APPROXIMATION PROPERTIES: TOPOLOGICAL ASPECT
(WITH APPLICATION TO A GROTHENDIECK CONJECTURE)

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In 1932, Stefan Banach posed his famous problem on the existence of a (Schauder) basis in any separable Banach space (the first example of a concrete basis in nonhilbertian space, namely, in $C[0,1]$ was given by Schauder). Since that time many mathematicians tried to find bases and did it for most classical Banach spaces (A. Haar, J. Schauder, J. Marcinkiewicz, and others). But the general Problem was still unsolved. For a long time, there were even no thoughts how to deal with this very important Problem. The positive solution would yield a lot of applications, would lead to positive solutions of most of well known open problems in the corresponding fields.

In 1955, there appeared the famous fundamental work of Alexander Grothendieck "Produits tensoriels topologiques et espaces nucléaires" in Mem. Amer. Math. Soc. 16 (1955). In this treatise, A. Grothendieck has introduced and investigated a lot of new notions, especially, the notions of different approximation properties of spaces and operators (so-called AP and BAP; they generalize the property of a Banach space to have a basis). One of the nice consequence: if X has the AP then every compact operator in X is in the closure of the set of all finite rank operators.

In his great work, A. Grothendieck has posed (in the very end of the treatise) more than 15 main unsolved Problems. Among them, Problems: 1) of existence of a Banach

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space without AP or BAP; 2) of existence of a space with AP, but without BAP; 3) of equivalence of AP and BAP for all spaces. Concerning the last question, he had showed that for Banach spaces with weakly compact identity maps (for reflexive spaces) these two properties are the same. This led him to ask: 4) whether any weakly compact operator with AP possesses also BAP?

Recall some definitions. The Banach space X has the *approximation property* AP if for each compact $K \subset X$ and for every $\varepsilon > 0$ there exists a finite rank operator R in X such that $||x - Rx|| \leq \varepsilon$ for any $x \in K$. In other words, X has the AP iff the identity map in X can be uniformly, on each compact, approximated by finite rank operators. X has the *bounded approximation property* BAP if there is a constant $C > 0$ so that for each compact $K \subset X$ and for every $\varepsilon > 0$ there exists a finite rank operator R in X such that $||x - Rx|| \leq \varepsilon$ for any $x \in K$ and $||R|| \leq C$. In his work, A. Grothendieck was able to reformulate these properties in terms of introduced by him the so–called projective and injective tensor products, as well as in terms of his "nuclear" and "integral" operators.

The problems of A. Grothendieck were solved by different mathematicians, firstly (and mainly) by P. Enflo (the famous example of a reflexive Banach space without AP; se, e.g., [10]).

The natural generalizations of AP and BAP (the properties $\mathrm{AP}_p, BAP_p, AP_p^{dual}$ etc.) were considered by P. Saphar [21], the author [11–13], [16] and others. All these properties were firstly defined in terms of some tensor products (generalizations of Grothendieck ones). One of the question of P. Saphar was to describe the AP_p in terms of something like "compact convergence" for absolutely p-summing operators. It was done in [15] by the author of this paper. The further investigations in this direction can be found in [1], [11–13] etc.

This paper is devoted, mainly, to the proceeding in the same direction for the class (ideal) Π_p^d of dually absolutely-p-summing operators, and giving the connections between some notions of "compact convergence" of type π_p^{dual} and the properties of tensor products, with applications. We will apply the results to the investigation of a natural question: when a Banach space has the generalized approximation properties (such as introduced below the properties AP^p)? We give here some sufficient and some necessary conditions for the space to have AP^p as well as construct some (counter)examples to the AP^p-approximation problems. In the very end of the paper, we apply our results to give a new proof of a much more stronger result than one from my paper [14] (concerning a conjecture of A. Grothendieck from his fundamental work [3]). Answering in negative to the Grothendieck question whether every weakly compact operator with the AP has also the BAP, we have constructed in [14] the example of a *compact* operator with AP, but without BAP. Here, in Theorem 10, we show that there exist the operators of such a kind, belonging even to the classes of dually quasi-p-nuclear (hence, compact) operators.

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§1. Preliminaries

We use standard notation of the theories of Banach spaces, of linear operators and of operator ideals (see, e.g., [10]). The Banach spaces under consideration will be denoted usually by X, Y, Z, E, etc. Elements of these spaces are denoted by corresponding small letters, for example, $x \in X, y \in Y, x_0 \in X_1, y_k \in Y etc.$ Banach duals to the spaces X, Y etc. are denoted by X^*, Y^* etc. The value of a functional $f \in X^*$ or of $x' \in X^*$ at an element $x \in X$ is denoted by $\langle f, x\rangle$ or $f(x)$; by $\langle x', x\rangle$ or $\langle x, x'\rangle$ (everywhere, we consider the space X as a subspace of its second dual X^{**} without introducing a special notation for the canonical isometric injection $X \to X^{**}$). $\mathrm{L}(X, Y)$ is the space of all (linear continuous) operators from X to Y with the usual operator norm. If $A \subset X$, then $\Gamma(A)$ is a absolutely convex hull of A, $\overline{A}$ is the closure of the set A in X. If we need to show that the closure is taken in a topology τ on X, we write $\overline{A}^{\tau}$. For an absolutely convex closed bounded set $K \subset X$, we denote by X_K the Banach space $\cup_{n=1}^{\infty} nK$ with he unit ball K; $\Phi_K : X_K \to X$ is then the canonical injection. The linear space of all finite rank operators from X to Y is considered as a tensor product $X^* \otimes Y$. Finely, if needed, the norm of an $x \in X$ in the space X is denoted also by $||x||_X$.

Recall that every Banach space X is isometric to a factor-space of a space $l_1(\Gamma)$. Recall also that an operator $T \in \mathrm{L}(X, Y)$ is weakly compact iff $T(X^{**}) \subset Y$.

