characterize Pólya processes in the class of all mixed Poisson processes. Several characterization results can be found in [4].

For example (see [4, th. 2.2]), if $\mathbb{E}\tau < \infty$ and $\mathbb{E}(\tau \,|\, N_{t_0} = n) = a_{t_0}n + b_{t_0}$ for some $t_0 > 0$ and any $n > 0$, then $N_\bullet$ is either a Poisson process or a Pólya process. Other characterizations of Pólya/Poisson processes are given in terms of their transition intensities and in terms of time-transformation of the corresponding mixed Poison processes (see [4, th. 4.1 and th. 4.3]).

The aim of this paper is to characterize Pólya processes in martingale terms. The interest to martingale characterizations can be explained as follows. By definition, the process $\xi_\bullet$ is the martingale relative to the *natural filtration* $\mathcal{F}_t^\xi = \sigma(\xi_s, 0 \leq s \leq t)$, if $\mathbb{E}|\xi_t| < \infty$ for any $t \geq 0$ and

$$\mathbb{E}(\xi_T \,|\, \mathcal{F}_t^\xi) = \xi_t \qquad a.s. \tag{1.2}$$

for any $T \geq t \geq 0$.

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If additionally $\mathbb{E}\xi_t^2 < \infty$ for any $t \geq 0$, then the left side of (1.2) has the interpretation in terms of prediction: $\mathbb{E}(\xi_T \,|\mathcal{F}_t^{\xi})$ is the best least-square approximation of ξ_T in the class $L^2_{\mathcal{F}_t^{\xi}}$ of all random variables ζ such that a) $\mathbb{E}\zeta^2 < \infty$ and b) ζ depends only on the trajectory $\{\xi_s,\, 0 \leq s \leq t\}$ of the process $\xi_{\bullet}$.

In other words, (1.2) means that ξ_t is the best (least-square) prediction of ξ_T for any $T > t$, if the approximation is based only on the knowledge of the trajectory of the process $\xi_{\bullet}$ up to time t. Evidently, such a prediction seems to be somewhat degenerate. Wiener process is the most well-known example of this situation.

A bit less trivial case occurs if some non-degenerate linear transformation of $\xi_{\bullet}$ is a martingale (rather than the process $\xi_{\bullet}$ itself). Indeed, assume that $\mathbb{E}\xi_t^2 < \infty$ for any $t \geq 0$ and consider the process $\zeta_t = b(t)(\xi_t + a(t))$ with $b(t) \neq 0$ for any $t \geq 0$.

Evidently, $\mathcal{F}_t^{\zeta} \stackrel{\text{def}}{=} \sigma(\zeta_s, 0 \leq s \leq t) = \mathcal{F}_t^{\xi} \stackrel{\text{def}}{=} \sigma(\xi_s, 0 \leq s \leq t)$. If $\zeta_{\bullet}$ is the martingale relative to the natural filtration $\mathcal{F}_t^{\zeta}$, then for any $T > t$

$$\mathbb{E}\big(\xi_T \,|\, \mathcal{F}_t^{\xi}\big) = \frac{b(t)}{b(T)}\,\xi_t + \frac{b(t)a(t)}{b(T)} - a(T). \tag{1.3}$$

(Of course, this equality is valid only almost surely. Starting from this point, we ignore these technicalities.) Therefore, the best least-square prediction of ξ_T is equal to the right side of (1.3).

The good example of such a situation is the Poisson process with the parameter λ: then $b(t) \equiv 1$, $a(t) = -\lambda t$ and the best prediction of ξ_T is $\xi_t + \lambda(T-t)$. Other processes with independent increments and finite second moments produce analogous examples.

Our aim is to prove that a non-degenerate linear transformation of a mixed Poisson process is the martingale iff this process is either a Poisson process or a Pólya process. (Note that a Pólya process has by no means independent increments.) Analogous proposition is demonstrated for the so-called Pólya sequences, common in some popular models of consumer behavior.

2 Characterization of Pólya processes

Let us start with the simple lemma.

Lemma 2.1. *Let $N_{\bullet}$ be a mixed Poison process with the structure distribution $\mathcal{P}$. Denote $\mathcal{F}_t^N = \sigma(N_s, 0 \leq s \leq t)$. Then*

$$\mathbb{E}(N_T \,|\, \mathcal{F}_t^N) = N_t + (T-t)\,\frac{\int_{(0,\infty)} \lambda^{N_t+1} e^{-\lambda t} \mathcal{P}(d\lambda)}{\int_{(0,\infty)} \lambda^{N_t} e^{-\lambda t} \mathcal{P}(d\lambda)} \tag{2.1}$$

for any $T > t \geq 0$.

