Trace Hardy–Sobolev inequalities in cones

*A.I. Nazarov**,
Saint-Petersburg University,
e-mail: an@AN4751.spb.edu

Abstract

We discuss the values of sharp constants and their attainability for the trace Hardy–Sobolev inequalities in cones.

Dedicated to V.G. Maz'ya
with great respect

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be an open cone, and let $G = \Omega \cap \mathbb{S}^{n-1}$ be a domain with a strictly Lipschitz boundary in the unit sphere. Denote by $\dot{\mathcal{C}}^1(\Omega)$ the set of smooth functions with bounded, separated from the origin support. For $1 \leqslant p \leqslant \infty$ denote by $\dot{W}^1_p(\Omega)$ the completion of $\dot{\mathcal{C}}^1(\Omega)$ with respect to the norm $\|\nabla v\|_{p,\Omega}$.

Consider a scale of weighted trace spaces $L_{q,\sigma}(\partial\Omega)$ with the norms

$$\|v\|_{q,\sigma,\partial\Omega} = \|r^{\sigma-1}v\|_{L_q(\partial\Omega)},$$

where $r = |x|$.

For $1 < p < \infty$, $p \neq n$ and $\frac{1}{p} \leqslant \sigma \leqslant \min\{1, \frac{n}{p}\}$ we denote by $p^{**}_\sigma = \frac{(n-1)p}{n-\sigma p}$ the critical exponent for the trace embedding $\dot{W}^1_p(\Omega) \hookrightarrow L_{q,\sigma}(\partial\Omega)$:

$$\lambda(p,\sigma,\Omega) = \inf_{v \in \dot{W}^1_p(\Omega)\setminus\{0\}} J_\sigma(v) \equiv \inf_{v \in \dot{W}^1_p(\Omega)\setminus\{0\}} \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^{**}_\sigma,\sigma,\partial\Omega}} > 0. \qquad (I_\sigma)$$

The case $p < n$, $\sigma = 1$ leads to the trace Sobolev inequality, see, e.g., [M, Sec. 1.4.5]. The case $\sigma = \frac{1}{p}$ is natural to be called the trace Hardy inequality. The intermediate cases can be obtained from these ones by the Hölder inequality. For $p > n$ one should use the Morrey inequality, see [M, Sec. 1.4.5], instead of the trace Sobolev inequality.

We call (I_σ) *the trace Hardy–Sobolev inequality.*

Remark 1. For $p = n$ the infimum in (I_σ) equals zero. For $\sigma = 1$ it is a classical fact, for $\sigma < 1$ it is sufficient to consider a sequence

$$v_k(x) = \begin{cases} 0, & r \leqslant \frac{1}{k} \ \text{ or } \ r \geqslant 2k; \\ kr-1, & \frac{1}{k} < r < \frac{2}{k}; \\ 1, & \frac{2}{k} \leqslant r \leqslant k; \\ 1-\frac{r}{k}, & k < r < 2k. \end{cases}$$

*Partially supported by RFBR grant 08-01-00748 and by grant NSh.227.2008.1.

Since the embedding operator $\dot{W}_p^1(\Omega) \hookrightarrow L_{p_\sigma^{**},\sigma}(\partial\Omega)$ is noncompact, the problem of attainability of the norm of this operator (i.e. the problem of existence of an extremal function in the embedding theorem) is nontrivial. Conventional Hardy–Sobolev inequality connected with embedding $\dot{W}_p^1(\Omega) \hookrightarrow L_{p_\sigma^*,\sigma}(\Omega)$ (here $p_\sigma^* = \frac{np}{n-\sigma p}$, $0 \leqslant \sigma \leqslant \min\{1, \frac{n}{p}\}$) was treated in many papers, see, e.g., the recent survey [N] and further references therein. The problem for the trace embedding is considerably less investigated.

The structure of our paper is as follows. §§2 and 3 are devoted to the values of sharp constants, respectively, in the trace Hardy inequality and in the trace Sobolev inequality. In §4 we discuss the existence of the minimizer in the intermediate cases.

Let us recall some notation. For $p < n$ we put $p^* = p_1^* = \frac{np}{n-p}$ (this is conventional Sobolev conjugate exponent to p) and $p^{**} = p_1^{**} = \frac{(n-1)p}{n-p}$. $\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the area of the unit sphere in $\mathbb{R}^n$. $\mathcal{B}$ is the Euler beta-function while $\mathrm{P}_\nu^\mu(x)$ is the Legendre function (see, e.g., [GR, 8.7]).

2 “Hardy” case $\sigma = \frac{1}{p}$

Our first result provides the sharp constants in the trace Hardy inequality in cones.

Theorem 1. Let $1 < p < \infty$, $p \neq n$, $\sigma = \frac{1}{p}$. Then the infimum in (I_σ) is not attained and equals $(\Lambda^{(p)}(G))^{\frac{1}{p}}$, where ($\nabla'$ stands for the tangential gradient on $\mathbb{S}^{n-1} \subset \mathbb{R}^n$)

$$\Lambda^{(p)}(G) = \min_{v \in W_p^1(G)\setminus\{0\}} \frac{\displaystyle\int_G \left(\left(\frac{n-p}{p}\right)^2 v^2 + |\nabla' v|^2\right)^{\frac{p}{2}} d\Theta}{\displaystyle\int_{\partial G} |v|^p \, dS}. \tag{1}$$

Proof. First, the minimum in (1) is attained due to the compactness of trace embedding $W_p^1(G) \hookrightarrow L_p(\partial G)$ and is positive due to $p \neq n$. Denote by V the minimizer of (1) normalized in $L_p(\partial G)$. By standard argument, V is positive in $\overline{G}$.

