Decompositions of Trigonometric Polynomials Related to Multivariate Subdivision Schemes

Nira Dyn and M. Skopina [1]

1.Introduction

In this paper we study decompositions of multivariate trigonometric polynomials, satisfying a set of constraints known as Strung-Fix conditions of order m, multiplied by a factor of the form $1 - exp(q(x))$, with $q(x)$ a multilinear polynomial with integer coefficients. The decomposition is a representation of such a polynomial as a sum of trigonometric polynomials, satisfying the Strung-Fix conditions of order $m-1$, multiplied by factors of the above form. This type of decompositions is of interest in the analysis of convergence and smoothness of multivariate subdivision schemes. The algebraic approach to the analysis of such schemes is investigated in [1] relative to the expansion matrix $2I$, and in [6], [7], relative to general expansion matrices. The analogous decompositions for multivariate algebraic polynomials are based on the rich structure of ideals of multivariate polynomials. The trigonometric decompositions are based on the rather simple idea of the representation of a trigonometric polynomial in terms of its unique polyphase trigonometric polynomials relative to an expansion matrix[4]. The outline of the papers is as follows: in Section 2 we introduce our notation, and bring basic results related to polyphase representations of trigonometric polynomials. In Section 3 properties of trigonometric polynomials satisfying the Strang-Fix conditions of order $m = 0$ relative to a general expansion matrix are derived, in terms of its polyphase trigonometric polynomials. These results are used in Section 4 to prove the decomposition for the case $m = 0$. An algorithm for the computation of the decomposition for $m = 0$ is presented, and the decomposition is used to derive that for the case $m = 1$. An example of this case is also presented, with all the computational details. The general case $m > 1$, which is much more complicated is investigated in Section 5, and an algorithm for this case is given. In the last section it is proved that trigonometric polynomials satisfying the Strang-Fix conditions of order m are determined by the value at the origin of the derivatives of their polyphase trigonometric polynomials. This result provides a tool for the construction of such polynomials. The

[1]The paper is supported by RFBR, project N 06-01-00457

application to the smoothness analysis of multivariate subdivision schemes will be done elsewhere.

2. Notation and preliminary information

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$ are its elements (vectors), $(x, y) = x_1y_1 + \dots + x_dy_d$, $|x| = \sqrt{(x, x)}$, $\mathbf{e}_j = (0, \dots, 1, \dots, 0)$ is the j-th unit vector in $\mathbb{R}^d$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$; $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$. For $x, y \in \mathbb{R}^d$, we write $x > y$ if $x_j > y_j$, $j = 1, \dots, d$; $\mathbb{Z}^d_+ = \{x \in Z^d : \ x \geq \mathbf{0}\}$. If $\alpha, \beta \in \mathbb{Z}^d_+$, $a, b \in \mathbb{R}^d$, we set $\alpha! = \prod\limits_{j=1}^{d} \alpha_j!$, $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$, $a^b = \prod\limits_{j=1}^{d} {a_j}^{b_j}$, $[\alpha] = \sum\limits_{j=1}^{d} \alpha_j$, $D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_d} x_d}$; δ_{ab} denotes Kronecker delta; $z := (z_1, \dots, z_d)$, where $z_k := e^{2\pi i x_k}$.

Let M be a non-degenerate $d \times d$ integer matrix whose eigenvalues are bigger than 1 in module, M^* is the conjugate matrix to M, I_d denotes the unit $d \times d$ matrix. We say that numbers $k, n \in \mathbb{Z}^d$ are congruent modulo M (write $k \equiv n \pmod{M}$) if $k - n = M\ell$, $\ell \in \mathbb{Z}^d$. The integer lattice $\mathbb{Z}^d$ is splitted into cosets with respect to the introduced relation of congruence. The number of cosets is equal to $|\det M|$ (see, e.g., [2, p. 107]). Let us take an arbitrary representative from each coset, call them digits and denote the set of digits by $D(M)$. Throughout the paper we consider that such a matrix M is fixed, $m = |\det M|$, $D(M) = \{s_0, \dots, s_{m-1}\}$, $D(M^*) = \{s^*_0, \dots, s^*_{m-1}\}$, $s_0 = s^*_0 = \mathbf{0}$, $r_k = M^{-1}s_k$, $k = 0, \dots, m-1$.

Proposition A *The matrix* $\left\{\frac{1}{\sqrt{m}}\, e^{2\pi i (r_k, s^*_l)}\right\}_{k,l=0}^{m-1}$ *is unitary. In particular,*

$$\frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (r_k, s^*_l)} = \delta_{0l}. \tag{1}$$

A proof of this statement can be found, e.g., in [3].

We will consider 1-periodic trigonometric polynomials in d variables

$$t(x) = \sum_{n \in \mathbb{Z}^d} \widehat{t}(n) e^{2\pi i (n, x)}.$$

For any trigonometric polynomial t, there exists a unique set of trigonometric polynomials τ_ν, $\nu = 0, \dots, m-1$, (polyphase functions of t) so that

$$t(x) = \sum_{\nu=0}^{m-1} e^{2\pi i (s_\nu, x)} \tau_\nu(M^* x). \tag{2}$$

It is clear that

$$\tau_\nu(x) = \sum_{n \in \mathbb{Z}^d} \widehat{t}(Mn + s_\nu) e^{2\pi i (n,x)}.$$

For any $n \in \mathbb{Z}_+$, denote by SF^n the set of trigonometric polynomials t such that $D^\beta t(M^{*-1}x)|_{x=s} = 0$ for all $s \in D(M^*)$, $s \neq \mathbf{0}$, and for all $\beta \in \mathbb{Z}_+^d$, $[\beta] \leq n$. Such a notation is reasonable because the refinable function corresponding to a polynomial $t \in SF^n$ satisfies Strang-Fix condition of order n. It will be convenient for us to define SF^{-1} as the set of all trigonometric polynomials.

3. Auxiliary results

Proposition 1 *A trigonometric polynomial t belongs to SF^0 if and only if*

$$\tau_\nu(\mathbf{0}) = \frac{t(\mathbf{0})}{m}, \quad \nu = 0, \dots, m-1, \tag{3}$$

where $\tau_0, \dots, \tau_{m-1}$ are the polyphase functions of t.