If X is a Banach space and μ is a measure then by $L_p(\mu; X)$, $0 < p \leqslant \infty$ we understand the L_p-space of all (equivalent classes of) strongly μ-measurable p-summable functions. In the case where the measures are discrete, we use also the notations of type $l_p(X)$, $l_p(\Gamma; X)$, $c_0(X)$, etc. For the quasinorm of a sequence $(x_k)_k$ from the space $l_p(X)$, we use the notation $\alpha_p(x_k) := \left(\sum_k \|x_k\|^p\right)^{1/p}$ (if $p = \infty$, the corresponding changes are needed). Recall that $l_p(\Gamma; X)^* = l_{p'}(\Gamma; X^*)$ and $c_0(\Gamma; X)^* = l_1(\Gamma; X)$ for all finite $p \geqslant 1$. A family $(x_k)_{k=1}^\infty \subset X$, for which the value $\alpha_p(x_k)$ is finite, is said to be *absolutely p-summable.* A family $(x_k)_{k=1}^\infty$ is called *weakly p-summable,* if $(\langle x_k, x' \rangle) \in l_p$ for all $x' \in X^*$. We set

$$\varepsilon_p(x_j) := \sup_{\|x'\| \leqslant 1} \left(\sum_j |\langle x_j, x' \rangle|^p\right)^{1/p}.$$

This is a quasinorm (a norm if $p \geqslant 1$) in the space $l_p\{X\}$ of all weakly p-summable sequences in X.

We use the following notations for the classical operator ideals (all the information on the theory of operator ideals can be found in [10]; in our work, we follow, however, the other notation and terminology; see [8, 9]):

$[\Pi_p, \pi_p]$ — the ideal of the absolutely p-summing operators;

$[\mathrm{QN}_p, \pi_p]$ — the ideal of quasi-p-nuclear operators;

$[\mathrm{N}_p, \nu_p]$ — the ideal of p-nuclear operators;

$[\mathrm{I}_p, i_p]$ — the ideal of (strictly) p-integral operators;

$[\Pi_p^d, \pi_p^d]$ — the ideal of the dually absolutely p-summing operators;

$[\mathrm{QN}_p^d, \pi_p^d]$ — the ideal of dually quasi-p-nuclear operators;

$[\mathrm{N}^p, \nu^p]$ — the ideal of dually p-nuclear operators;

$[\mathrm{I}^p, i^p]$ — the ideal of (strictly) dually p-integral operators.

Recall some important definitions (see [8–10,17–18]). Let $T \in \mathrm{L}(X,Y)$. For $0 < p \leqslant \infty$, the operator $T : X \to Y$ is called *absolutely p-summing*, if there is a constant $C > 0$ such that, for any finite family $\{x_n\}_{n=1}^M \subset X$, the following inequality holds $\alpha_p(Tx_n) \leqslant C\varepsilon_p(x_n)$; corresponding norm ($\inf C$) is denoted by $\pi_p(T)$. Note that $[\Pi_\infty, \pi_\infty]$ is exactly the operator ideal $[\mathrm{L}, \|\cdot\|]$. For $0 < p \leqslant \infty$, the operator $T : X \to Y$ is called *quasi-p-nuclear*, $T \in \mathrm{QN}_p(X,Y)$, if for some isometric embedding $i : Y \to L_\infty(\mu)$ the composition iT is in $\mathrm{N}_p(X, L_\infty(\nu))$. The quasinorm (the norm if $p \geqslant 1$) in $\mathrm{QN}_p(X,Y)$ is induced from the space $\Pi_p(X, L_\infty(\nu))$.

Generally, let A be an operator ideal with a norm a. By $\mathrm{A}^{\mathrm{dual}}$, we denote the dual (to A) ideal: $T \in \mathrm{A}^{\mathrm{dual}}$ iff $T^* \in \mathrm{A}$. The ideal $\mathrm{A}^{\mathrm{dual}}$ with the norm $a^{\mathrm{dual}}(T) := a(T^*)$ is a normed operator ideal. Let us denote by a^* the norm in the conjugate ideal $[\mathrm{A}, a]^* = [\mathrm{A}^*, a^*]$ [10, p. 9.1] . For example, $\Pi_p^d = \Pi_p^{\mathrm{dual}}$, $\mathrm{QN}_p^d = \mathrm{QN}_p^{\mathrm{dual}}$, $[\Pi_p^d, \pi_p^d]^* = [\mathrm{I}^{p'}, i^{p'}]$. Here and everywhere, $1/p + 1/p' = 1$. Consider some important examples.

Let $p \in [1, \infty]$. We say that an operator $T \in \mathrm{L}(X,Y)$ *belongs to the ideal* N^p, if it can be represented in the form

$$T := \sum_{n=1}^{\infty} x'_n \otimes y_n,$$

where the sequences (x'_n) and (y_n) are such that $\varepsilon_{p'}(x'_n) < \infty$, $\alpha_p(y_n) < \infty$. With the norm $\nu^p(T) := \inf \varepsilon_{p'}(x'_n)\alpha_p(y_n)$ the class N^p is a normed operator ideal. This is almost dual ideal to the ideal N_p; the ideal, which is dual to N_p, is exactly the ideal $(\mathrm{N}^p)^{\mathrm{reg}}$ (see [10] for notation), and $T \in \mathrm{N}_p^{\mathrm{dual}}(X,Y)$ iff $T \in \mathrm{N}^p(X, Y^{**})$. Let us give other characteristics of the operators T from $\mathrm{N}^p(X,Y)$. An operator T is an N^p-operator iff it factors in the following way:

$$X \xrightarrow{A} l_{p'} \xrightarrow{\Delta} l_1 \xrightarrow{B} Y,$$

where Δ is a diagonal operator with a diagonal from l_p, A and B are the operators of norm 1; moreover, $\nu^p(T)$ is just the norm of the diagonal in l_p.

Let us say that an operator $T \in \mathrm{L}(X,Y)$ *belongs to the ideal* I^p, if it admits a factorization of the kind

$$X \xrightarrow{A} L_{p'}(\mu) \overset{j}{\hookrightarrow} L_1(\mu) \xrightarrow{B} Y,$$

where μ is a probability measure, j is the identity injection, A and B are continuous operators. We put $i^p(T) = \inf \|A\|\|B\|$, where the inf is taken over all the factorizations of T of the mentioned kind.