Proof. It is easy to show that

$$\mathbb{P}(N_{t_1} = m_1, \ldots, N_{t_n} = m_n) = \prod_{i=1}^{n-1} \frac{(t_{i+1} - t_i)^{m_{i+1} - m_i}}{(m_{i+1} - m_i)!} \int\limits_{(0,\infty)} \lambda^{m_n} e^{-\lambda t_n} \mathcal{P}(d\lambda)$$

for any $n \geq 1$, $0 = t_0 < t_1 < \ldots < t_n$ and integer $0 = m_0 \leq m_1 \leq \ldots \leq m_n$ Therefore, $N_{\bullet}$ is the Markov process and $\mathbb{E}(N_T \,|\, \mathcal{F}_t^N) = \mathbb{E}(N_T \,|\, N_t)$. The transition function of $N_{\bullet}$ has the form

$$\mathbb{P}(N_T = m \,|\, N_t = l) = \frac{(T-t)^{m-l}}{(m-l)!}\,\frac{\int_{(0,\infty)} \lambda^m e^{-\lambda T} \mathcal{P}(d\lambda)}{\int_{(0,\infty)} \lambda^l e^{-\lambda t} \mathcal{P}(d\lambda)}\,, \tag{2.2}$$

where $m \geq l \geq 0$ and $T > t > 0$. Analogously, if $t = 0$ and $m \geq 0$, then

$$\mathbb{P}(N_T = m \,|\, N_0 = 0) = \mathbb{P}(N_T = m) = \frac{T^m}{m!} \int\limits_{(0,\infty)} \lambda^m e^{-\lambda T} \mathcal{P}(d\lambda). \tag{2.3}$$

In view of (2.2) and (2.3), we get (2.1). $\square$

Proposition 2.1. *1. Consider a Pólya process $N_{\bullet}$ with the parameters k, μ and denote $\zeta_t = (N_t + k)/(t + \mu)$. Then $\zeta_{\bullet}$ is the martingale relative to the natural filtration $\mathcal{F}_t^{\zeta}$.*
2. Let $N_{\bullet}$ be a mixed Poisson process with the structure distribution $\mathcal{P}$. Suppose that there exist functions $b(\cdot)$ and $a(\cdot)$ such that the process $\zeta_t = b(t)(N_t + a(t))$ is the martingale relative to the natural filtration $\mathcal{F}_t^{\zeta}$. Suppose additionally that $b(t) \neq 0$ for any $t \geq 0$. Then either $\mathcal{P} = \delta_{\alpha_0}$ for some $\alpha_0 > 0$ or $\mathcal{P} = \mathrm{G}(k, \mu)$ with certain $k, \mu > 0$.

Proof. 1. As it was already mentioned, $\mathcal{F}_t^{\zeta} = \mathcal{F}_t^N$. Since $\mathcal{P} = \mathrm{G}(k, \mu)$, then

$$\int\limits_{(0,\infty)} \lambda^l e^{-\lambda t} \mathcal{P}(d\lambda) = \frac{\mu^k}{\Gamma(k)} \frac{\Gamma(k + l)}{(\mu + t)^{k+l}} \tag{2.4}$$

for any $l \geq 0$. In view of (2.4), we obtain from (2.1) that

$$\mathbb{E}(N_T \,|\, \mathcal{F}_t^N) = N_t + (T - t)\, \frac{k + N_t}{\mu + t} = \frac{N_t(\mu + T) + k(T - t)}{\mu + t} \,.$$