Let us define $U(r, \Theta) = r^{1-\frac{n}{p}} V(\Theta)$, where (r, Θ) are spherical coordinates in Ω. Then for any $h \in \dot{\mathcal{C}}^1(\Omega)$ we have

$$\begin{aligned}
\int\limits_\Omega |\nabla U|^{p-2} \nabla U \cdot \nabla h \, dx &= \int\limits_\Omega |\nabla U|^{p-2} (U_r h_r + \frac{1}{r^2} \nabla' U \cdot \nabla' h) \, dx = \\
&= \int\limits_G \left(\left(\frac{n-p}{p}\right)^2 V^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} V \cdot \int\limits_0^\infty \frac{p-n}{p} \, r^{\frac{n}{p}-1} h_r \, dr d\Theta + \\
&+ \int\limits_0^\infty \int\limits_G \left(\left(\frac{n-p}{p}\right)^2 V^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} \nabla' V \cdot \nabla' h \, r^{\frac{n}{p}-2} \, d\Theta dr =
\end{aligned} \tag{2}$$

$$= \int\limits_0^\infty \int\limits_G \left(\left(\frac{n-p}{p}\right)^2 V^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} \left(\left(\frac{n-p}{p}\right)^2 V h + \nabla' V \cdot \nabla' h\right) r^{\frac{n}{p}-2} \, d\Theta dr \overset{*}{=}$$

$$\overset{*}{=} \Lambda^{(p)}(G) \int\limits_0^\infty \int\limits_{\partial G} V^{p-1} h \, r^{\frac{n}{p}-2} \, dS dr = \Lambda^{(p)}(G) \int\limits_{\partial\Omega} \frac{U^{p-1}}{r^{p-1}} h \, d\Sigma$$

(the equality $(*)$ follows from the weak Euler–Lagrange equation for V). Thus, U is a positive weak solution of the Steklov type problem

$$-\Delta_p u \equiv -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in} \quad \Omega; \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \mathbf{n}} = \Lambda^{(p)}(G)\,\frac{u^{p-1}}{r^{p-1}} \quad \text{on} \quad \partial\Omega.$$

The relation $\Lambda^{(p)}(G) \leqslant \lambda^p(p, \frac{1}{p}, \Omega)$ follows now from the generalized Picone identity (see [AH]). For any $u \in \dot{\mathcal{C}}^1(\Omega)$ we set $h = \frac{|u|^p}{U^{p-1}}$. Then (2) implies

$$\begin{aligned}
\Lambda^{(p)}(G)\int\limits_{\partial\Omega}\frac{|u|^p}{r^{p-1}}\,d\Sigma &= \Lambda^{(p)}(G)\int\limits_{\partial\Omega}\frac{U^{p-1}}{r^{p-1}}h\,d\Sigma = \int\limits_{\Omega}|\nabla U|^{p-2}\nabla U\cdot\nabla h\,dx = \\
&= \int\limits_{\Omega}\Big(p|\nabla U|^{p-2}\nabla U\cdot\nabla u\frac{|u|^{p-2}u}{U^{p-1}} - (p-1)|\nabla U|^p\frac{|u|^p}{U^p}\Big)\,dx \overset{**}{\leqslant} \\
&\overset{**}{\leqslant} \int\limits_{\Omega}\Big(p|\nabla u|\cdot|\nabla U|^{p-1}\frac{|u|^{p-1}}{U^{p-1}} - (p-1)|\nabla U|^p\frac{|u|^p}{U^p}\Big)\,dx \leqslant \int\limits_{\Omega}|\nabla u|^p\,dx.
\end{aligned} \tag{3}$$

Here $(**)$ is the Cauchy inequality while the last inequality follows from

$$x^p - pxy^{p-1} + (p-1)y^p \geqslant 0, \qquad x, y > 0. \tag{4}$$

By approximation, (3) holds true for $u \in \dot{W}^1_p(\Omega)$.

To prove $\Lambda^{(p)}(G) = \lambda^p(p, \frac{1}{p}, \Omega)$ we consider the sequence

$$u_\delta(r,\Theta) = \begin{cases} r^{1-\frac{n}{p}+\delta}V(\Theta), & r \leqslant 1; \\ r^{1-\frac{n}{p}-\delta}V(\Theta), & r \geqslant 1. \end{cases}$$

Clearly, $u_\delta \in \dot{W}^1_p(\Omega)$. Direct computation gives

$$\begin{aligned}
\Lambda^{(p)}(G)\int\limits_{\partial\Omega}\frac{|u_\delta|^p}{r^{p-1}}\,d\Sigma &= \frac{2}{p\delta}\cdot\Lambda^{(p)}(G)\int\limits_{\partial G}V^p\,dS = \frac{2}{p\delta}\int\limits_{G}\Big(\Big(\tfrac{n-p}{p}\Big)^2V^2 + |\nabla' V|^2\Big)^{\frac{p}{2}}dS = \\
= \frac{1}{p\delta}\int\limits_{G}\Big[\Big(\Big(\tfrac{n-p}{p}+\delta\Big)^2V^2 &+ |\nabla' V|^2\Big)^{\frac{p}{2}} + \Big(\Big(\tfrac{n-p}{p}-\delta\Big)^2V^2 + |\nabla' V|^2\Big)^{\frac{p}{2}}\Big]dx + O(\delta) = \\
&= \int\limits_{\Omega}|\nabla u_\delta|^p\,dx + O(\delta),
\end{aligned} \tag{5}$$

and the statement follows.