Now, finally, some important words on the notions of the Banach tensor products. We consider, mainly, the tensor norms on the tensor products of the kind $X^* \otimes Y$. In this case, the tensor product $X^* \otimes Y$ can be identified naturally with the linear space of all finite dimensional operators from X to Y. On the class $\mathfrak{T}$ of all such tensor products there is a maximal (the strongest) and a minimal (the weakest) tensor norms [3]. The strongest tensor norm ν_1^0 on $X^* \otimes Y$ generates (after completion with respect to this norm) the *projective tensor product of Grothendieck* $X^* \widehat{\otimes} Y$ [3], and the weakest one — $\|\cdot\|$ — *injective tensor product* $X^* \widehat{\widehat{\otimes}} Y$, which can be considered as the completion of the linear space of all finite dimensional operators from X to Y, equipped with the usual operator norm $\|\cdot\|$. Thus, $X^* \widehat{\widehat{\otimes}} Y$ can be identified with the closed linear subspace of $\mathrm{L}(X, Y)$. Therefore, for any tensor norm α between ν_1^0 and $||\cdot||$, the natural mapping $X^* \otimes_\alpha Y \to \mathrm{L}(X, Y)$ can be extended to the *canonical map* from $X^* \widehat{\otimes}_\alpha Y$ to $\mathrm{L}(X, Y)$.

Let us give the main examples of the tensor products we will working with.

The finite p-nuclear tensor norm $\|\cdot\|_p$ for $p \in [1, +\infty]$ is defined on the product $X \otimes Y$ by the following way: if $z \in X \otimes Y$, then

$$(**) \qquad \|z\|_p := \inf\left(\sum_{k=1}^N \|x_k\|^p \right)^{1/p} \sup_{\|y'\| \leqslant 1} \left\{ \left(\sum_{k=1}^N |\langle y_k, y' \rangle|^{p'} \right)^{1/p'} \right\},$$

where $1/p + 1/p' = 1$ and the infimum is taken over all representations of the tensor element z in the space $X \otimes Y$ in the form $z = \sum_{k=1}^N x_k \otimes y_k$ (formally, (**) has sense only for finite exponents $p > 1$, and for the case $p = 1$ and $p = +\infty$, the definition have to be modified). The completion of the tensor product $X \otimes Y$ with respect to the norm $\|\cdot\|_p$, $1 \le p \leqslant \infty$, is denoted by $X \widehat{\otimes}_p Y$. When $p \in [1, \infty]$, the completion of the tensor product $X \otimes Y$ with respect to the dual to $\|\cdot\|_p$ norm is denoted by $X \widehat{\otimes}^p Y$.

If $p \in [1, \infty]$, then the conjugate space to the tensor product $X \widehat{\otimes}_p Y$ is equal to $\Pi_{p'}(Y, X^{**})$ (with the natural duality defined by trace), and the conjugate space to the tensor product $X\widehat{\otimes}^p Y$ is identified with $\Pi^d_{p'}(Y,X^{**})$ (again, the duality are defined naturally, with the help of the trace of the superposition of a tensor element and the operator which generates a functional) All the tensor products under consideration are closely related to corresponding operator ideals. For example, for any Banach space X and Y and for each $p\geq 1$, the space $\mathrm{N}_p(X,Y)$ is a factor-space of the tensor product $X^*\widehat{\otimes}_p Y$ and $\mathrm{N}^p(X,Y)$ is a factor-space of the tensor product $X^*\widehat{\otimes}^p Y$.

§2. Topological aspect of π approximation.

Now, let us consider in the space $\Pi^d_p(Y,X)$ a *topology* τ^d_p *of* π^d_p*-compact convergence*, a local base of which (in zero) is defined by sets of type

$$\omega_{K,\varepsilon}=\left\{U\in\Pi^d_p(Y,X):\ \pi^d_p(U\Phi_K)<\varepsilon\right\},$$

where $\varepsilon>0, K=\overline{\Gamma(K)}$ is a compact absolutely convex subset of Y.

Theorem 1. *Let* R *be a linear subspace in* $\Pi^d_p(Y,X)$, *containing* $Y^*\otimes X$. *Then* $(\mathrm{R},\tau^d_p)'$ *is isomorphic to a factor space of the space* $X^*\widehat{\otimes}^{p'}Y$. *More precisely, if* $\varphi\in(\mathrm{R},\tau^d_p)'$, *then there exists an element* $z=\sum_1^\infty x'_n\otimes y_n\in X^*\widehat{\otimes}^{p'}Y$ *such that*

$$(*)\qquad \varphi(U)=\mathrm{trace}\ U\circ z,\ \ U\in\mathrm{R}.$$

On the other hand, for every $z\in X^*\widehat{\otimes}^{p'}Y$ *the relation* $(*)$ *defines a linear continuous functional on* (R,τ^d_p).

Proof. Let φ be a linear continuous functional on (R,τ^d_p). Then one can find a neighborhood of zero $\omega_K=\omega_{K,\varepsilon}$, such that φ is bounded on it: $\forall\, U\in\omega_K,\ |\varphi(U)|\leqslant 1$. We may assume that $\varepsilon=1$. Consider the operator $U\Phi_K:\ Y_K\xrightarrow{\Phi_K}Y\xrightarrow{U}X$. Since the mapping Φ_K is compact, $U\Phi_K\in\mathrm{QN}^d_p(Y_K,X)$. Put $\varphi_K(U\Phi_K)=\varphi(U)$ for $U\in\mathrm{R}$. On the linear subspace $\mathrm{R}_K=\left\{V\in\mathrm{QN}^d_p(Y_K,X):\ V=U\Phi_K\right\}$ of the space $\mathrm{QN}^d_p(Y_K,X)$, the linear functional φ_K is bounded: if $V=U\Phi_K\in\mathrm{R}_K$ and $\pi^d_p(V)\leqslant 1$, then $|\varphi_K(V)|=|\varphi(U)|\leqslant 1$. Therefore, φ_K can be extended to a linear continuous functional $\widetilde{\varphi}$ on the whole $\mathrm{QN}^d_p(Y_K,X)$; moreover, because of the surjectivity of the ideal

QN_p^d, considering Y_K as a factor-space of some space $L = l_1(\Gamma)$, we may assume that $\widetilde{\varphi} \in \mathrm{QN}_p^d(L, X)^*$. Let us mention that

$$\widetilde{\varphi}(U\Phi_K l) = \varphi_K(U\Phi_K) = \varphi(U) \tag{1}$$

(here l is a factor map from L onto Y_K).