Therefore,

$$\mathbb{E}(\zeta_T \,|\, \mathcal{F}_t^{\zeta}) = \frac{\mathbb{E}(N_T \,|\, \mathcal{F}_t^N) + k}{\mu + T} + \frac{k}{\mu + T} = \frac{N_t}{\mu + t} + \frac{k(T - t)}{(\mu + t)(\mu + T)} + \frac{k}{\mu + T} = \zeta_t$$

and the first assertion is proved.
2. Since $\zeta_{\bullet}$ is the martingale,

$$\mathbb{E}(\zeta_{t+\Delta} \,|\, \mathcal{F}_t^{\zeta}) = b(t + \Delta)\Big(\mathbb{E}(N_{t+\Delta} \,|\mathcal{F}_t^N) + a(t + \Delta)\Big) = \zeta_t = b(t)(N_t + a(t)) \tag{2.5}$$

for any $t, \Delta \geq 0$. In view of Markov property of the process $N_{\bullet}$, we obtain from (2.5) that

$$\mathbb{E}(N_{t+\Delta} \,|N_t = l) = l\, \frac{b(t)}{b(t + \Delta)} + \frac{b(t)a(t)}{b(t + \Delta)} - a(t + \Delta) \tag{2.6}$$

for any $t > 0$ and integer $l \geq 0$. Denote

$$f_1(t, \Delta) = \frac{b(t) - b(t + \Delta)}{b(t + \Delta)} \quad \text{and} \quad f_2(t, \Delta) = \frac{b(t)a(t)}{b(t + \Delta)} - a(t + \Delta).$$

It follows from (2.1) and (2.6) that functions $f_1(t, \Delta)$ and $f_2(t, \Delta)$ are both proportional to Δ: $f_1(t, \Delta) = c_1(t)\Delta$ and $f_2(t, \Delta) = c_2(t)\Delta$ with $c_2(t) > 0$ and $c_1(t) \geq 0$. Moreover, if $c_1(t_0) = 0$ for some $t_0 > 0$, then $c_1(t) = 0$ for any $t > 0$. As the result, we come to the equality

$$\frac{\int_{(0,\infty)} \lambda^{l+1} e^{-\lambda t} \mathcal{P}(d\lambda)}{\int_{(0,\infty)} \lambda^l e^{-\lambda t} \mathcal{P}(d\lambda)} = c_1(t)l + c_2(t), \quad t > 0, \quad l \geq 0. \tag{2.7}$$

Let us fix $t_0 > 0$ and denote $c_1 = c_1(t_0)$ and $c_2 = c_2(t_0)$ for brevity. Additionally, denote

$$\gamma = \int\limits_{(0,\infty)} e^{-\lambda t_0} \mathcal{P}(d\lambda)$$

and $\mathcal{Q}(d\lambda) = e^{-\lambda t_0}\mathcal{P}(d\lambda)/\gamma$. Evidently, Q is the probability distribution. Now the equality (2.7) obtains the form

$$\frac{\int_{(0,\infty)} \lambda^{l+1}\mathcal{Q}(d\lambda)}{\int_{(0,\infty)} \lambda^{l}\mathcal{Q}(d\lambda)} = c_1 l + c_2. \tag{2.8}$$

Let $c_1 = 0$. Then $\int_{(0,\infty)} \lambda^l \mathcal{Q}(d\lambda) = c_2^l$ for any integer $l \geq 0$ and $\mathcal{Q} = \delta_{c_2}$. Therefore $\mathcal{Q} = \delta_\alpha$ with $\alpha = c_2$ and $N_{\boldsymbol{.}}$ is the Poisson process with the parameter α.

Suppose that $c_1 > 0$. If $\mathcal{R}$ stands for the gamma-distribution $\mathrm{G}(k', \mu')$, then

$$\frac{\int_{(0,\infty)} \lambda^{l+1}\mathcal{R}(d\lambda)}{\int_{(0,\infty)} \lambda^{l}\mathcal{R}(d\lambda)} = (l + k')/\mu'$$

in view of (2.4). The latter equality is of the same form as (2.8). Since any gamma-distribution is determined by its moments, then $\mathcal{Q} = \mathrm{G}(k_1, \mu_1)$ with $k_1 = c_2/c_1$ and $\mu_1 = 1/c_1$. Therefore, the distribution $\mathcal{P}$ has the density

$$p(\lambda) = \gamma \frac{\mu_1^{k_1}}{\Gamma(k_1)} \lambda^{k_1 - 1} e^{-(\mu_1 - t_0)\lambda},$$

where $\lambda > 0$ and $\mu_1 > t_0$. In other words, $\mathcal{P} = \mathrm{G}(k, \mu)$ with $k = k_1$ and $\mu = \mu_1 - t_0$. Since these parameters do not depend on t_0, we get the explicit expressions for functions $c_1(\,\cdot\,)$, $c_2(\,\cdot\,)$: $c_1(t) = 1/(\mu + t)$ and $c_2(t) = k/(\mu + t)$. □

3 Characterization of Pólya sequences

The so-called "NBD-Dirichlet" model of consumer behavior (see [3] for the detailed discussion) uses a Pólya process N_t to describe the number of purchase occasions up to time t. The second part of the model deals with a so-called Pólya sequence of d-dimensional random vectors. This sequence explains how a buyer makes a choice of one among d interchangeable goods. To be short, we restrict ourselves to the case $d = 2$.