Finally, the equality sign in $(**)$ means $\nabla u \parallel \nabla U$ while the equality in (4) means $x = y$. These two facts imply

$$\frac{\nabla u}{u} = \frac{\nabla U}{U} \qquad \Longrightarrow \qquad u = cU$$

on the set $\{u \neq 0\}$ and, therefore, in the whole Ω. Since $U \notin \dot{W}^1_p(\Omega)$, the equality in (3) is impossible. $\square$

Note that the minimizer of (1) is unique up to a multiplicative constant. There are many ways to prove this. The simplest one is by convexity, see, e.g., [LL, Theorem 7.8] and [Kw, Proposition 4] for particular cases.

Lemma 1. Let $F(t,x)$ be a function on $\mathbb{R}_+\times\mathbb{R}^n$ positive homogeneous of degree $p>1$ and convex. If $1\leqslant q\leqslant p$, then $F(v,\nabla v)$ is convex with respect to v^q on positive functions.

Proof. Setting $f=v^q$, we are to prove that $\Phi(f)=F(f^{\frac{1}{q}},\frac{1}{q}f^{\frac{1}{q}-1}\nabla f)$ is convex. Note that the homogeneity of F implies the relations

$$\begin{aligned} D^2_{tt}F(s,y)\,s &+ \sum_k D^2_{tx_k}F(s,y)\,y_k &= (p-1)D_tF(s,y); \\ D^2_{tx_j}F(s,y)\,s &+ \sum_k D^2_{x_kx_j}F(s,y)\,y_k &= (p-1)D_{x_j}F(s,y), \qquad j=1,\dots,n. \end{aligned} \tag{6}$$

By direct calculation we obtain, subject to (6),

$$d^2\Phi(f;\ h)=\frac{f^{\frac{p}{q}-2}}{q^2}\cdot\left[\frac{p-q}{p-1}\cdot d^2F\Big(1,\frac{\nabla f}{qf};\ h,z^{(1)}\Big)+\frac{q-1}{p-1}\cdot d^2F\Big(1,\frac{\nabla f}{qf};\ 0,z^{(2)}\Big)\right], \tag{7}$$

where

$$z^{(1)}=\nabla h-\tfrac{(q-1)\nabla f}{qf}h;\qquad z^{(2)}=\nabla h-\tfrac{\nabla f}{f}h.$$

By convexity of F, the expression in (7) is nonnegative, and the statement follows. $\square$

The uniqueness in (1) follows now from the fact that a function $F(t,x)=(\alpha^2t^2+|x|^2)^{\frac{p}{2}}$ is strictly convex. Thus, vanishing of (7) implies $\frac{\nabla h}{h}=\frac{\nabla f}{f}$, i.e. $h=cf$.

In particular, this means that for G being a spherical "hat"

$$G=\{(\theta,\phi_1,\dots,\phi_{n-2})\in\mathbb{S}^{n-1}:\ 0<\theta<\theta_*\} \tag{8}$$

the minimizer in (1) depends only on θ. For $p=2$ this gives the opportunity to calculate $\Lambda^{(2)}(G)$ explicitly.

Theorem 2. Let G be a spherical "hat" (8). Then

$$\Lambda^{(2)}(G)=\frac{(n-2)^2}{4}\cdot\frac{\mathrm{P}^{-\frac{n-1}{2}}_{-\frac{1}{2}}\big(\cos(\theta_*)\big)}{\mathrm{P}^{-\frac{n-3}{2}}_{-\frac{1}{2}}\big(\cos(\theta_*)\big)}. \tag{9}$$

In particular, for the case $\theta_*=\frac{\pi}{2}$, which corresponds $\Omega=\mathbb{R}^n_+=\{x=(x',x_n):\ x_n>0\}$, one has

$$\Lambda^{(2)}(G)=\frac{2\Gamma^2\big(\frac{n}{4}\big)}{\Gamma^2\big(\frac{n-2}{4}\big)}. \tag{10}$$

Proof. By standard variational argument, the minimizer in (1) for $p=2$ is an eigenfunction of the Steklov problem (Δ' stands for the Beltrami operator on $\mathbb{S}^{n-1}$)

$$-\Delta'V+\tfrac{(n-2)^2}{4}V=0\quad\text{in}\ \ G;\qquad \frac{\partial V}{\partial\mathbf{n}}=\Lambda^{(2)}(G)\ V\quad\text{on}\ \ \partial G.$$

Since the minimizer is a "one-dimensional" function, this problem is reduced to

$$V'' + (n-2)\,\mathrm{ctg}\,(\theta)V' - \frac{(n-2)^2}{4}V = 0; \qquad |V(0)| < \infty; \quad V'(\theta_*) = \Lambda^{(2)}(G)V(\theta_*).$$