Furthermore, since $\mathrm{QN}_p^d(L, X)^* = \mathrm{I}^{p'}(X, L^{**})$, we can find an operator $\Psi : X \to L^{**}, \Psi : X \xrightarrow{A} L_p \xrightarrow{i} L_1 \xrightarrow{B} L^{**}$ (here i is an injection), for which

$$\widetilde{\varphi}(A) = \mathrm{trace}\ \Psi A,\ \ A \in L^* \otimes X.$$

Let $A_n \in L^* \otimes X,\ \pi_p^d(A_n - U\Phi_K l) \to 0$. Then

$$\widetilde{\varphi}(U\Phi_K l) = \lim \mathrm{trace}\ \Psi A_n. \tag{2}$$

Consider the operator $\Phi_K^{**} l^{**}\Psi : X \xrightarrow{A} L_p \xrightarrow{i} L_1 \xrightarrow{B} L^{**} \xrightarrow{l^{**}} (Y_K)^{**} \xrightarrow{\Phi_K^{**}} Y$. Since $i \in \mathrm{I}^{p'}$ and Φ_K is compact then $\Phi_K^{**} l^{**}\Psi \in X^*\widehat{\otimes}^{p'} Y$. Let $\sum_{n=1}^{\infty} \mu_n \otimes y_n$ be a representation of the operator $\Phi_K^{**} l^{**} Bi$ in $L_p^*\widehat{\otimes}^{p'} Y$. Put $z = \sum A^*(\mu_n) \otimes y_n$. The element z generates an operator $\Phi_K^{**} l^{**}\Psi$ from X to Y. We will show that trace $U \circ z = \widetilde{\varphi}(U\Phi_K l)$ (note that $U \circ z$ is an element of the space $X^*\widehat{\otimes} X$, so the trace is well defined). We have:

$$\begin{aligned} \mathrm{trace}\ U \circ z &= \mathrm{trace} \left(\sum A^*(\mu_n) \otimes Uy_n\right) = \sum \langle A^*(\mu_n), Uy_n\rangle = \\ &= \sum \langle \mu_n, AUy_n\rangle = \mathrm{trace}\ AU\Phi_K^{**} l^{**} Bi = \mathrm{trace}\,(U\Phi_K)^{**} l^{**}\Psi, \end{aligned} \tag{3}$$

where $(U\Phi_K)^{**} l^{**}\Psi : X \xrightarrow{\Psi} L^{**} \xrightarrow{l^{**}} Y^{**} \xrightarrow{\Phi_K^{**}} Y \xrightarrow{U} X$. Since $\pi_p^d(A_n - U\Phi_K l) \to 0$, then $\pi_p^d\,(A_n^{**} - (U\Phi_K l)^{**}) \to 0$. Moreover, if $A_n = \sum_1^N w_m \otimes f_m \in L^* \otimes X$, then

$$\mathrm{trace}\ A_n^{**}\Psi = \sum_m \langle \Psi^* w_m, f_m\rangle = \sum_m \langle w_m, \Psi f_m\rangle = \mathrm{trace}\ \Psi A_n.$$

Hence, $\mathrm{trace}\,(U\Phi_K l)^{**}\Psi = \lim \mathrm{trace}\ A_n^{**}\Psi = \lim \mathrm{trace}\ \Psi A_n$. Now, it follows from (3) and (2) that $\widetilde{\varphi}(U\Phi_K l) = \mathrm{trace}\ U \circ z$. Finally, from (1): $\varphi(U) = \mathrm{trace}\ U \circ z$. Thus, the functional φ is defined by an element of $X^*\widehat{\otimes}^{p'} Y$.

Inversely, if $z \in X^*\widehat{\otimes}^{p'} Y$, put $\varphi(U) = \mathrm{trace}\ U \circ z$ for $U \in \mathrm{R}$ (the trace is defined since $U \circ z \in X^*\widehat{\otimes} X$). We have to show that the linear functional φ is bounded on a neighborhood $\omega_{K,\varepsilon}$ of zero in τ_p^d. For this, we need

Lemma 1. *If $z \in X^*\widehat{\otimes}^q Y$, then $z \in X^*\widehat{\otimes}^q Y_K$, where $K = \overline{\Gamma(K)}$ is a compact in Y.*

Proof of the lemma. Let $z = \sum x'_n \otimes y_n$, $\{c_n\} \in c_0$ and $\sum \|y_n\|^q\, c_n^{-1} < +\infty$. Consider the operator $A_1 : l_{q'} \to l_1$, $A_1\{a_n\} = \sum a_n c_n^{-1}\, ||y_n||\, e_n$ (e_n are orths in l_1). Let $A_0 : l_1 \to Y$, $A_0\{b_n\} = \sum b_n c_n ||y_n||^{-1} y_n$. Since this operator is compact, one can find a compact $K \subset Y$ and an operator $A_2 : l_1 \to Y_K$, for which $A_0 = \Phi_K A_2$. Put $z_0 = \sum x'_n \otimes e_n$ (e_n are orths in l_q). Then $\overline{z} := (\mathbf{1} \otimes [A_2 A_1])(z_0) \in X^*\widehat{\otimes}^q Y_K$, and $\Phi_K(\overline{z}) = z$.

Let us continue the proof of Theorem. Let K be a compact in Y, with $z \in X^*\widehat{\otimes}^{p'} Y_K$. If $U \in \omega_{K,1}$, then $\pi_p^d(U\Phi_K) < 1$ and $|\operatorname{trace} U\Phi_K \circ z| \leqslant \|z\|_{X^*\widehat{\otimes}^{p'} Y_K}\, \pi_p^d(U\Phi_K) \leqslant C$.