Consider the sequence $\varepsilon_{\boldsymbol{.}} = \{\varepsilon_j,\, j \geq 1\}$ of random variables such that $\varepsilon_j \in \{0, 1\}$. Assume that the sequence $\varepsilon_{\boldsymbol{.}}$ is permutable. This means that for any n all permutations of random variables $\varepsilon_1, \ldots, \varepsilon_n$ have identical n-dimensional distributions.

Then (e.g., [1, ch. 7]) there exists a *structure distribution* $\mathcal{P}$ such that $\mathcal{P}([0, 1]) = 1$ and

$$\mathbb{P}(\varepsilon_1 = \delta_1, \ldots, \varepsilon_n = \delta_n) = \int\limits_{[0,1]} z^m (1 - z)^{n-m} \mathcal{P}(dz) \quad \text{with} \quad m = \delta_1 + \ldots + \delta_n \tag{3.1}$$

for any $n \geq 1$ and any $\delta_i \in \{0, 1\}$.

If $\mathcal{P}$ is a beta-distribution with some parameters (ν_1, ν_2), then the sequence $\varepsilon_{\boldsymbol{.}}$ can be described in terms of Pólya urn models. Therefore it is natural to call $\varepsilon_{\boldsymbol{.}}$ as the two-dimensional *Pólya sequence* with parameters (ν_1, ν_2). (Various results on Pólya urn models can be found in [2].)

Our aim is to give a martingale characterization of Pólya sequences among all permutable random sequences $\varepsilon_{\boldsymbol{.}} = \{\varepsilon_j,\, j \geq 1\}$. More precisely, we consider random variables $S_n = \varepsilon_1 + \ldots + \varepsilon_n$ and their non-degenerate linear transformations. Evidently, it is sufficient to restrict ourselves to the case $\mathcal{P}((0, 1)) = 1$.

Let us start with the proposition analogous to Lemma 2.1.

Lemma 3.1. *Suppose that the distribution of the sequence* $\varepsilon_{\bullet} = \{\varepsilon_j,\, j \ge 1\}$ *is defined by* (3.1) *with* $\mathcal{P}((0,1)) = 1$. *Denote* $S_n = \varepsilon_1 + \ldots + \varepsilon_n$. *Then*

$$\mathbb{E}(S_N \,|\, S_1, \ldots, S_n) = S_n + (N-n)\, \frac{\int_{(0,1)} z^{S_n+1}(1-z)^{n-S_n}\mathcal{P}(dz)}{\int_{(0,1)} z^{S_n}(1-z)^{n-S_n}\mathcal{P}(dz)} \tag{3.2}$$

for any $N > n \ge 0$.

Proof. Let $\delta_j \in \{0,1\}$ and $m_j = \delta_1 + \ldots + \delta_j$. It follows from (3.1) that

$$\mathbb{P}(S_{n+1} = m_{n+1} \,|\, S_1 = m_1, \ldots, S_n = m_n) = \frac{\int_{(0,1)} z^{m_{n+1}}(1-z)^{n+1-m_{n+1}}\mathcal{P}(dz)}{\int_{(0,1)} z^{m_n}(1-z)^{n-m_n}\mathcal{P}(dz)}\,.$$

Therefore, the sequence $\{S_n,\, n \ge 0\}$ is the Markov chain with the transition function

$$\mathbb{P}(S_{n+1} = m+\delta \,|\, S_n = m) = \frac{\int_{(0,1)} z^{m+\delta}(1-z)^{n+1-m-\delta}\mathcal{P}(dz)}{\int_{(0,1)} z^{m}(1-z)^{n-m}\mathcal{P}(dz)}\,,$$

where $\delta \in \{0,1\}$ and $m \in \{0, \ldots, n\}$. As the result, $\mathbb{E}(S_N \,|\, S_1, \ldots, S_n) = \mathbb{E}(S_N \,|\, S_n)$. Analogously,