A solution bounded at the origin is proportional to $\sin^{-\frac{n-3}{2}}(\theta) \cdot \mathrm{P}_{-\frac{1}{2}}^{-\frac{n-3}{2}}\big(\cos(\theta)\big)$, see, e.g., [Km, 2.171]. Formula (9) follows now from [GR, 8.733], and formula (10) – from [GR, 8.756.1]. $\square$

For example, we list the sharp constants in the trace Hardy inequality for the half-spaces of small dimensions.

n	3	4	5	6
$\lambda(2, \frac{1}{2}, \mathbb{R}^n_+)$	$\dfrac{2\pi}{\Gamma^2(\frac{1}{4})}$	$\sqrt{\frac{2}{\pi}}$	$\dfrac{\Gamma^2(\frac{1}{4})}{4\pi}$	$\sqrt{\frac{\pi}{2}}$

It is easy to see that $\lambda(2, \frac{1}{2}, \mathbb{R}^n_+) \cdot \lambda(2, \frac{1}{2}, \mathbb{R}^{n+2}_+) = \frac{n-2}{2}$.

3 "Sobolev" case $\sigma = 1$

The next theorem deals with the trace Sobolev inequality in circular cones.

Theorem 3. Let Ω be a convex circular cone, i.e. let G be a spherical "hat" (8), $\theta_* \leqslant \frac{\pi}{2}$. Suppose $1 < p < n$ and $\sigma = 1$. Then the function

$$w(x) = |x - \mathbf{e}|^{-\frac{n-p}{p-1}}, \tag{11}$$

with $\mathbf{e} = (0, \dots, 0, -1)$, provides minimum in (I_σ).

Remark 2. For the case of half-space ($\theta_* = \frac{\pi}{2}$), this statement was proved in the remarkable paper [Nt], see also [E] for $p = 2$.

Proof. We repeat almost literally the proof of [Nt, Theorem 1]. The way is based on the mass transportation approach (generalized Monge–Kantorovich problem). Consider two probability measures in Ω with smooth densities F and G and bounded, separated from the origin supports. Then, see [Br] and [MC], there exists the so-called *Brenier map* $T = \nabla\varphi$ such that for all measurable functions ψ

$$\int\limits_\Omega \psi(x)G(x)\,dx = \int\limits_\Omega \psi(T(x))F(x)\,dx. \tag{12}$$

Moreover, the function φ is convex and satisfies the Monge–Ampère equation

$$F(x) = G(\nabla\varphi(x)) \cdot \det(D^2\varphi(x)). \tag{13}$$

almost everywhere w.r.t. the measure $F dx$. Here $D^2\varphi$ is a.e.-Hessian matrix of φ which exists by A.D. Aleksandrov theorem.

By (12),

$$\int\limits_\Omega G^{1-\frac{1}{n}}(x)\,dx = \int\limits_\Omega G^{-\frac{1}{n}}(\nabla\varphi(x))F(x)\,dx.$$

Using (13) and the Hadamard inequality, we obtain

$$\int\limits_{\Omega} G^{1-\frac{1}{n}}(x)\,dx = \int\limits_{\Omega} \det^{\frac{1}{n}}(D^2\varphi(x))F^{1-\frac{1}{n}}(x)\,dx \leqslant \frac{1}{n}\int\limits_{\Omega} \Delta\varphi(x)F^{1-\frac{1}{n}}(x)\,dx. \tag{14}$$

Since φ is convex, we can change $\Delta\varphi$, understood as a.e.-Laplacian in the right-hand side of (14), to the full distributional Laplacian.

Integrating by parts, we get

$$n\int\limits_{\Omega} G^{1-\frac{1}{n}}(x)\,dx \leqslant \int\limits_{\partial\Omega} F^{1-\frac{1}{n}}(x)\langle\nabla\varphi(x),\mathbf{n}\rangle\,d\Sigma - \int\limits_{\Omega}\langle\nabla\varphi(x),\nabla(F^{1-\frac{1}{n}})(x)\rangle\,dx.$$

By definition of the Brenier map, $\nabla\varphi(x)\in\Omega$ for all $x\in\Omega$. Therefore, $\langle\nabla\varphi(x),\mathbf{n}\rangle\leqslant 0$ on $\partial\Omega$, and

$$n\int\limits_{\Omega} G^{1-\frac{1}{n}}(x)\,dx \leqslant -\int\limits_{\Omega}\langle\nabla\varphi(x),\nabla(F^{1-\frac{1}{n}})(x)\rangle\,dx. \tag{15}$$

Adding to both parts of (15) the integral

$$\int\limits_{\Omega}\langle\mathbf{e},\nabla(F^{1-\frac{1}{n}})(x)\rangle\,dx = \int\limits_{\partial\Omega} F^{1-\frac{1}{n}}(x)\langle\mathbf{e},\mathbf{n}\rangle\,d\Sigma = \sin(\theta_*)\int\limits_{\partial\Omega} F^{1-\frac{1}{n}}(x)\,d\Sigma,$$

we arrive at

$$\sin(\theta_*)\int\limits_{\partial\Omega} F^{1-\frac{1}{n}}(x)\,d\Sigma + n\int\limits_{\Omega} G^{1-\frac{1}{n}}(x)\,dx \leqslant \int\limits_{\Omega}\langle\mathbf{e}-\nabla\varphi(x),\nabla(F^{1-\frac{1}{n}})(x)\rangle\,dx. \tag{16}$$