Proof. Let $s \in D(M^*)$, using (2) we have

$$t(M^{*-1}s) = \sum_{\nu=0}^{m-1} e^{2\pi i (r_\nu, s)} \tau_\nu(s) = \sum_{\nu=0}^{m-1} e^{2\pi i (r_\nu, s)} \tau_\nu(\mathbf{0}).$$

So, the relation $t \in SF^0$ can be rewritten as

$$\sum_{\nu=0}^{m-1} e^{2\pi i (r_\nu, s_l^*)} \tau_\nu(\mathbf{0}) = t(\mathbf{0}) \delta_{\mathbf{0}l}, \quad l = 0, \dots, m-1.$$

Consider these equalities as a linear system with unknowns $\tau_0(\mathbf{0}), \dots, \tau_{m-1}(\mathbf{0})$. Due to Proposition A, the system has a unique solution given by (3). $\Diamond$

Lemma 2 *Let t be a trigonometric polynomial, $t(\mathbf{0}) = 0$, then*

$$t(x) = \sum_{k=1}^{d} t_k(x)(1 - e^{2\pi i(x,\mathbf{e}_k)}),$$

where t_k, $k = 1, \dots, d$, are trigonometric polynomials.

Proof. There exists $N \in \mathbb{Z}^d$ such that $t(x)e^{2\pi i(N,x)} = p(z)$, where p is an algebraic polynomial. It is clear that $P(1, \dots, 1) = 0$. By Taylor formula it follows that

$$p(z) = \sum_{k=1}^{d} p_k(z)(1 - z_k).$$

where p_k, $k = 1, \dots, d$, are algebraic polynomials. It remains to set $t_k(x) := p_k(z)e^{-2\pi i(N,x)}$.$\Diamond$

Lemma 3 *Let $t, \tilde{t} \in SF^0$, $t(\mathbf{0}) = \tilde{t}(\mathbf{0})$, then*

$$t(x) - \tilde{t}(x) = \sum_{k=1}^{d} t_k(x)\left(1 - e^{2\pi i(M^*x,\mathbf{e}_k)}\right),$$

where t_k, $k = 1, \dots, d$, are trigonometric polynomials.

Proof. Using (2) for $t, \tilde{t}$, we have

$$t(x) - \tilde{t}(x) = \sum_{\nu=0}^{m-1} e^{2\pi i(s_\nu,x)}(\tau_\nu(M^*x) - \tilde{\tau}_\nu(M^*x)). \tag{4}$$

Due to Proposition 1, $\tau_\nu(\mathbf{0}) - \tilde{\tau}_\nu(\mathbf{0}) = 0$, $\nu = 1, \dots, m-1$. Hence, by Lemma 2,

$$\tau_\nu(y) - \tilde{\tau}_\nu(y) = \sum_{l=1}^{d} \tau_{\nu k}(y)\left(1 - e^{2\pi i(y,\mathbf{e}_k)}\right), \tag{5}$$

where $\tau_{\nu k}$, $k = 1, \dots, d$, are trigonometric polynomials. It remains to set $y = M^*x$ and combine (5) with (4). $\Diamond$

Lemma 4 *Let $t \in SF^0$, $r \in \mathbb{Z}^d$, $\tilde{t} := e^{2\pi i(r,\cdot)}t$, then $\tilde{t} \in SF^0$.*

Proof. By (2),

$$\tilde{t}(x) = e^{2\pi i(r,x)} \sum_{\nu=0}^{m-1} e^{2\pi i(s_\nu,x)} \tau_\nu(M^*x) = \sum_{\nu=0}^{m-1} e^{2\pi i(s_\nu+r,x)} \tau_\nu(M^*x).$$

It is clear that $s_\nu + r = s_{l_\nu} + Mn_\nu$, $n_\nu \in \mathbb{Z}^d$, $\{l_0, \dots, l_{m-1}\} = \{0, \dots, m-1\}$. So,

$$\tilde{\tau}_{l_\nu}(x) = \sum_{\nu=0}^{m-1} e^{2\pi i(s_{l_\nu},x)} e^{2\pi i(n_\nu,M^*x)} \tau_\nu(M^*x),$$

and the trigonometric polynomial $\tilde{\tau}_{l_\nu}(x) := e^{2\pi i(n_\nu,x)}\tau_\nu(x)$ is the l_ν-th polyphase function of $\tilde{t}$. If $s \in D(M^*)$, $s \neq \mathbf{0}$, due to Proposition 1,

$$\tilde{\tau}_{l_\nu}(s) = e^{2\pi i(n_\nu,s)}\tau_\nu(s) = \tau_\nu(s) = \tau_\nu(\mathbf{0}) = \frac{t(\mathbf{0})}{m} = \frac{\tilde{t}(\mathbf{0})}{m}.$$

Again by Proposition 1, it follow that $\tilde{t} \in SF^0$. $\Diamond$

Lemma 5 *Let $n \in \mathbb{Z}^d$, $1 - e^{2\pi i(M^{-1}n,\mathbf{e}_j)} = 0$ for each $j = 1, \dots, d$. Then $n \equiv \mathbf{0} \pmod{M}$.*

Proof. Since $e^{2\pi i n} = 1$ if and only if $n \in \mathbb{Z}$, we have $(M^{-1}n)_j = (M^{-1}n, \mathbf{e}_j) \in \mathbb{Z}$ for each $j = 1, \dots, d$. Hence, $M^{-1}n = l \in \mathbb{Z}^d$, i.e. $n = Ml$. $\Diamond$

4. Decomposition of masks

Proposition 6 *Let $t \in SF^0$, $k = 1, \dots, d$, then*

$$(1 - e^{2\pi i(x,\mathbf{e}_k)})t(x) = \sum_{j=1}^{d} t_{jk}(x)(1 - e^{2\pi i(M^*x,\mathbf{e}_j)}), \tag{6}$$

where t_{jk}, $j = 1, \dots, d$, are trigonometric polynomials. Inversely, if (6) holds for a trigonometric polynomial t, then $t \in SF^0$.

Proof. If $t \in SF^0$, then (6) follows immediately form Lemmas 3 and 4. Now let t be an arbitrary trigonometric polynomial such that (6) holds, $s \in D(M^*)$, $s \neq \mathbf{0}$. It follows from (6) that

$$t(M^{*-1}s)\left(1 - e^{2\pi i(M^{*-1}s,\mathbf{e}_k))}\right) = \sum_{j=1}^{d} t_{jk}(M^{*-1}s)(1 - e^{2\pi i(s,\mathbf{e}_j)}) = 0.$$

Taking into account Lemma 5, we obtain $t(M^{*-1}s)$, which was to be proved. $\Diamond$

Analyzing the proofs of Lemmas 3 and 4 it is not difficult to describe an algorithm for finding polynomials t_{jk}, $j,k=1,\dots,d$, in decomposition (6). We will consider that the polyphase functions τ_ν of t are given, and the algorithm derives the polyphase functions $\tau_{jk\nu}$ of each polynomial t_{jk}.