Corollary 1. $(\mathrm{R}, \tau_p^d)' = (\mathrm{R}, \sigma)'$, *where* $\sigma = \sigma(\mathrm{R}, X^*\widehat{\otimes}^{p'} Y)$. *Thus, the closures of convex subsets of the space* $\Pi_p^d(Y, X)$ *in* τ_p^d *and in* σ *are the same.*

Denote by $X^*\widetilde{\otimes}^{p'} Y$ the closure of the set $X^* \otimes Y$ in the space $\mathrm{I}^{p'}(X, Y^{**})$ (dual to $Y^*\widehat{\widehat{\otimes}}^p X$ — the closure of $Y^* \otimes X$ in the space $\mathrm{QN}_p^d(Y, X)$)

Theorem 2. *Let A be the intersection of the unit ball of the space $G = G(Y, X^{**})$ (dual to $X^*\widetilde{\otimes}^{p'} Y$) with the subspace $Y^* \otimes X$. $*$-weak closure of the set A in $G \cap \Pi_p^d(Y, X)$ coincides with the closure of A in $(\Pi_p^d(Y, X), \tau_p^d)$.*

Proof. Let us consider the canonical mappings

$$\begin{array}{ccc} X^*\widehat{\otimes}^{p'} Y & \xrightarrow{\ j\ } & X^*\widetilde{\otimes}^{p'} Y, \\ \Pi_p^d(Y, X^{**}) & \xleftarrow{\ j^*\ } & G(Y, X^{**}). \end{array}$$

Since j^* is one-to-one, then the closures of the bounded sets in $G(Y, X^{**})$, in topologies $\sigma(G, X^*\widetilde{\otimes}^{p'} Y)$ and $\sigma(G, X^*\widehat{\otimes}^{p'} Y)$, are the same. Therefore, if B is a convex bounded set in $G \cap \Pi_p^d(Y, X)$, then the closure of the set B in $\left(\Pi_p^d(Y, X) \cap G, \sigma(G, X^*\widetilde{\otimes}^{p'} Y)\right)$ coincides with the closure of the set B in the space $\left(\Pi_p^d(Y, X), \sigma(\Pi_p^d(Y, X), X^*\widehat{\otimes}^{p'} Y)\right)$ and therefore, by Corollary 1, with the closure of B in $\left(\Pi_p^d(Y, X), \tau_p^d\right)$.

Corollary 2. *With notations of the theorem 2, the closure of the set A in τ_p coincides with the closure of A in the space $\mathrm{L}(Y, X)$ in the topology of compact convergence.*

For the *proof* it is enough to use the previous assertion, considering the canonical mapping from $X^*\widehat{\otimes}_1 Y$ into $X^*\widetilde{\otimes}^{p'} Y$ instead of the map j from the proof of the theorem 2 (and to apply either Theorem 1 for $p=+\infty$, or results on dualities from [3]).

Corollary 3. *Let $C>0$ and $T\in \Pi_p^d(Y,X)$. The following assertions are equivalent:*

1) *there ia a net $\{T_\alpha\}, T_\alpha\in Y^*\otimes X$, converging to T in the topology τ_p^d such that $\pi_p^d(T_\alpha)\leqslant C$;*

2) *there is a net $\{T_\alpha\}, T_\alpha\in Y^*\otimes X$, converging to T in the topology of compact convergence, such that $\pi_p^d(T_\alpha)\leqslant C$;*

3) *the canonical mapping*

$$X^*\widehat{\otimes}^{p'} Y \xrightarrow{\ \ j\ \ } X^*\widetilde{\otimes}^{p'} Y$$

is C-isometric.

Theorem 3. *For an operator $T\in\Pi_p^d(Y,X), \overline{T(Y)}=X$, the following are the same:*

1) $T\in\overline{Y^*\otimes X}^{\tau_p}$;

2) *there is a net of operators $R_\alpha\in Y^*\otimes Y$ such that $TR_\alpha\to T$ in the topology τ_p^d.*

Proof. Assuming that 2) is not valid, we (by Corollary 1) get:

$$\nexists\, R_\alpha\in Y^*\otimes Y:\ TR_\alpha\to T\ \text{ in }\ \left(\Pi_p^d(Y,X), \sigma(\Pi_p^d(Y,X), X^*\widehat{\otimes}^{p'} Y)\right). \tag{4}$$

Consider the associated with T mappings:

$$\begin{array}{ccc} X^*\widehat{\otimes}^{p'} Y & \xrightarrow{\ \ \widetilde{T}\ \ } & Y^*\widehat{\otimes}_1 Y, \\ \Pi_p^d(Y,X^{**}) & \xleftarrow{\ \ \widetilde{T}^*\ \ } & \mathrm{L}(Y,Y^{**}). \end{array}$$

where $\widetilde{T}(z)=z\circ T$ for $z\in X^*\widehat{\otimes}^{p'} Y$. Let $Z=\overline{\widetilde{T}^*(Y^*\otimes Y)}^{\,*}$ (the closure is taken in the space $\Pi_p^d(Y,X)$ in *-weak topology). It follows from (4) that T is not zero on the subspace $Z^\perp\subset X^*\widehat{\otimes}^{p'} Y$, i.e. there exists an $A\in Z^\perp$ such that $\langle T,A\rangle=\operatorname{trace} AT=1$. But $AT=\widetilde{T}(A)$, and if $R\in Y^*\otimes Y$, then $\langle\widetilde{T}(A),R\rangle=\langle A,(\widetilde{T})^*(R)\rangle=0$. Hence, the element $\widetilde{T}(A)$ of the projective tensor product $Y^*\widehat{\otimes} Y$ is not zero (since trace $\widetilde{T}(A)=1$), but generates a null-operator in Y. For any $y'\in Y^*$ and $Ty\in T(Y)$ we have:

$\langle A, Ty \otimes y' \rangle = \langle ATy, y' \rangle = 0$. Since $\overline{T(Y)} = X$, we obtain that a non-zero tensor element $A \in X^* \widehat{\otimes}^{p'} Y$ which generates a zero-operator. Again, by using the equality trace $AT = 1$, we conclude that T can not be approximated in *-weak topology by finite rank operators. Now, it follows from Corollary 1 that the condition 1) is not fulfilled.