Put $F=v^{p^*}$, $G=u^{p^*}$. Then $\|v\|_{p^*}=\|u\|_{p^*}=1$, and (16) becomes

$$\sin(\theta_*)\|v\|_{p^{**},\partial\Omega}^{p^{**}} \leqslant \frac{(n-1)p}{n-p}\int\limits_{\Omega} v^{\frac{n(p-1)}{n-p}}(x)\langle\mathbf{e}-\nabla\varphi(x),\nabla v(x)\rangle\,dx - n\|u\|_{p^{**},\Omega}^{p^{**}}.$$

By the Hölder inequality and (12),

$$\begin{aligned}\sin(\theta_*)\|v\|_{p^{**},\partial\Omega}^{p^{**}} &\leqslant \\ &\leqslant \frac{(n-1)p}{n-p}\,\|\nabla v\|_{p,\Omega}\Bigg(\int\limits_{\Omega} v^{p^*}(x)|\mathbf{e}-\nabla\varphi(x)|^{p'}\,dx\Bigg)^{\frac{1}{p'}} - n\|u\|_{p^{**},\Omega}^{p^{**}} = \\ &= \frac{(n-1)p}{n-p}\,\|\nabla v\|_{p,\Omega}\Bigg(\int\limits_{\Omega} u^{p^*}(x)|\mathbf{e}-x|^{p'}\,dx\Bigg)^{\frac{1}{p'}} - n\|u\|_{p^{**},\Omega}^{p^{**}}.\end{aligned} \tag{17}$$

Note that both sides of (17) do not contain the Brenier map. Hence, by approximation, this inequality remains valid for $u,v\in\dot{W}_p^1(\Omega)$.

Now we specify (17) by setting $u=Cw$, where w is defined in (11) while $C=\|w\|_{p^*,\Omega}^{-1}$ is the normalization constant. Then for any $v\in\dot{W}_p^1(\Omega)$ such that $\|v\|_{p^*,\Omega}=1$, we have

$$\|v\|_{p^{**},\partial\Omega}^{p^{**}} \leqslant A\|\nabla v\|_{p,\Omega} - B, \tag{18}$$

where

$$A=\frac{(n-1)p\,C^{\frac{n(p-1)}{n-p}}}{(n-p)\sin(\theta_*)}\cdot\mathcal{I}^{\frac{1}{p'}},\quad B=\frac{n\,C^{p^{**}}}{\sin(\theta_*)}\cdot\mathcal{I};\qquad \mathcal{I}=\|w\|^{p^{**}}_{p^{**},\Omega}=\int\limits_{\Omega}\frac{dx}{|x-\mathbf{e}|^{(n-1)p'}}.$$

For arbitrary $v\in\dot W^1_p(\Omega)$, without normalization, (18) can be rewritten as follows:

$$\left(\frac{K(v)}{J_1(v)}\right)^{p^{**}}\leqslant AK(v)-B\qquad\Longleftrightarrow\qquad J_1^{p^{**}}(v)\geqslant\mathcal{F}(K(v))\equiv\frac{K^{p^{**}}(v)}{AK(v)-B},$$

where

$$J_1(v)=\frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^{**},\partial\Omega}},\qquad K(v)=\frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^*,\Omega}}.$$

By elementary calculus, the function $\mathcal{F}$ achieves its minimum at the point

$$\frac{p(n-1)B}{n(p-1)A}=\frac{n-p}{p-1}\,C\mathcal{I}^{\frac{1}{p}}=K(w),$$

and therefore,

$$J_1^{p^{**}}(v)\geqslant\frac{K^{p^{**}}(w)}{AK(w)-B}=\left(\frac{n-p}{p-1}\right)^{\frac{n(p-1)}{n-p}}\mathcal{I}^{\frac{p-1}{n-p}}\sin(\theta_*).\tag{19}$$

If $v=Cw$, then the Brenier map is the identity. Direct calculation shows that all the inequalities become equalities, and the statement follows. $\square$

Thus, the right-hand side of (19) equals $\lambda^{p^{**}}(p,1,\Omega)$. By [GR, 3.252.10], [GR, 8.733.1] and [GR, 8.335.1],

$$\mathcal{I}=\omega_{n-2}\int\limits_0^{\theta_*}\int\limits_0^{\infty}\frac{r^{n-1}\sin^{n-2}(\theta)\,drd\theta}{(r^2+2r\cos(\theta)+1)^a}=$$

$$=\omega_{n-2}2^{a-\frac12}\Gamma\Big(a+\frac12\Big)\mathcal{B}(n,2a-n)\int\limits_0^{\theta_*}\sin^{n-a-\frac32}(\theta)\mathrm{P}^{\frac12-a}_{n-a-\frac12}\big(\cos(\theta)\big)\,d\theta=$$

$$=\pi^{\frac{n}{2}-1}2^{a-\frac32}\Gamma\Big(a-\tfrac{n-1}{2}\Big)\mathcal{B}\Big(\tfrac{n}{2},a-\tfrac{n}{2}\Big)\sin^{n-a-\frac12}(\theta_*)\mathrm{P}^{\frac12-a}_{n-a-\frac32}\big(\cos(\theta_*)\big),\tag{20}$$

where $a=\frac{(n-1)p}{2(p-1)}$.