ALGORITHM 1

Input: $\{\widehat{\tau}_\nu(l),\ l\in\mathbb{Z}^d,\ \nu=0,\dots,m-1\}$.

Output: $\{\widehat{\tau}_{jk\nu}(l),\ l\in\mathbb{Z}^d,\ \nu=0,\dots,m-1,\ j,k=1,\dots,d\}$.

Step 0. For each $\nu=0,\dots,m-1$ find the sets

$$\Omega_\nu := \{l\in\mathbb{Z}^d:\ \widehat{\tau}_\nu(l)\neq 0\}.$$

For each $k=1,\dots,d$ **do**

For each $\nu=0,\dots,m-1$ **do**

Step 1. Compute $M^{-1}(\mathbf{e}_k-s_\nu+s_n)=:l_n$ for all $n=0,\dots,m-1$ and denote by n^* the unique n such that $l_n\in\mathbb{Z}^d$.

Step 2. Find the set $\tilde{\Omega}_{k\nu}:=\{l=r+l_{n^*}, r\in\Omega_{n^*}\}$, and for each $l\in\tilde{\Omega}_{k\nu}$ put $\widehat{\tilde{\tau}}_\nu(l):=\widehat{\tau}_{n^*}(l-l_{n^*})$,

$$p_{k\nu}(z):=\sum_{l\in\Omega_\nu\cup\tilde{\Omega}_{k\nu}}(\widehat{\tau}_\nu(l)-\widehat{\tilde{\tau}}_\nu(l))z^l.$$

Step 3. Put

$$\tau_{1k\nu}(x)=\frac{1}{1-z_1}\left(p_{k\nu}(z)-p_{k\nu}(1,z_2,\dots,z_d)\right),$$
$$\tau_{2k\nu}(x)=\frac{1}{1-z_2}\left(p_{k\nu}(1,z_2,z_3,\dots,z_d)-p_{k\nu}(1,1,z_3,\dots,z_d)\right),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\tau_{dk\nu}(x)=\frac{1}{1-z_d}\left(p_{k\nu}(1,1,\dots,1,z_d)-p_{k\nu}(1,1,\dots,1,1)\right).\Diamond$$

Theorem 7 *If $t\in SF^1$, then in any decomposition (6) the trigonometric polynomials t_{jk}, $j,k=1,\dots,d$, belong to SF^0.*

Proof. Let $l = 1, \ldots, d$, $s \in D(M^*)$, $s \neq \mathbf{0}$. Since $t \in SF^{(1)}$,

$$\frac{\partial}{\partial x_l}\left(\left(1 - e^{2\pi i(M^{*-1}x, \mathbf{e}_k))}\right) t(M^{*-1}x)\right)\Big|_{x=s} = 0, \quad k = 1, \ldots, d. \tag{7}$$

On the other hand, it follows from (6) that for each $k = 1, \ldots, d$ we have

$$\frac{\partial}{\partial x_l}\left(\left(1 - e^{2\pi i(M^{*-1}x, \mathbf{e}_k))}\right) t(M^{*-1}x)\right)\Big|_{x=s} =$$

$$\frac{\partial}{\partial x_l}\left(\sum_{j=1}^{d} t_{jk}(M^{*-1}x)(1 - e^{2\pi i x_j})\right)\Big|_{x=s} =$$

$$\sum_{j=1}^{d} t_{jk}(M^{*-1}s)\frac{\partial}{\partial x_l}(1 - e^{2\pi i x_j})\Big|_{x=s} = -2\pi i\, t_{lk}(M^{*-1}s).$$

So, we proved that $t_{lk}(M^{*-1}s) = 0$ for all $s \in D(M^*)$, $s \neq \mathbf{0}$, which means $t_{lk} \in SF^{(0)}$. $\Diamond$

Example. Let $M = \begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}$,

$$s_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, s_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, s_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, s_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Define the polyphase functions of a polynomial t by

$$\begin{aligned}
\tau_0(x) &= \frac{1}{16}(4 + 4z_1 + 4z_2 + 4z_1z_2),\\
\tau_1(x) &= \frac{1}{16}(5 + 4z_1 + z_1^{-1} + 2z_2 + 3z_2^{-1} + z_1z_2),\\
\tau_2(x) &= \frac{1}{16}(4 + z_1 + 2z_1^{-1} + 5z_2 + z_2^{-1} + 3z_1z_2),\\
\tau_3(x) &= \frac{1}{16}(5 + z_1 + 4z_1^{-1} + z_2 + 3z_2^{-1} + z_1z_2 + z_1^{-1}z_2).
\end{aligned}$$

It is not difficult to see that $t \in SF^0$. Using Algorithm 1, we find

$$\begin{aligned}
&\tau_{110}(x) = \tfrac{1}{16}(-1 + 2z_2 + z_1z_2 + 2z_2^2 + z_1z_2^2), && \tau_{210}(x) = \tfrac{1}{16}(5 + 3z_2),\\
&\tau_{111}(x) = \tfrac{1}{16}(z_1^{-1} + 3z_2), && \tau_{211}(x) = \tfrac{1}{16}(5 + 3z_2^{-1}),\\
&\tau_{112}(x) = \tfrac{1}{16}(-1 + 2z_1^{-1} + z_2 + z_2^2 - z_1^{-1}z_2), && \tau_{212}(x) = \tfrac{1}{16}(5 + 3z_2 + z_2^{-1}),\\
&\tau_{113}(x) = \tfrac{1}{16}(2z_1^{-1} + 2z_2 + z_1^{-1}z_2), && \tau_{213}(x) = \tfrac{1}{16}(5 + 2z_2^{-1}),\\
&\tau_{120}(x) = \tfrac{1}{16}(1 + z_1 + 4z_2 + z_2^{-1} + 3z_1z_2), && \tau_{220}(x) = \tfrac{1}{16}z_2^{-1},\\
&\tau_{121}(x) = \tfrac{1}{16}(2 + z_1 + z_1^{-1} + z_2 + 3z_2^{-1} + z_1z_2), && \tau_{221}(x) = 0,\\
&\tau_{122}(x) = \tfrac{1}{16}(3 + z_2 + 2z_1^{-1}), && \tau_{222}(x) = \tfrac{1}{16}z_2^{-1},\\
&\tau_{123}(x) = \tfrac{1}{16}(3 + 3z_1^{-1} + z_1^{-1}z_2), && \tau_{223}(x) = 0.
\end{aligned}$$