$$\mathbb{P}(S_N = l+m \,|\, S_n = l) = \frac{\mathrm{C}_{N-n}^{m} \int_{(0,1)} z^{l+m}(1-z)^{N-l-m}\mathcal{P}(dz)}{\int_{(0,1)} z^{l}(1-z)^{n-l}\mathcal{P}(dz)} \tag{3.3}$$

for $N > n \ge 0$, $l \in \{0, \ldots, n\}$ and $m \in \{0, \ldots, N-n\}$, Using (3.3), we obtain that

$$\mathbb{E}(S_N \,|\, S_n = l) = l + (N-n)\, \frac{\int_{(0,1)} z^{l+1}(1-z)^{n-l}\mathcal{P}(dz)}{\int_{(0,1)} z^{l}(1-z)^{n-l}\mathcal{P}(dz)} \tag{3.4}$$

and the proof is complete. □

Proposition 3.1. *1. Let* $\varepsilon_{\bullet} = \{\varepsilon_j,\, j \ge 1\}$ *be a* Pólya sequence *with parameters* (ν_1, ν_2). *Denote* $\zeta_n = (S_n + \nu_1)/(n + \nu_1 + \nu_2)$. *Then* $\{\zeta_n,\, n \ge 1\}$ *is the martingale relative to the natural filtration* $\mathcal{F}_n^{\zeta} = \sigma(\zeta_1, \ldots, \zeta_n)$.
2. Let the conditions of Lemma 3.1 be fulfilled. Suppose that there exist sequences $b_n \ne 0$ *and* a_n *such that the random sequence* $\zeta_n = b_n(S_n + a_n)$ *is a martingale relative to the natural filtration* $\mathcal{F}_n^{\zeta} = \sigma(\zeta_1, \ldots, \zeta_n)$. *Then either* $\mathcal{P} = \delta_\alpha$ *for some* $\alpha \in (0,1)$ *or* $\mathcal{P}$ *is a beta-distribution with a certain parameters* (ν_1, ν_2).

Proof. 1. Evidently, $\mathcal{F}_n^{\zeta} = \mathcal{F}_n^{S} \overset{\text{def}}{=} \sigma(S_1, \ldots, S_n)$. Besides, if $\mathcal{P}$ is the beta-distribution with parameters (ν_1, ν_2), then

$$\int\limits_{(0,1)} z^k(1-z)^m\mathcal{P}(dz) = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\, \frac{\Gamma(\nu_1+k)\Gamma(\nu_1+m)}{\Gamma(\nu_1+\nu_2+k+m)} \tag{3.5}$$

for $k > -\nu_1$ and $m > -\nu_2$. Using (3.5) and (3.2), we see that

$$\mathbb{E}(S_{n+1} \,|\, \mathcal{F}_n^S) = S_n + \frac{S_n + \nu_1}{\nu_1 + \nu_2 + n}\,.$$

Therefore,

$$\mathbb{E}(\zeta_{n+1} \,|\, \mathcal{F}_n^{\zeta}) = \frac{\mathbb{E}(S_{n+1} \,|\, \mathcal{F}_n^S) + \nu_1}{n+1+\nu_1+\nu_2} = (S_n + \nu_1)\frac{1 + 1/(\nu_1+\nu_2+n)}{n+1+\nu_1+\nu_2} = \zeta_n.$$

2. Without loss of generality we can take $b_0 = 1$. Let $N \ge n \ge 0$ and denote $\Delta = N - n$. Since $\{\zeta_n,\, n \ge 0\}$ is a martingale, then

$$\mathbb{E}(S_{n+\Delta}\,|S_n = l) = \frac{b_n(l + a_n)}{b_{n+\Delta}} - a_{n+\Delta} \tag{3.6}$$

for any $\Delta, n \ge 0$ and $0 \le l \le n$. Denote

$$A(n,l) = \frac{\int_{(0,1)} z^{l+1}(1-z)^{n-l}\mathcal{P}(dz)}{\int_{(0,1)} z^{l}(1-z)^{n-l}\mathcal{P}(dz)}\,.$$

Then (see Lemma 3.1) the equality (3.6) can be rewritten in the form

$$\Delta A(n,l) = \frac{b_n - b_{n+\Delta}}{b_{n+\Delta}}\, l + \frac{b_n - b_{n+\Delta}}{b_{n+\Delta}}\, a_n + a_n - a_{n+\Delta}. \tag{3.7}$$