In particular, for $\theta_*=\frac{\pi}{2}$ we obtain

$$\lambda(p,1,\mathbb{R}^n_+)=\left(\frac{n-p}{p-1}\right)^{\frac{1}{p'}}\left(\frac{\omega_{n-2}}{2}\cdot\mathcal{B}\Big(\frac{n-1}{2},\frac{n-1}{2(p-1)}\Big)\right)^{\frac{1}{(n-1)p'}}.$$

Theorem 4. Let Ω be a nonconvex circular cone, i.e. let G be a spherical "hat" (8), $\theta_*>\frac{\pi}{2}$. Suppose $1<p<n$ and $\sigma=1$. Then the function (11) does not provide minimum in (I_σ).

Proof. Direct calculation shows that w is a positive solution of the Neumann problem

$$-\Delta_p u=0\quad\text{in}\quad\Omega;\qquad|\nabla u|^{p-2}\frac{\partial u}{\partial\mathbf{n}}=\gamma u^{p^{**}-1}\quad\text{on}\quad\partial\Omega,\tag{21}$$

where $\gamma = J_1^{p^{**}}(w)/\|\nabla w\|_{p,\Omega}^{p^{**}-p}$.

Let $\Omega_1 \subset \Omega$ be a half-space such that $\partial\Omega_1$ is a hyperplane tangent to $\partial\Omega$. Then w solves the same problem (21) in Ω_1.

By the dilation invariance of the quotient (I_1), the function w provides minimum to the functional J_1 in Ω_1. Thus, $\gamma = \lambda^{p^{**}}(p, 1, \mathbb{R}^n_+)/\|\nabla w\|_{p,\Omega_1}^{p^{**}-p}$, and $\Omega_1 \subset \Omega$ implies $J_1(w) > \lambda(p, 1, \mathbb{R}^n_+)$.

Now we claim that $\lambda(p, 1, \Omega) \leqslant \lambda(p, 1, \mathbb{R}^n_+)$. Indeed, consider $w_k(x) = |x - x^k|^{-\frac{n-p}{p-1}}$, where $x^k \notin \overline{\Omega}$, $x^k \to x^0 \neq 0$, and $x^0 \in \partial\Omega$. Since the neighborhood of x^0 in the large scale looks like a half-space, by the dilation invariance of the quotient (I_σ) we obtain $\lim\limits_k J_1(w_k) = \lambda(p, 1, \mathbb{R}^n_+)$, and the statement follows. $\square$

Remark 3. One can see from the proof that Theorem 3 remains valid for any convex cone Ω, if its supporting hyperplanes at almost every point have a constant angle with the axis x_n. The simplest example of such cone is a dyhedral angle less than half-space. Another interesting example is a cone with G being *arbitrary* simplex in $\mathbb{S}^{n-1}$.

Similarly, Theorem 4 remains valid for Ω being a complement of such convex cone.

Remark 4. The statements of Theorems 3 and 4 hold true also for a more general definition of $\|\nabla v\|_{p,\Omega}$ in the numerator of J_1. Namely, consider an arbitrary norm $|\!|\!|x|\!|\!|$ in $\mathbb{R}^n$ and replace $\|\nabla v\|_{p,\Omega}$ by

$$|\!|\!|\nabla v|\!|\!|_{p,\Omega} = \Big(\int\limits_\Omega |\!|\!|\nabla v|\!|\!|^p dx\Big)^{\frac{1}{p}}.$$

Then Theorems 3 and 4 remain valid, if we replace $|x - \mathbf{e}|$ in (11) by the dual norm

$$|\!|\!|x|\!|\!|_o = \sup_{|\!|\!|\xi|\!|\!| \leqslant 1} \langle x, \xi\rangle = \sup_{\xi \neq 0} \frac{\langle x, \xi\rangle}{|\!|\!|\xi|\!|\!|}.$$

The proofs run without changes.

4 Intermediate case

Now we consider the case $\frac{1}{p} < \sigma < \min\{1, \frac{n}{p}\}$. For the conventional Hardy–Sobolev inequality corresponding sharp constants in cones are attained (see [N]). This is the case also for the trace embedding.

Theorem 5. Let $1 < p < \infty$, $p \neq n$, $\frac{1}{p} < \sigma < \min\{1, \frac{n}{p}\}$. Then the infimum in (I_σ) is attained.

Remark. Standard arguments show that under suitable normalization the minimizer of J_σ is a positive solution of the Neumann problem

$$-\Delta_p u = 0 \quad \text{in} \quad \Omega; \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \mathbf{n}} = \frac{u^{p^{**}_\sigma - 1}}{r^{(1-\sigma)p^{**}_\sigma}} \quad \text{on} \quad \partial\Omega. \tag{22}$$

Proof. Consider a minimizing sequence $\{v_k\}$ for the quotient I_σ. Since this quotient is invariant with respect to dilations and to multiplication v by constant, we can ensure the fulfilment of the relations

$$\int\limits_\Omega |\nabla v_k|^p \, dx = 1, \qquad \int\limits_{B_1 \cap \Omega} |\nabla v_k|^p \, dx = \frac{1}{2}. \tag{23}$$

Without loss of generality we can assume that $v_k \rightharpoonup v$ in $\dot{W}_p^1(\Omega)$.