Due to Theorem 2 of [5], to prove that the subdivision scheme S_t is convergent, it sufficient to check that the norm the matrix subdivision operator S_T from ℓ^2_∞ to ℓ^2_∞ is strictly less than 1, and that M has a self-similar tile, i.e. there exists a set $E \subset \mathbb{R}^2$ such that $ME = \bigcup\limits_{s\in D(M)}(E+s)$, $\bigcup\limits_{l\in\mathbb{Z}^2}(E+l) = \mathbb{R}^2$, $\mathrm{mes}E = 1$. The first requirement is fulfilled because

$$\|S_T\|_{\ell^2_\infty\to\ell^2_\infty} \le \max_{(\nu,k)}(\|\tau_{1k\nu}\|_{\ell_1} + \|\tau_{2k\nu}\|_{\ell_1}) \le \frac{15}{16}$$

(here we identify a trigonometric polynomial with its sequence of Fourier coefficients). Set

$$E = \left\{ x \in \mathbb{R}^2 : \quad x = \sum_{k=1}^{\infty} M^{-k}x_k,\ x_k \in D(M) \right\}.$$

To show that E provides a self-similar tile it suffices to check (see, e.g., [2]) that the function

$$m_0(x) = \frac{1}{4}(1 + e^{2\pi i x_1} + e^{2\pi i x_2} + e^{2\pi i(x_1+x_2)})$$

is such that $m_0(M^{*-k}x)$ does not vanish on $[-1/2, 1/2]^2$ for all $k \in \mathbb{N}$. Since

$$|m_0(M^{*-1}x)|^2 = \left(1 + \cos 2\pi\left(\frac{x_1}{4} + \frac{x_2}{2}\right)\right)\left(1 + \cos 2\pi\frac{x_1}{2}\right),$$

we see that all zeros of this function are on the lines $\frac{x_1}{4}+\frac{x_2}{2}=\pm\frac{1}{2}+l$, $\frac{x_1}{2}=\pm\frac{1}{2}+l$, $l\in\mathbb{Z}$, and it is clear that the cube $[-1/2,1/2]^2$ does not intersect with these lines. It remains to note that $\|M^{*-1}x\|_\infty\le\frac{3}{4}\|x\|_\infty$, and so $M^{*-1}[-1/2,1/2]^2\subset[-1/2,1/2]^2$.

Proposition 6 and Theorem 7 state that for $n=0,1$ condition $t\in SF^n$ implies $t_{jk}\in SF^{n-1}$. This fact can not be extended to the case $n>1$. If $t\in SF^2$, there exist decompositions (6) whose elements are not in SF^1. Nevertheless, it will be shown that decomposition of $t\in SF^n$ can be fixed up to provide $t_{jk}\in SF^{n-1}$.

Theorem 8 *Let $n\in\mathbb{Z}_+$. A trigonometric polynomial t belongs to SF^n if and only if there exists a decomposition (6) with $t_{jk}\in SF^{n-1}$, $j,k=1,\dots,d$.*

Proof. Set

$$\begin{aligned} a(x)&=t(M^{*-1}x), & a_{jk}(x)&=t_{jk}(M^{*-1}x),\\ b_k(x)&=1-e^{2\pi i(M^{*-1}x,\mathbf{e}_k)}, & c_k(x)&=1-e^{2\pi i(x,\mathbf{e}_k)}. \end{aligned}$$

In these notation, (6) can be rewritten as

$$b_k(x)a(x)=\sum_{j=1}^{d}a_{jk}(x)c_j(x). \tag{8}$$

Note the following trivial properties of c_j:

$$c_j(l)=0\ \forall l\in\mathbb{Z}^d, \tag{9}$$
$$D^\delta c_j\equiv 0\ \forall\delta\ne r\mathbf{e}_j, r\in\mathbb{N},\delta\ne\mathbf{0}, \tag{10}$$
$$\frac{\partial c_j}{\partial x_j}(l)\ne 0\ \forall l\in\mathbb{Z}^d. \tag{11}$$

We will prove the theorem by induction on n. We have a base for $n=0$ due to Proposition 6. Let us prove the inductive step: $n\to n+1$.

Assume that each polynomial t_{jk} belongs to SF^n, i.e. $D^\beta a_{jk}(s_l^*)=0$, $l=1,\dots,m-1$, for all $\beta\in\mathbb{Z}_+^d$, $[\beta]\le n$. Let $\alpha\in\mathbb{Z}^d$, $[\alpha]=n+1$, $s\in D(M^*)$, $s\ne\mathbf{0}$. It follows from (8) and Leibniz formula that

$$\sum_{\mathbf{0}\le\beta\le\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}b_k(s)D^\beta a(s)=D^\alpha a_{jk}(s)c_j(s).$$

By the inductive hypotheses, $D^\beta a(s) = 0$ whenever $\beta < \alpha$. Hence, taking into account (9), we have

$$b_k(s) D^\alpha a(s) = c_j(s) D^\alpha a_{jk}(s) = 0.$$

Due to Lemma 5, $b_k(s) \neq 0$ for at least one $k = 1, \dots, d$. It follows that $D^\alpha a(s) = 0$.

Now we assume that $t \in SF^{n+1}$, i.e. $D^\alpha a(s_\nu^*) = 0$, $\nu = 1, \dots, m-1$, for all $\alpha \in \mathbb{Z}_+^d$, $[\alpha] \leq n+1$. Due to the inductive hypotheses, there exist polynomials $t_{jk} \in SF^{n-1}$ satisfying (6), i.e. $D^\delta a_{jk}(s_\nu^*) = 0$, $\nu = 1, \dots, m-1$, for all $\delta \in \mathbb{Z}_+^d$, $[\delta] \leq n-1$. We will construct new trigonometric polynomials satisfying (6) and belonging to SF^n. Note that for $n = 0$ any trigonometric polynomials t_{jk} satisfying (6) belong to SF^0, due to Theorem 7.

Let us prove the following statement. If $D^\delta a_{jk}(s_\nu^*) = 0$, $\nu = 1, \dots, m-1$, for all $\delta \in \mathbb{Z}_+^d$, $[\delta] \leq n$ and for all $j = 1, \dots, l-1$ $(l = 1, \dots, d)$, then there exist polynomials $\tilde{t}_{jk}$, $j, k = 1, \dots, d$, such that $\sum\limits_{j=1}^{d} \tilde{a}_{jk} c_j = \sum\limits_{j=1}^{d} a_{jk} c_j$, where $\tilde{a}_{jk}(x) = \tilde{t}_{jk}(M^{*-1}x)$, and $D^\delta \tilde{a}_{jk}(s_\nu^*) = 0$, $\nu = 1, \dots, m-1$, $k = 1, \dots, d$, for all $\delta \in \mathbb{Z}_+^d$, $[\delta] \leq n$, and for all $j = 1, \dots, l$.