In other terms,

$$\Delta A(n,l) = \frac{b_n - b_{n+\Delta}}{b_{n+\Delta}}\, l + \Delta A(n,0).$$

This means that $(b_n - b_{n+\Delta})/b_{n+\Delta}$ is proportional to Δ: $(b_n - b_{n+\Delta})/b_{n+\Delta} = c_1(n)\Delta$. In other words, $b_{n+\Delta} = b_n/(1 + c_1(n)\Delta)$ and $b_n = 1/(1 + c_1\, n)$ with $c_1 = c_1(0)$. Besides,

$$(b_n - b_{n+\Delta})/(b_{n+\Delta}) = c_1\Delta/(1 + c_1 n). \tag{3.8}$$

Analogously, $a_{n+\Delta} - a_n$ is proportional to Δ: $a_{n+\Delta} - a_n = c_2(n)\Delta$. Therefore

$$a_n = a_0 + c_2 n \quad \textit{and} \quad a_{n+\Delta} - a_n = c_2\Delta \tag{3.9}$$

with $c_2 = c_2(0)$. Applying (3.7), (3.8) and (3.9), we obtain that

$$\frac{\int_{(0,1)} z^{l+1}(1-z)^{n-l}\mathcal{P}(dz)}{\int_{(0,1)} z^{l}(1-z)^{n-l}\mathcal{P}(dz)} = \frac{c_1 l}{1 + c_1 n} + \frac{c_1(a_0 + c_2 n)}{1 + c_1 n} - c_2 = \frac{c_1 l + c_1 a_0 - c_2}{1 + c_1 n} \tag{3.10}$$

and

$$\frac{\int_{(0,1)} z^{l+1}\mathcal{P}(dz)}{\int_{(0,1)} z^{l}\mathcal{P}(dz)} = \frac{c_1 a_0 - c_2 + c_1 l}{1 + c_1 l}\,. \tag{3.11}$$

Suppose now that $c_1 = 0$. Then the right side of (3.11) is a constant $\alpha = -c_2 \in (0,1)$ and $\mathcal{P} = \delta_\alpha$. Consider the case $c_1 \ne 0$.

If we take $l = 0$ and $l = 1$ in (3.11), then under denotation $x = c_1 a_0 - c_2$

$$0 < \int\limits_{(0,1)} z\mathcal{P}(dz) = x < 1 \quad \text{and} \quad \int\limits_{(0,1)} z^2\mathcal{P}(dz) = x\,\frac{x + c_1}{1 + c_1}\,.$$

Since $\mathcal{P} \ne \delta_\alpha$, then $x(x + c_1)/(1 + c_1) > x^2$ and either $c_1 > 0$ or $c_1 < -1$. On the other hand, if we choose $l = 0$ in (3.10), we see that the supposition $c_1 < -1$ leads to the contradiction: in this case the left side of (3.10) is negative. Therefore $c_1 > 0$.

If $\mathcal{R}$ stands for the beta-distribution with parameters (ν_1, ν_2), then

$$\frac{\int_{(0,1)} z^{l+1}\mathcal{R}(dz)}{\int_{(0,1)} z^{l}\mathcal{R}(dz)} = \frac{\nu_1 + l}{\nu_1 + \nu_2 + l}\,. \tag{3.12}$$

in view of (3.5).

Let us denote $\nu_1 = (c_1 a_0 - c_2)/c_1$ and $\nu_2 = 1/c_1 - \nu_1$. Then (3.11) has the same form as (3.12). Since any beta-distribution is determined by its moments, we obtain the result. $\square$

References

[1] Feller, W (1971). An Introduction to Probability Theory and Its Applications, 2-nd ed., V.2, Wiley, New York e.a.

[2] Johnson N.L., Kotz S. (1977) Urn Models and their Application. Wiley, New York e.a.

[3] Ehrenberg A.S.C. (1988) Repeat-buying: Facts, Theory and Applications. 2-nd ed. London: Charles Griffin & Company LTD; New York: Oxford University Press.

[4] Grandell J. (1997) Mixed Poisson Processes. Chapman & Hall, London e.a.

[5] Cook, R. J., Wei, W. (2003) Conditional Analysis of Mixed Poisson Processes with Baseline Counts: Implications for Trial Design and Analysis. Biostatistics. 2003. Vol. 4, P. 479 - 494.