By the P.-L. Lions concentration-compactness principle ([Ls, Part 1]; see also [N1, Lemma 3.1]) we have

$$|r^{\sigma-1}v_k|^{p_\sigma^{**}} \rightharpoonup |r^{\sigma-1}v|^{p_\sigma^{**}} + \alpha_0\delta_0(x) + \alpha_\infty\delta_\infty(x),$$
$$|\nabla v_k|^p \rightharpoonup \mu \geqslant |\nabla v|^p + \beta_0\delta_0(x) + \beta_\infty\delta_\infty(x), \quad (24)$$

where

$$\alpha_0, \alpha_\infty \geqslant 0, \qquad \beta_0 = \lambda^p(p,\sigma,\Omega)\alpha_0^{p/p^{**}}, \qquad \beta_\infty = \lambda^p(p,\sigma,\Omega)\alpha_\infty^{p/p^{**}}.$$

The second convergence in (24) is in the sense of measures on the compact $\mathcal{K} = \overline{\Omega} \cup \{\infty\}$, and the first one is in the sense of measures on $\partial\mathcal{K}$.

Since $\{v_k\}$ is a minimizing sequence, by verbatim repetition of arguments from Theorem 3.1 [N1] we obtain the alternative – either v is a minimizer of $J_\sigma(v)$ and $\alpha_0 = \alpha_\infty = 0$, or $v = 0$; in the last case $\beta_0 = 1$ or $\beta_\infty = 1$. However, the second relation in (23) keeps only the first possibility, and the statement follows. $\square$

Let $\Omega = \mathbb{R}^n_+$. Then we could expect the symmetry of the least energy solution of (22). For $p < n$ this is the case, the minimizer of $J_\sigma(u)$ is radially symmetric in x', i.e. $u = u(|x'|, x_n)$. This follows from the properties of the Schwarz symmetrization with respect to x'-variables (or from the properties of the Steiner symmetrization with respect to x_1 for $n = 2$). Indeed, this transformation reduces the numerator in (I_σ), see, e.g., [PS, Ch.7], and enlarges the denominator, see, e.g., [LL, Ch.3]. Moreover, u is symmetrically decreasing in x' for every $x_n > 0$ and decreasing in x_n for any $x' \in \mathbb{R}^{n-1}$ (the last statement is proved by the monotone rearrangement with respect to x_n).

For $p > n$ the symmetrization arguments do not work because of condition $u(0) = 0$. It turns out that symmetry of minimizer really breaks in this case. For conventional Hardy–Sobolev inequality in $\mathbb{R}^n \setminus \{0\}$ this effect was established in [N1, Theorem 4.1].

Theorem 6. Let G be a spherical "hat" (8). Then for any $p > n$ there exists $\widehat{\sigma}(n,p,\theta_*) < \frac{n}{p}$ such that for $\sigma > \widehat{\sigma}$ no symmetric function $u(|x'|, x_n)$ give the minimum to the functional J_σ.

Proof. Let $u(|x'|, x_n)$ be a function providing the minimum to the functional J_σ over the set of symmetric functions in $\dot{W}_p^1(\Omega)$. Without loss of generality, we assume that $\|u\|_{p_\sigma^{**},\sigma,\partial\Omega} = 1$. By the principle of symmetric criticality, see [P], $dJ_\sigma(u; h) = 0$ for any variation $h \in \dot{W}_p^1(\Omega)$.

Similarly to [N2, Theorem 1.3], the second differential of J_σ at the point u can be written as follows:

$$J_\sigma^{p-1}(u)\cdot d^2J_\sigma(u;\ h) = \int\limits_\Omega |\nabla u|^{p-4}\big((p-2)\langle\nabla u, \nabla h\rangle^2 + |\nabla u|^2|\nabla h|^2\big)\, dx -$$
$$-J_\sigma^p(u)\cdot\Big[(p - p_\sigma^{**})\cdot\Big(\int\limits_{\partial\Omega}\frac{|u|^{p_\sigma^{**}-2}uh}{r^{(1-\sigma)p_\sigma^{**}}}\, d\Sigma\Big)^2 + (p_\sigma^{**} - 1)\cdot\int\limits_{\partial\Omega}\frac{|u|^{p_\sigma^{**}-2}h^2}{r^{(1-\sigma)p_\sigma^{**}}}\, d\Sigma\Big]. \quad (25)$$

Now we set $h(x) = u(|x'|, x_n)f(x)$ where $f(x) = \frac{x_1}{r}$. By symmetry of u, $\int\limits_{\partial\Omega}\frac{|u|^{p_\sigma^{**}-2}uh}{r^{(1-\sigma)p_\sigma^{**}}}\, d\Sigma = 0$. Next,

$$\langle\nabla u, \nabla h\rangle^2 = |\nabla u|^2\langle\nabla u, \nabla(uf^2)\rangle + u^2\langle\nabla u, \nabla f\rangle^2, \qquad |\nabla h|^2 = \langle\nabla u, \nabla(uf^2)\rangle + u^2|\nabla f|^2.$$