Let $s \in D(M^*)$, $s \neq \mathbf{0}$, $\beta \in \mathbb{Z}_+^d$, $[\beta] = n$, $\sum\limits_{i=l+1}^{d} \beta_i = 0$. Set $\alpha = \beta + \mathbf{e}_l$ and note that $[\alpha] = n+1$. It follows from (8) that

$$D^\alpha \left(\sum_{j=1}^{d} a_{jk}(x) c_j(x) \right) \Bigg|_{x=s} = D^\alpha (b_k(x) a(x))|_{x=s} = 0. \tag{12}$$

On the other hand, due to (9), (10) and the assumption of the statement, we have

$$D^\alpha \left(\sum_{j=1}^{d} a_{jk}(x) c_j(x) \right) \Bigg|_{x=s} = \sum_{j=1}^{l} D^{\alpha - \mathbf{e}_j} a_{jk}(s) \frac{\partial c_j}{\partial x_j}(s) = D^\beta a_{lk}(s) \frac{\partial c_l}{\partial x_l}(s).$$

Combining this with (11) and (12), we get $D^\beta a_{lk}(s) = 0$. is proved already for $l = d$ (in this case $\tilde{t}_{jk} = t_{jk}$, $j, k = 1, \dots, d$).

Next let $j = l+1, \dots, d$, $\beta \in \mathbb{Z}_+^d$, $[\beta] = n$, $\sum\limits_{i=l+1}^{d} \beta_i > 0$. Define the functions

$$q_{\beta j}(x) = \frac{1}{-2\pi i} g_{n-1, \beta - \mathbf{e}_j}(x) \sum_{\nu=1}^{m-1} h_\nu(x) D^\beta a_{lk}(s_\nu^*),$$

where $g_{N\delta}$ is a trigonometric polynomial such that $D^\gamma g_{N\delta}(\mathbf{0}) = 0$ for all $\gamma \in \mathbb{Z}_+^d$, $[\gamma] \le N$, $\gamma \ne \delta$ and $D^\delta g_{N\delta}(\mathbf{0}) = 1$;

$$h_\nu(x) = \frac{1}{m}\sum_{\mu=0}^{m-1} e^{2\pi i (x - s_\nu^*, M^{-1}s_\mu)}.$$

Since, by Proposition A, $h_\nu(s_\mu^*) = \delta_{\mu\nu}$, due to (9), (10) and Leibniz formula, it is not difficult to see that for each $\nu = 1, \dots, m-1$ we have

$$D^\beta(c_j(x)q_{\beta j}(x))\Big|_{x=s_\nu^*} = D^\beta a_{lk}(s_\nu^*); \tag{13}$$

$$D^\delta(c_j(x)q_{\beta j}(x))\Big|_{x=s_\nu^*} = 0 \quad \forall \delta \in \mathbb{Z}_+^d,\ [\delta] \le n,\ \delta \ne \beta; \tag{14}$$

$$D^\delta(c_i(x)q_{\beta j}(x))\Big|_{x=s_\nu^*} = 0 \quad \forall \delta \in \mathbb{Z}_+^d,\ [\delta] \le n-1, \forall i = 1, \dots, d. \tag{15}$$

Set

$$\begin{aligned}
\tilde a_{lk}(x) &:= a_{lk}(x) - \sum_{j=l+1}^{d} \sum_{\substack{[\beta]=n,\ \beta_j>0 \\ \beta_{l+1}=\dots=\beta_{j-1}=0}} c_j(x)q_{\beta j}(x), \\
\tilde a_{jk}(x) &:= a_{jk}(x) + \sum_{\substack{[\beta]=n,\ \beta_j>0 \\ \beta_{l+1}=\dots=\beta_{j-1}=0}} c_l(x)q_{\beta j}(x), \quad j = l+1, \dots, d.
\end{aligned}$$

Because of construction, $\sum\limits_{j=l}^{d} \tilde a_{jk}c_j = \sum\limits_{j=l}^{d} a_{jk}c_j$, and, taking into account (13), (14), for each $\nu = 1, \dots, m-1$ we have $D^\beta \tilde a_{lk}(s_\nu^*) = 0$ whenever $[\beta] = n$, $\sum\limits_{i=l+1}^{d} \beta_i > 0$; $D^\beta \tilde a_{lk}(s_\nu^*) = D^\beta a_{lk}(s_\nu^*) = 0$ whenever $[\beta] = n$, $\sum\limits_{i=l+1}^{d} \beta_i = 0$. At last, due to (15), $D^\delta \tilde a_{jk}(s_\nu^*) = 0$, $j = l, \dots, d$ for all $\delta \in \mathbb{Z}_+^d$, $[\delta] \le n-1$. To complete the proof of the statement it remains to put $\tilde t_{jk} = \tilde a_{jk}(M^*x)$ for $j = l, \dots, d$ and $\tilde t_{jk} = t_{jk}$ for $j = 1, \dots, l-1$.

So, we described one step for improvement of decomposition (6). Starting with $l = 1$, after $(d-1)$ steps we will obtain required polynomials. $\Diamond$

Analyzing the proof of Theorem 8 it is not difficult to describe an algorithm for finding polynomials $t_{jk} \in SF^{n-1}$, $j, k = 1, \dots, d$, in decomposition (6). To realize this algorithm we will need functions $g_{N\delta}$. Explicit recursive formulas for these functions are presented in [4].

ALGORITHM 2

Input: $t \in SF^N$, $N > 1$.

Output: $t_{jk} \in SF^{N-1}, j, k = 1, \dots, d$.

Step 1. Using Algorithm 1, find t_{jk}, $j, k = 1, \dots, d$, satisfying (6).