Next two statements give us sufficient (but not necessary, as we will see below) conditions for the density of the set of all finite rank operators in the space of operators $\Pi_p^d(Y,X)$ in the topology τ_p^d of π_p^d-compact convergence.

Theorem 4. *If* $\mathrm{QN}_p^d(Y,X) = \overline{Y^* \otimes X}^{\pi_p^d}$ *for every Banach space* Y*, then for each* Y $\Pi_p^d(Y,X) = \overline{Y^* \otimes X}^{\tau_p^d}$.

Proof. Let $U \in \Pi_p^d(Y,X)$, $\varepsilon > 0$, $K = \overline{\Gamma(K)}$ be a compact in Y. By assumptions, there is an operator $V \in (Y_K)^* \otimes X$, such that $\pi_p^d(V - U\Phi_K) < \varepsilon$. We need to change successfully V by an operator $\widetilde{V}\Phi_K$, where $\widetilde{V} \in Y^* \otimes X$. Let $V = \sum_{n=1}^N z_n \otimes x_n$. Note that we can consider only the case where Φ_K^{**} is one-to-one (else, with the help of the construction of [2] we change Y_K by a space Y_{K_0}, for which the operator Φ_{K_0} is compact and the operator $\Phi_{K_0}^{**}$ is one-to-one). In this case Y^* is norm dense in $(Y_K)^*$ and, therefore, for every positive number sequence $\{\varepsilon_n\}$ there exist the elements $y_n' \in Y^*$, for which $\|y_n' - z_n\|_{Y_K^*} < \varepsilon_n$. Put $\widetilde{V} = \sum_1^N y_n' \otimes x_n \in Y^* \otimes X$. Let $\{a_n\}$ be a sequence of the elements of the space X^*, such that $\sup \{\sum |\langle a_n, a' \rangle|^p : \ \|a'\|_X \leqslant 1\} \leqslant 1$. We have:

$$\sum_{i=1}^m \|(\widetilde{V}\Phi_K - V)^* a_i\|^p = \sum_{i=1}^m \| \sum_{n=1}^N (y_n' - z_n) \langle a_i, x_n \rangle \|^p \leqslant$$

$$\sum_{i=1}^m \left(\sum_{n=1}^N \|y_n' - z_n\|^{p'} \right)^{p/p'} \sum_{n=1}^N |\langle x_n, a_i \rangle|^p \leqslant$$

$$\sum_{n=1}^N \|x_n\|^p \left(\sum_{n=1}^N \|y_n' - z_n\|_{Y_K^*}^{p'} \right)^{p/p'} \leqslant \sum_{n=1}^N \|x_n\|^p \left(\sum_{n=1}^N \varepsilon_n^{p'} \right)^{p/p'} .$$

If we take ε_n small enough then the last number is less then ε, and, from the inequality $\pi_p\left((V - U\Phi_K)^*\right) \leqslant \varepsilon$, we get that $\pi_p^d(\widetilde{V}\Phi_K - U\Phi_K) \leqslant \varepsilon + \pi_p^d(V - \widetilde{V}\Phi_K) \leqslant 2\varepsilon$. Hence, $\widetilde{V} - U \in \omega_{K,2\varepsilon}$. Thus, we have shown that for every neighborhood $\omega_{K,\varepsilon}$ there exists an operator $\widetilde{V} \in Y^* \otimes X$, for which $\widetilde{V} - U \in \omega_{K,\varepsilon}$. Therefore $U \in \overline{Y^* \otimes X}^{\tau_p^d}$.

Theorem 5. *If the canonical mapping* $j : X^*\widehat{\otimes}^{p'} Y \to \mathrm{N}^{p'}(X,Y)$ *is one-to-one then* $\Pi_p^d(Y,X) = \overline{Y^*\otimes X}^{\tau_p^d}$.

Proof. If the map j is one-to-one then the annihilator $j^{-1}(0)^\perp$ of its kernel in the space, dual to $X^*\widehat{\otimes}^{p'} Y$, coincides with $\Pi_p^d(Y,X^{**})$. On the other hand, in any case $j^{-1}(0)^\perp = \overline{Y^*\otimes X}^*$ (the closure in *-weak topology of the space $\Pi_p^d(Y,X^{**})$); by Corollary 1,

$$\Pi_p^d(Y,X) \cap \overline{Y^*\otimes X}^* = \overline{Y^*\otimes X}^{\tau_p^d}.$$

Therefore, $\Pi_p^d(Y,X) = \overline{Y^*\otimes X}^{\tau_p^d}$.

For a reflexive space X, the dual space to $X^*\widehat{\otimes}^{p'} Y$ is equal to $\Pi_p^d(Y,X)$. Consequently, by Corollary 1 and Theorem 5, we get

Corollary 4. *For a reflexive pace* X *the canonical mapping* $j : X^*\widehat{\otimes}^{p'} Y \to \mathrm{N}^{p'}(X,Y)$ *is one-to-one iff the set of finite rank operators is dense in the space* $\Pi_p^p(Y,X)$ *in the topology* τ_p^d *of* π_p^d*-compact convergence.*

To conclude this part of our considerations, let us give an assertion which shows that it follows from the existing of an dually absolutely p-summing operator, non-approximated in the topology τ_p^d, the existing of a non-approximated in the same topology dually quasi-p-nuclear operator (with values in the same space).

Theorem 6. *If there exists an operator* $T \in \Pi_p^d(Y,X) \setminus \overline{Y^*\otimes X}^{\tau_p^d}$, *then there exist a reflexive space* Z *and an operator* $U \in \mathrm{QN}_p^d(Z,X)$, *which is not in the closure* $\overline{Y^*\otimes X}^{\tau_p^d}$.

For the *proof*, it is enough to remember the definition of the topology τ_p^d and to use the following two facts:

a) if V is a compact operator then it can be represented as a composition of two compact operators;

b) the product AB of a compact operator B and dually absolutely p-summing operator A is a dually quasi-p-nuclear map (concerning a), see [4] ; b) is a evident).

Now, we fix the space of images of operators (or the spaces where operators are defined), and investigate the conditions under which it is possible to approximate (by finite rank operators) *all* dually absolutely p-summing mappings with values in a given space (or acting from a given space).