Substituting these relations into (25), we obtain, subject to $dJ_\sigma(u; uf^2) = 0$,

$$J_\sigma^{p-1}(u) \cdot d^2 J_\sigma(u;\ h) = \\ = \int\limits_\Omega |\nabla u|^{p-4} u^2 \left((p-2)\langle \nabla u, \nabla f\rangle^2 + |\nabla u|^2 |\nabla f|^2\right) dx - \\ - J_\sigma^p(u) \cdot (p_\sigma^{**} - p) \cdot \int\limits_{\partial\Omega} \frac{|u|^{p_\sigma^{**}} f^2}{r^{(1-\sigma)p_\sigma^{**}}}\, d\Sigma.$$

Note that

$$\int\limits_{\partial\Omega} \frac{|u|^{p_\sigma^{**}}}{r^{(1-\sigma)p_\sigma^{**}}}\ \frac{x_1^2}{r^2}\, d\Sigma = \frac{1}{n-1} \int\limits_{\partial\Omega} \frac{|u|^{p_\sigma^{**}}}{r^{(1-\sigma)p_\sigma^{**}}}\ \frac{|x'|^2}{r^2}\, d\Sigma = \frac{\sin^2(\theta_*)}{n-1} \int\limits_{\partial\Omega} \frac{|u|^{p_\sigma^{**}}}{r^{(1-\sigma)p_\sigma^{**}}}\, d\Sigma = \frac{\sin^2(\theta_*)}{n-1}$$

(the last equality is the normalization condition for u). Since $|\nabla f| \leqslant \frac{1}{r}$, we obtain

$$J_\sigma^{p-1}(u) \cdot d^2 J_\sigma(u;\ h) \leqslant (p-1) \cdot \int\limits_\Omega |\nabla u|^{p-2} \frac{u^2}{r^2}\, dx - J_\sigma^p(u) \cdot (p_\sigma^{**} - p) \cdot \frac{\sin^2(\theta_*)}{n-1}. \tag{26}$$

Finally, we estimate the integral in (26) by Hölder and Hardy inequalities and arrive at

$$d^2 J_\sigma(u;\ h) \leqslant J_\sigma(u) \cdot \left[(p-1)\left(\tfrac{p}{p-n}\right)^2 - (p_\sigma^{**} - p) \cdot \frac{\sin^2(\theta_*)}{n-1}\right].$$

The quantity in square brackets is negative for σ close to $\frac{n}{p}$, and the statement follows. □

Corollary. For $p > n$ and $\widehat{\sigma} < \sigma < \frac{n}{p}$ the problem (22) in circular cone has at least two nonequivalent positive solutions.

Proof. The first solution is a global minimizer of J_σ, the second one is a minimizer over the set of symmetric functions. □

References

[AH] W. Allegretto & Y.X. Huang, *A Picone's identity for p-Laplacian and applications*, Nonlin. Anal. **32** (1998), 819–830.

[Br] Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. **44** (1991), N4, 375–417.

[E] J.F. Escobar, *Sharp constant in a Sobolev trace inequality*, Indiana Univ. Math. J. **37** (1988), 687–698.

[GR] I.S. Gradshteyn & I.M. Ryzhik, *Tables of integrals, sums, series and products*, 5th ed., Nauka, Moscow, 1971 (Russian). English transl.: *Table of integrals, series, and products. Corr. and enl. ed. by Alan Jeffrey*, Acad. Press, New York–London–Toronto, 1980.

[Km] E. Kamke, *Differentialgleichungen. Lösungsmethoden und Lösungen. I*, 6th ed., Leipzig, 1959.

[Kw] B. Kawohl, *Symmetry results for functions yelding best constants in Sobolev-type inequalities*, Discr. and Contin. Dynam. Systems, **6** (2000), 683–690.

[LL] E. Lieb & M. Loss, *Analysis*, Grad. Studies in Math. **14**, 2nd ed., AMS, 2001.

[Ls] P.L. Lions, *The concentration-compactness principle in the Calculus of Variations. The locally compact case*, Ann. Inst. H.Poincaré. Anal. Non Linéaire **1** (1984), 109–145, 223–283.

[M] V.G. Maz'ya, Sobolev spaces, Leningrad University Publishing, Leningrad, 1985 (Russian). English transl.: Springer, Berlin, 1985.

[MC] R.J. McCann, *Existence and uniqueness of monotone measure-preserving maps*, Duke Math. J. **80** (1995), N2, 309–323.

[Nt] B. Nazaret, *Best constant in Sobolev trace inequalities on the half-space*, Nonlin. Anal., **65** (2006), N10, 1977–1985.

[N] A.I. Nazarov, *Dirichlet and Neumann problems to critical Emden–Fowler type equations*, J. Global Optim. **40** (2008), 289–303.

[N1] A.I. Nazarov, *Hardy–Sobolev inequalities in a cone*, Probl. Math. Anal. **31** (2005), 39–46 (Russian). English transl.: J. Math. Sci. **132** (2006), N4, 419–427.

[N2] A.I. Nazarov, *The one-dimensional character of an extremum point of the Friedrichs inequality in spherical and plane layers*, Probl. Math. Anal. **20** (2000), 171–190 (Russian). English transl.: J. Math. Sci. **102** (2000), N5, 4473–4486.

[P] R.S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979), 19–30.

[PS] G. Pólya & G. Szegö, *Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, 1951.