Step 2. For $n = 1, \dots, N-1$ do

Set $t_{jk}^{(0)} := t_{jk}$, $j, k = 1, \dots, d$;

For each $l = 1, \dots, d-1$ do

For each $k = 1, \dots, d$ do

$a_{lk}(x) = t_{lk}^{(l-1)}(M^{*-1}x)$;

$$t_{lk}^{(l)}(x) := t_{lk}^{(l-1)}(x) - \sum_{j=l+1}^{d} \left(1 - e^{2\pi i(M^*x, \mathbf{e}_j)}\right) \sum_{\substack{[\beta]=n,\ \beta_j>0 \\ \beta_{l+1}=\dots=\beta_{j-1}=0}} q_{\beta j}(M^*x);$$

For each $j = 1, \dots, l$ do

$t_{jk}^{(l)}(x) := t_{jk}^{(l-1)}(x)$;

For each $j = l+1, \dots, d$ do

$$t_{jk}^{(l)}(x) := t_{jk}^{(l-1)}(x) + \left(1 - e^{2\pi i(M^*x, \mathbf{e}_l)}\right) \sum_{\substack{[\beta]=n,\ \beta_j>0 \\ \beta_{l+1}=\dots=\beta_{j-1}=0}} q_{\beta j}(M^*x). \Diamond$$

In the case $d = 2$, Step 2 of Algorithm 2 does not look so frightening, it is reduced to the following.

For $n = 1, \dots, N-1$ do

Set $l = 1$, $t_{jk}^{(0)} := t_{jk}$, $j, k = 1, 2$;

For each $k = 1, 2$ do

$$\begin{aligned}
a_{1k}(x) &= t_{1k}^{(0)}(M^{*-1}x); \\
t_{1k}(x) &:= t_{1k}^{(0)}(x) - \left(1 - e^{2\pi i(M^*x, \mathbf{e}_2)}\right) \sum_{[\beta]=n,\ \beta_2>0} q_{\beta j}(M^*x); \\
t_{2k}(x) &:= t_{2k}^{(0)}(x) + \left(1 - e^{2\pi i(M^*x, \mathbf{e}_1)}\right) \sum_{[\beta]=n,\ \beta_2>0} q_{\beta j}(M^*x).
\end{aligned}$$

In particular, for $d = 2$, $N = 2$, Step 2 may be realized as follows.

Set $t_{jk}^{(0)} := t_{jk}$, $j, k = 1, 2$;

For each $k = 1, 2$ do

$$t_{1k}(x) := t_{lk}^{(0)}(x) + \frac{1}{2\pi i}\left(1 - e^{2\pi i (M^* x)_2}\right)\sum_{\nu=1}^{m-1} h_\nu(M^* x)\frac{\partial}{\partial x_2} t_{1k}^{(0)}(M^{*-1}u)\Big|_{u=s_\nu^*};$$

$$t_{2k}(x) := t_{2k}^{(0)}(x) - \frac{1}{2\pi i}\left(1 - e^{2\pi i (M^* x)_1}\right)\sum_{\nu=1}^{m-1} h_\nu(M^* x)\frac{\partial}{\partial x_2} t_{1k}^{(0)}(M^{*-1}u)\Big|_{u=s_\nu^*}.$$

For each $n \in \mathbb{N}$ we introduce the set

$$\Gamma^n := \{k \in \mathbb{R}^n : \quad k_l = 1, \dots, d, \ l = 1, \dots, n\}.$$

Theorem 9 *Let $n, n_0 \in \mathbb{N}$, $n \le n_0$, $t \in SF^{n_0 - 1}$, $k \in \Gamma^n$, then there exist trigonometric polynomials $t_{jk} \in SF^{n_0 - n - 1}$, $j \in \Gamma^n$, such that*

$$\prod_{l=1}^{n}\left(1 - e^{2\pi i (x, \mathbf{e}_{k_l})}\right) t(x) = \sum_{j_1=1}^{d} \dots \sum_{j_n=1}^{d} t_{jk}(x) \prod_{l=1}^{n}\left(1 - e^{2\pi i (M^* x, \mathbf{e}_{j_l})}\right), \quad (16)$$

$$t_{jk}(\mathbf{0}) = \prod_{l=1}^{n} (M^{*-1})_{j_l k_l}\, t(\mathbf{0}). \quad (17)$$

Proof. We will proof by induction on n.

Base: $n = 1$. Let $k \in \Gamma^1$. Due to Proposition 6 and Theorem 8, there exist trigonometric polynomials $t_{jk} \in SF^{n_0 - 2}$, $j \in \Gamma^1$, such that (6) holds. Rewrite identity (6) as follows

$$(1 - e^{2\pi i (x, M^{-1}\mathbf{e}_k)}) t(M^{*-1}x) = \sum_{j=1}^{d} t_{jk}(M^{*-1}x)(1 - e^{2\pi i (x, \mathbf{e}_j)}),$$

Differentiating by x_l, we have

$$-2\pi i\,(M^{-1}\mathbf{e}_k, \mathbf{e}_l) e^{2\pi i (x, M^{-1}\mathbf{e}_k)}) t(M^{*-1}x) + (1 - e^{2\pi i (x, M^{-1}\mathbf{e}_k)})\frac{\partial}{\partial x_l} t(M^{*-1}x) =$$

$$\sum_{j=1}^{d} \frac{\partial}{\partial x_l} t_{jk}(M^{*-1}x)(1 - e^{2\pi i (x, \mathbf{e}_j)}) - 2\pi i\, t_{jk}(M^{*-1}x)\, e^{2\pi i (x, \mathbf{e}_j)}.$$

Substituting $x = \mathbf{0}$, we obtain

$$-2\pi i\,(M^{-1}\mathbf{e}_k, \mathbf{e}_l) t(\mathbf{0}) = -2\pi i\, t_{jk}(\mathbf{0}).$$

To prove the base it remains to note that

$$(M^{-1}\mathbf{e}_k, \mathbf{e}_l) = (M^{*-1}\mathbf{e}_l, \mathbf{e}_k) = (M^{*-1})_{kl}.$$

Inductive step: $n-1 \to n$. Let $1 < n \le n_0$, $k \in \Gamma^n$, $k' := (k_1, \dots, k_{n-1})$. By the inductive hypotheses there exist $t_{j'k'} \in SF^{n_0-n}$, $j' \in \Gamma^{n-1}$, such that

$$\prod_{l=1}^{n-1} \left(1 - e^{2\pi i (x, \mathbf{e}_{k_l})}\right) t(x) = \sum_{j_1=1}^{d} \dots \sum_{j_{n-1}=1}^{d} t_{j'k'}(x) \prod_{l=1}^{n-1} \left(1 - e^{2\pi i (M^* x, \mathbf{e}_{j_l})}\right), \quad (18)$$

$$t_{jk}(\mathbf{0}) = \prod_{l=1}^{n-1} (M^{*-1})_{j_l k_l}\, t(\mathbf{0}). \quad (19)$$