The next statement, among other things, gives us a partial inversion of Theorem 4.

Theorem 7. *For a Banach space X the following are equivalent:*

1) *for every Banach space Y the equality* $\mathrm{QN}_p^d(Y,X) = \overline{Y^*\otimes X}^{\pi_p^d}$ *holds;*

2) *for every Banach space Y the equality* $\Pi_p^d(Y,X) = \overline{Y^*\otimes X}^{\tau_p^d}$ *holds;*

3) *for every Banach space Y the equality* $\mathrm{QN}_p^d(Y,X) = \overline{Y^*\otimes X}^{\tau_p^d}$ *holds;*

4) *for every reflexive Banach space Y the equality* $\mathrm{QN}_p^d(Y,X) = \overline{Y^*\otimes X}^{\tau_p^d}$ *holds.*

Proof. Implications 2) $\implies$ 3) $\implies$ 4) are evident; 1) $\implies$ 2) by Theorem 4. For the proof of the implication 4) $\implies$ 1), consider an operator $V\in \mathrm{QN}_p^d(Y,X)$. By using the results of [2], we can factorize the operator V by the following way: $V = UA$, where $A\in \mathrm{K}(Y,Z)$ (compact), $U\in \mathrm{QN}_p^d(Z,X)$, and, moreover, the space Z is reflexive. Put $K=\overline{A(B_Y)}$. From 4), it follows that

$$\forall\varepsilon>0\ \exists\, \widetilde{V}\in Z^*\otimes X:\quad \widetilde{V}-U\in \omega_{K,\varepsilon},$$

i.e. $\pi_p^d(\widetilde{V}\Phi_K - U\Phi_K)<\varepsilon$. Hence, $\pi_p^d(V-\widetilde{V}A)\leqslant C_A\,\pi_p^d(U\Phi_K-\widetilde{V}\Phi_K)\leqslant C_A\varepsilon$.

Corollary 5. *If for every reflexive space Y the canonical mapping $X^*\widehat{\otimes}^{p'}Y\to \mathrm{N}^{p'}(X,Y)$ is one-to-one then for each Banach space Y the equality* $\mathrm{QN}_p^d(Y,X) = \overline{Y^*\otimes X}^{\pi_p^d}$ *holds.*

For the *proof*, Theorem 5 can be applied; then use the implication 2) $\implies$ 1) of the previous result.

It follows from Corollaries 4 and 5 and Theorem 7.

Corollary 6. *For a reflexive Banach space X the following are equivalent:*

1) *for each space Y the canonical mapping $X^*\widehat{\otimes}^{p'}Y\to \mathrm{N}^{p'}(X,Y)$ is one-to-one;*

2) *for each space Y the set of finite rank operators is dense in the Banach space* $\mathrm{QN}_p^d(Y,X)$.

The next statement, among other things, gives us a partial inversion of Theorem 5.

Corollary 7. *For every Banach space X the following are equivalent:*

1) *for each space Y the canonical mapping $Y^*\widehat{\otimes}^p X \to \mathrm{N}^p(Y,X)$ is one-to-one;*

2) *for each reflexive Banach space Y the canonical mapping $Y^*\widehat{\otimes}^p X \to \mathrm{N}^p(Y,X)$ is one-to-one;*

3) *for each (reflexive) Banach space Y $\overline{X^* \otimes^{p'} Y}^{\tau^d_{p'}} = \Pi^d_{p'}(X,Y)$.*

Proof. Concerning the equivalence 1) $\iff$ 2), see [18]; the implications 1) $\implies$ 3) and 3) $\implies$ 2) follow from Theorem 5 and Corollary 4, respectively.

The previous statement yields the well known result:

Corollary 8 (P. Saphar, O.I. Reinov)**.** *If a Banach space X has the AP then for any $p>1$ and any space Y the canonical mapping $Y^*\widehat{\otimes}^p X \to \mathrm{N}^p(Y,X)$ is one-to-one.*

The above results lead us to the following definition which is equivalent to corresponding definition, e.g., in [17],[18].

Definition 1. Let $p \geqslant 1$. A Banach space X has the property AP^p (the approximation property of order p^{dual}), if every dually absolutely p'-summing operator, acting from the space X, can be approximated in the topology of $\pi^d_{p'}$-compact convergence $\tau^d_{p'}$ by operators of finite rank.

It follows from Theorem 4 and Corollary 7 that

Corollary 9. *If for each Banach space Y the equality $\mathrm{QN}^d_{p'}(X,Y) = \overline{X^* \otimes Y}^{\pi^d_{p'}}$ holds then the space X has the property AP^p in the sense of Definition 1.*

Now, let us consider some counterexamples in connection with the theme of our investigations (a lot of counterexamples concerning AP's can be found in [5], [7], [11–20]. Bellow, we make use of results of [12], [18]. Namely: for any $q \geqslant 1$, $q \neq 2$, there exists a separable reflexive Banach space without the property AP^q . Moreover, for every $q \geqslant 1$, $q \neq 2$, there exist a separable reflexive E, an operator $R \in \Pi^d_{q'}(E,E)$ and a tensor element $t \in E^*\widehat{\otimes}^q E$, so that trace $t \circ R = 1$ and trace $t \circ A = 0$ for each finite rank operator $A \in E^* \otimes E$.

It is often very useful to apply the following fact when constructing some counterexamples ([6]): *For every separable Banach space E there exist a separable conjugate Banach space $H = Y^*$ with a basis and operators $Q : H \to E$ and $U : H^* \to E^*$ so that $Q(H) = E$, $U(H^*) = E^*$, $\|U\| \leqslant 1$, $\|Q\| \leqslant 1$, $UQ^* = \mathrm{id}_{E^*}$ and the space $(\mathrm{id}_{E^*} - Q^*U)(H^*)$ is isomorphic to the space Y.*

Now we are ready for constructions of our counterexamples. Firstly, we will show that the inversion of Theorem 5 and Corollary 5 are not valid.