Since $n_0 - n \ge 0$ and the theorem is proved already for $n = 1$, for each $t_{j'k'}$, there exist trigonometric polynomials $t_{j_n k_n} \in SF^{n_0-n-1}$, $j \in \Gamma^1$, such that

$$\begin{aligned} \left(1 - e^{2\pi i (x, \mathbf{e}_{k_n})}\right) t_{j'k'}(x) &= \sum_{j_n=1}^{d} t_{j_n k_n}(x) \left(1 - e^{2\pi i (M^* x, \mathbf{e}_{j_n})}\right), & (20) \\ t_{j_n k_n}(\mathbf{0}) &= (M^{*-1})_{j_n k_n} t_{j'k'}(\mathbf{0}). & (21) \end{aligned}$$

Combining (18), (19) with (20), (21) we comlete the proof. $\Diamond$

To impart a more compact form to (16), (17), we introduce the following notations. Set

$$\begin{aligned} \Delta_k(x) = \Delta_k = \left(1 - e^{2\pi i (\mathbf{e}_k, x)}\right), &\quad \Delta = (\Delta_1, \dots, \Delta_d), \\ \delta_k(x) = \delta_k = \left(1 - e^{2\pi i (\mathbf{e}_k, M^* x)}\right), &\quad \delta = (\delta_1, \dots, \delta_d)^T. \end{aligned}$$

Let $A^{[n]}$ denotes the n-th Kronecker degree of a matrix A, i.e. if A is a $M \times N$ matrix with entries a_{jk}, then

$$A^{[1]} := A, \quad A^{[n+1]} := \begin{pmatrix} a_{11} A^{[n]} & \dots & a_{1N} A^{[n]} \\ \vdots & \ddots & \vdots \\ a_{M1} A^{[n]} & \dots & a_{MN} A^{[n]} \end{pmatrix}.$$

Now Theorem 9 can be rewritten as

Theorem 8$'$ *Let $n, n_0 \in \mathbb{N}$, $n \le n_0$, $t \in SF^{n_0-1}$, $k \in \Gamma^n$, then there exists a $d^n \times d^n$ matrix T whose entries are trigonometric polynomials $t_{jk} \in SF^{n_0-n-1}$ such that*

$$(\Delta(x))^{[n]} t(x) = T(x) (\delta(x))^{[n]}, \quad T(\mathbf{0}) = (M^{*-1})^{[n]} t(\mathbf{0}).$$

5. Construction of desirable masks

A simple description of the classes SF^n is well known in the one-dimensional dyadic case. A general form is given by the formula $t(x) = (1 + e^{2\pi ix})^n T(x)$, where T is an arbitrary trigonometric polynomial. In the multidimensional case SF^n can not be described in a similar way. We will give a characterization of the class SF^n for arbitrary matrix dilation which allows to construct its elements in practice.

Theorem 10 *A trigonometric polynomial t belongs to SF^n if and only if the derivatives of its polyphase function τ_k, $k = 0, \dots, m-1$, up to order n are given by*

$$D^\alpha \tau_k(\mathbf{0}) = \frac{1}{m} \sum_{\mathbf{0} \le \beta \le \alpha} \lambda_\beta \binom{\alpha}{\beta} (-2\pi i r_k)^{\alpha - \beta}, \quad \alpha \in \mathbb{Z}_+^d, [\alpha] \le n, \tag{22}$$

where $\lambda_\alpha = D^\alpha t(M^{*-1}x)|_{x=0}$.

Remark. For polynomials t whose polyphase functions $\tau_0, \dots, \tau_{m-1}$ form a unimodular row (i.e. there exists a dual row of trigonometric polynomials $\tilde{\tau}_0, \dots, \tilde{\tau}_{m-1}$ such that $\sum_{k=0}^{m-1} \tau_k \tilde{\tau}_k \equiv 1$), the statement of Theorem 10 follows from combining the results of [3] and [4]. It was proved in these papers that both the conditions are equivalent to vanishing moments of the corresponding wavelet system.

Proof. Assume that (22) holds with some complex numbers λ_α. Let $s \in D(M^*)$. By Leibniz formula,

$$D^\alpha \left(e^{2\pi i (r_k, x)} \tau_k(x) \right) \Big|_{x=s} = \sum_{\mathbf{0} \le \beta \le \alpha} \binom{\alpha}{\beta} D^\beta \left(e^{2\pi i (r_k, x)} \right) \Big|_{x=s} D^{\alpha-\beta} \tau_k(\mathbf{0}) =$$

$$\sum_{\mathbf{0} \le \beta \le \alpha} \binom{\alpha}{\beta} e^{2\pi i (r_k, s)} (2\pi i r_k)^\beta D^{\alpha - \beta} \tau_k(\mathbf{0}) =$$

$$\frac{1}{m} e^{2\pi i (r_k, s)} \sum_{\mathbf{0} \le \beta \le \alpha} \binom{\alpha}{\beta} (2\pi i r_k)^\beta \sum_{\mathbf{0} \le \gamma \le \alpha - \beta} \lambda_\gamma \binom{\alpha - \beta}{\gamma} (-2\pi i r_k)^{\alpha - \beta - \gamma} =$$

$$\frac{1}{m} e^{2\pi i (r_k, s)} \sum_{\mathbf{0} \le \beta \le \alpha} \sum_{\mathbf{0} \le \gamma \le \alpha - \beta} \lambda_\gamma \binom{\alpha - \beta}{\gamma} \binom{\alpha}{\beta} (-2\pi i r_k)^{\alpha - \gamma} \prod_{j=1}^d (-1)^{-\beta_j} =$$

$$\frac{1}{m} e^{2\pi i (r_k, s)} \sum_{\mathbf{0} \le \gamma \le \alpha} \lambda_\gamma (-2\pi i r_k)^{\alpha - \gamma} \binom{\alpha}{\gamma} \sum_{\mathbf{0} \le \beta \le \alpha - \gamma} \binom{\alpha - \gamma}{\beta} \prod_{j=1}^d (-1)^{-\beta_j}.$$

Since

$$\sum_{\mathbf{0}\le\beta\le\alpha-\gamma}\binom{\alpha-\gamma}{\beta}\prod_{j=1}^{d}(-1)^{-\beta_j}=\prod_{j=1}^{d}\sum_{\mathbf{0}\le\beta_j\le\alpha_j-\gamma_j}\binom{\alpha_j-\gamma_j}{\beta_j}(-1)^{-\beta_j}=$$
$$\prod_{j=1}^{d}(1-1)^{\alpha_j-\gamma_j}=\begin{cases}0, & \alpha\ne\gamma,\\ 1, & \alpha=\gamma,\end{cases}$$

we have

$$D^\alpha\left(e^{2\pi i(r_k,x)}\tau_k(x)\right)\Big|_{x=s}=\frac{\lambda_\alpha}{m}e^{2\pi i(r_k,s)},\quad k=0,\dots,m-1.$$