Theorem 8. *For every $p \in [1, +\infty]$, $p \neq 2$, there exist a separable reflexive space E, a separable conjugate space H with a basis such that the canonical mapping $j : H^* \widehat{\otimes}^{p'} E \to \mathrm{N}^{p'}(H, E)$ is not one-to-one. On the other hand, since H has the Grothendieck AP, $\mathrm{QN}_p^d(E, H) = \overline{E^* \otimes H}^{\pi_p^d}$ and $\Pi_p^d(E, H) = \overline{E^* \otimes H}^{\tau_p^d}$.*

Proof. Let E, $t \in E^* \widehat{\otimes}^{p'} E$ and $R \in \Pi_p^d(E, E)$ be the mentioned above spaces, tensor element and operator so that trace $t \circ R = 1$ and $t = 0$ as an operator. Set $g = t \circ Q \in H^* \widehat{\otimes}^{p'} E$ and $V = U^* R \in \mathrm{QN}_p^d(E, H^{**})$. Then trace $V \circ g = 1$ (consequently, the tensor element g is not equal to zero) and $g = 0$ as an operator.

Corollary 10. *For every $p \in [1, +\infty]$, $p \neq 2$, there exist a Banach space Z, an element $z \in Z^* \widehat{\otimes}^{p'} Z$ and an operator $\Psi \in \Pi_p^d(Z, Z^{**})$ such that* trace $\Psi \circ z = 1$, *but* trace $\Phi \circ z = 0$ *for all $\Phi \in \Pi_p(Z, Z)$.*

Proof. Let us use notation introduced in the proof of Theorem 8. Let $\sum y_n' \otimes y_n$ be any representation of a tensor element $g \in H^* \widehat{\otimes}^{p'} E$. Note that trace $A \circ g = 0$ for every operator $A \in \Pi_p^d(E, H)$ (because of the space H has AP). Put $Z = H \oplus E$, $z = \sum (y_n', 0) \otimes (0, y_n)$ and define the operator $\Psi \in \mathrm{QN}_p^d(Z, Z^{**})$ by $\Psi(h, y) = (Vy, 0)$. We have:

$$\operatorname{trace} \Psi \circ z = \sum \langle (y_n', 0), (Vy_n, 0) \rangle = \sum \langle y_n', Vy_n \rangle = \operatorname{trace} V \circ g = 1.$$

On the other hand, denoting by P_H and P_E the natural projectors from Z onto H and E respectively, we have, for arbitrary operator $\Phi \in \Pi_p^d(Z, Z)$,

$$\Phi(h,y) = \Phi|_H(h) + \Phi|_E(y) = (P_H\Phi|_H(h) + P|_E\Phi|_H(h)) + (P_H\Phi|_E(y) + P|_E\Phi|_E(y)),$$

whence,

$$\text{trace } \Phi\circ z = \sum\langle (y'_n,0), [P_H\Phi(0,y_n)]\rangle = \sum\langle (y'_n,0), P_H\Phi|_E(y_n)\rangle.$$

Denoting by A the operator $P_H\Phi|_E$, we get:

$$A\in \Pi_p^d(E,H);\quad \text{trace } \Phi\circ z = \text{trace } A\circ g = 0.$$

Remark 1. For $p=+\infty$, we get again (cf. [15]) nonzero tensor element $z\in Z^*\widehat{\otimes}_1 Z$, vanishing on the subspace $\mathrm{L}(Z,Z)$ of the space $\mathrm{L}(Z,Z^{**})$. This is an answer to a question of Swedish mathematician Sten Kaijser.

Now we will show that the inversion of Theorem 4 and Corollary 9 are invalid too.

Theorem 9. *For every $p\in[1,+\infty]$, $p\neq 2$, there exist a separable reflexive space E, a separable conjugate space H with a basis (so, with the property* $\mathrm{AP}^{p'}$*) such that $\mathrm{QN}_p^d(H,E)\neq \overline{H^*\otimes E}^{\pi_p^d}$; on the other hand, $\Pi_p^d(H,E)=\overline{H^*\otimes E}^{\tau_p^d}$* (see Corollary 7).

Proof. With the notation of Theorem 8, set $L=RQ$. Since $\text{trace}\,(t\circ RQ^{**}Y^*)=1$ and $t=0$ as an operator, the map RQ^{**} can not be approximated by finite rank operators in the space $\mathrm{QN}_p^d(H^{**},E)$. Moreover, $L\not\subset \overline{H^*\otimes E}^{\pi_p^d}$.

We finish the paper with an application of the above results to a solution of A. Grothendieck's problem, posed him in [3, Chap. 2, p. 135, "Question non Résolues"]. A. Grothendieck had showed that for Banach spaces with weakly compact identity maps (for reflexive spaces) the approximation and the bounded approximation properties are the same. This led him to the following question: *whether any weakly compact operator with AP possesses also BAP*? As to this question on weakly compact operators, it is a paper of mine [14], where the first counterexample to this problem was constructed. There was shown ("by hands") that there is a compact operator (from a Banach space with AP), that can not be approximated, in the topology of the compact convergence, by finite rank operators with uniformly bounded norms.

The next striking enough result (proved early by me by using a rather different and difficult method) shows that such operators may be lie even in the classes of dually quasi-p-nuclear operators (which are evidently compact).

For the proof, we need the following assertion from my works [17] and [18, Theorem 4.21,(1)]: there exist a separable space E with the property AP, a separable reflexive space F such that for every $q \in [1, 2)$ the canonical mapping

$$F^* \widehat{\otimes}^q E \xrightarrow{j} F^* \widetilde{\otimes}^q E$$

is one-to-one but is not C-isometric, for any $C > 0$.

Theorem 10. *There exist a separable space Y with the property* AP, *a separable reflexive space X such that for every $p \in (2, +\infty]$ there exists an operator $T \in \mathrm{QN}_p^d(Y, X)$ for which:*

for any constant $C > 0$, there is no net $\{T_\alpha\}, T_\alpha \in Y^* \otimes X$, converging to T in the topology of compact convergence, and such that $\pi_p^d(T_\alpha) \leqslant C$;

Proof. Put $X := F$, $Y := E$ and $p = q'$ and apply just mentioned result from [18] and Corollary 3 (note that for a reflexive X one has $\Pi_p^d(Y, X) = \mathrm{QN}_p^d(Y, X)$; see [8], [18]).

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