It follows from (2) and Proposition A that

$$D^\alpha(t(M^{*-1}x)\Big|_{x=s}=\sum_{k=0}^{m-1}D^\alpha\left(e^{2\pi i(r_k,x)}\tau_k(x)\right)\Big|_{x=s}=$$
$$\frac{\lambda_\alpha}{m}\sum_{k=0}^{m-1}e^{2\pi i(r_k,s)}=\begin{cases}\lambda_\alpha, & \text{if } s=\mathbf{0},\\ 0, & \text{if } s\ne\mathbf{0}.\end{cases}$$

Now let us check that (22) follows from the relation $t\in SF^n$. We will prove by induction on n. The base for $n=0$ was established in Proposition 1. To prove the inductive step $n\to n+1$, we assume that $t\in SF^{n+1}$ and (22) holds. Let $\alpha\in\mathbb{Z}^d$, $[\alpha]=n+1$, $s\in D(M^*)$. By (2) and Leibniz formula,

$$D^\alpha(t(M^{*-1}x)\Big|_{x=s}=\sum_{k=0}^{m-1}\sum_{\mathbf{0}\le\beta\le\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}\left(e^{2\pi i(r_k,x)}\right)\Big|_{x=s}D^\beta\tau_k(\mathbf{0})=$$
$$\sum_{k=0}^{m-1}\left(D^\alpha\tau_k(0)+\sum_{\mathbf{0}\le\beta<\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}(2\pi ir_k)^{\alpha-\beta}D^\beta\tau_k(\mathbf{0})\right)e^{2\pi i(r_k,s)}.\quad(23)$$

Because of the inductive hypotheses, we have

$$\sum_{\mathbf{0}\le\beta<\alpha}\binom{\alpha}{\beta}(2\pi ir_k)^{\alpha-\beta}D^\beta\tau_k(\mathbf{0})=$$
$$\frac{1}{m}\sum_{\mathbf{0}\le\beta<\alpha}\sum_{\mathbf{0}\le\gamma\le\beta}\lambda_\gamma\binom{\alpha}{\beta}\binom{\beta}{\gamma}(2\pi ir_k)^{\alpha-\beta}(-2\pi ir_k)^{\beta-\gamma}=$$

$$\frac{1}{m}\sum_{\mathbf{0}\le\gamma<\alpha}\lambda_\gamma(2\pi i r_k)^{\alpha-\gamma}\sum_{\gamma\le\beta<\alpha}\binom{\alpha}{\beta}\binom{\beta}{\gamma}\prod_{j=1}^{d}(-1)^{\beta_j-\gamma_j}=$$

$$\frac{1}{m}\sum_{\mathbf{0}\le\gamma<\alpha}\binom{\alpha}{\gamma}\lambda_\gamma(2\pi i r_k)^{\alpha-\gamma}\sum_{\mathbf{0}\le\delta<\alpha-\gamma}\binom{\alpha-\gamma}{\delta}\prod_{j=1}^{d}(-1)^{\delta_j}=$$

$$\frac{1}{m}\sum_{\mathbf{0}\le\gamma<\alpha}\binom{\alpha}{\gamma}\lambda_\gamma(2\pi i r_k)^{\alpha-\gamma}\left(\prod_{j=1}^{d}(1-1)^{\alpha_j-\gamma_j}-\prod_{j=1}^{d}(-1)^{\alpha_j}\right)=$$

$$-\frac{1}{m}\sum_{\mathbf{0}\le\gamma<\alpha}\binom{\alpha}{\gamma}\lambda_\gamma(-2\pi i r_k)^{\alpha-\gamma}.$$

Combining this with (23), we obtain

$$\sum_{k=0}^{m-1}e^{2\pi i(r_k,s)}\left(D^\alpha\tau_k(0)-\frac{1}{m}\sum_{\mathbf{0}\le\gamma<\alpha}\binom{\alpha}{\gamma}\lambda_\gamma(-2\pi i r_k)^{\alpha-\gamma}\right)=D^\alpha(t(M^{*-1}x))\Big|_{x=s}.$$

Set $\lambda_\alpha = D^\alpha(t(M^{*-1}x))\big|_{x=0}$. Due to Proposition A, the linear system

$$\sum_{k=0}^{m-1}e^{2\pi i(r_k,s_l^*)}y_k=\lambda_\alpha\delta_{0l},\quad l=0,\dots,m-1,$$

has a unique solution $y_k=\frac{\lambda_\alpha}{m}$, $k=0,\dots,m-1$. It follows that

$$D^\alpha\tau_k(0)-\frac{1}{m}\sum_{\mathbf{0}\le\gamma<\alpha}\binom{\alpha}{\gamma}\lambda_\gamma(-2\pi i r_k)^{\alpha-\gamma}=\frac{\lambda_\alpha}{m},$$

which was to be proved. $\Diamond$

So, if we want a polynomial t to belong to SF^n, its polyphase functions should have derivatives at the origin given by (22). This can be easily realized for an arbitrary set of parameters λ_β, $[\beta]\le n$. General forms for all such polynomials t are presented in [4].

References

[1] Dyn N. *Subdivision schemes in Computer Aided Geometric Design*, In "Advances in Numerical Analysis-Vol. II, Wavelets, Subdivision Algorithms and Radial Basis Functions", W. Light (ed.), Clarendon Press, Oxford, (1992), 36-104.

[2] Wojtaszczyk P. *A mathematical introduction to wavelets*, London Math. Soc. Student texts 37, 1997.

[3] Jia R.Q. *Approximation properties of multivariate wavelets*, Math. Comp. **67** (1998), 647-655.

[4] Skopina M., *On Construction of Multivariate Wavelets with Vanishing Moments* ACHA **20** (2006), 3, 375-390

[5] Latour V., Müller J., Nickel W. *Stationry subdivision for general scaling matrices*, Math. Zeitschrift **227** (1998), 645-661.

[6] Sauer T. *Polynomial interpolation, ideals and approximation order of multivariate refinable functions* Proceedings of the American Mathematical Society **130** (2002), 11, 3335-3347.

[7] Möller H.M. and Sauer T. *Multivariate refinable functions of high approximation order via quotient ideals of Laurent polynomials*, Adv. Comput. Math. **20** (2004), No.1-3, 205-228.