Qualitative properties of solutions to elliptic and parabolic equations with unbounded lower-order coefficients

A.I. Nazarov, N.N. Ural'tseva[*],
Saint-Petersburg University,
e-mail: al.il.nazarov(at)gmail.com

1 Introduction

Qualitative theory of partial differential equations is in intensive development over last half of century. We discuss four classical and frequently used properties of solutions to elliptic and parabolic equations:
— strong maximum principle;
— the Harnack inequality;
— the Liouville theorem;
— the Hopf lemma.

For the Laplace operator these properties are known more then one hundred years and read as follows:

1°. Strong maximum principle:

$$-\Delta u = f \geq 0 \ \text{ in } \ \Omega; \qquad M := \inf_{\Omega} u = u(x^0), \quad x^0 \in \Omega \quad \Longrightarrow \quad u \equiv M.$$

2°. The Harnack inequality:

$$\Delta u \equiv 0, \quad u \geq 0 \ \text{ in } \ \Omega; \qquad \Omega' \Subset \Omega \quad \Longrightarrow \quad \sup_{\Omega'} u \leq N(\Omega, \Omega') \cdot \inf_{\Omega'} u.$$

3°. The Liouville theorem:

$$\Delta u \equiv 0, \quad u \geq -M \ \text{ in } \ \mathbb{R}^n \quad \Longrightarrow \quad u \equiv const.$$

4°. The Hopf lemma:

$$-\Delta u = f \geq 0 \ \text{ in } \ \Omega; \quad \partial\Omega \in \mathcal{C}^2; \quad x^0 \in \partial\Omega; \quad u > u(x^0) \ \text{ in } \ \Omega; \quad \Longrightarrow \quad \frac{\partial u}{\partial \mathbf{n}}(x^0) < 0.$$

We consider generalized solutions for elliptic and parabolic equations in divergence and nondivergence form. For *divergence* equations

$$\mathcal{L}u \equiv -D_i\big(a_{ij}(x)D_ju\big) + b_i(x)D_iu = f(x); \tag{\textbf{DE}}$$

[*]Partially supported by RFBR grant 09-01-00729 and by grant NSh.227.2008.1.

$$\mathcal{M}u \equiv \partial_t u - D_i\big(a_{ij}(x;t)D_j u\big) + b_i(x;t)D_i u = f(x;t) \tag{DP}$$

we deal with *weak* solutions. Namely, we assume $Du \in L_{2,loc}(\Omega)$ in (**DE**) and $u \in L_{2,\infty,loc}(Q)$, $Du \in L_{2,loc}(Q)$ in (**DP**).

For *nondivergence* equations

$$\mathcal{L}u \equiv -a_{ij}(x)D_iD_ju + b_i(x)D_iu = f(x); \tag{NDE}$$

$$\mathcal{M}u \equiv \partial_t u - a_{ij}(x;t)D_iD_ju + b_i(x;t)D_iu = f(x;t) \tag{NDP}$$

we assume that u is a *strong* solution, i.e. $D(Du) \in L_{n,loc}(\Omega)$ in (**NDE**) and $\partial_t u$, $D(Du) \in L_{n+1,loc}(Q)$ in (**NDP**) (in the parabolic case also some anisotropic spaces are admissible).

We always suppose that operators under consideration are uniformly elliptic (parabolic), i.e. for all values of arguments

$$\nu|\xi|^2 \le a_{ij}(\cdot)\xi_i\xi_j \le \nu^{-1}|\xi|^2, \qquad \xi \in \mathbb{R}^n, \tag{1}$$

where ν is a positive constant.

The problem of our interest is how "bad" may be lower-order coefficients b_i to ensure the properties **1°**– **4°** to hold true. The answers will be given in terms of Lebesgue spaces.

It is worth noting that all the key tools needed for our results are well known, though, to the best of our knowledge, most of final results were unknown yet. So, here we only formulate all the statements and will give the full proofs in the forthcoming survey.

Let us recall some notation. $x = (x_1,\dots,x_{n-1},x_n) = (x',x_n)$ is a vector in $\mathbb{R}^n$, $n \ge 2$, with the Euclidean norm $|x|$; $(x;t)$ is a point in $\mathbb{R}^{n+1}$.

Ω is a domain in $\mathbb{R}^n$ and $\partial\Omega$ is its boundary; $\mathbf{n} = (\mathbf{n}_i(x))$ is the unit vector of the outward normal to $\partial\Omega$ at the point x.

For a cylinder $Q = \Omega\times]-T,0]$ we denote by $\partial''Q = \partial\Omega\times]-T,0]$ its lateral surface and by $\partial'Q = \partial''Q \cup \{\overline{\Omega}\times\{-T\}\}$ its parabolic boundary.

We denote by $\Omega^+(Q^+)$ the part of Ω (of Q) lying in the halfspace $x_n > 0$.

We define $B_R(x^0) = \{x \in \mathbb{R}^n : |x-x^0| < R\}$, $\quad Q_R(x^0;t^0) = B_R(x^0)\times]t^0-R^2,t^0]$; $B_R = B_R(0)$, $\quad Q_R = Q_R(0;0)$.

The indices i,j vary from 1 to n. Repeated indices indicate summation.

The symbol D_i denotes the operator of differentiation with respect to x_i; in particular, $Du = (D_1u,\dots,D_nu) = (D'u,D_nu)$ is the gradient of u. $\partial_t u$ stands for the derivative of u with respect to t.

We denote by $\|\cdot\|_{p,\Omega}$ the norm in $L_p(\Omega)$. We introduce two scales of anisotropic spaces:

$L_{p,q}(Q) = L_q\big(]0,t[\to L_p(\Omega)\big)$ with the norm $\|f\|_{p,q,Q} = \big\|\|f(\cdot;t)\|_{p,\Omega}\big\|_{q,]-T,0[}$;

$\widetilde{L}_{p,q}(Q) = L_p\big(\Omega \to L_q(]0,t[)\big)$ with the norm $\|\!|f|\!\|_{p,q,Q} = \big\|\|f(x;\cdot)\|_{q,]-T,0[}\big\|_{p,\Omega}$.

Obviously, $L_{p,p}(Q) = \widetilde{L}_{p,p}(Q) = L_p(Q)$.

2 Operators in divergence form

2.1 Elliptic case

Recall that in this subsection we assume $Du \in L_{2,loc}(\Omega)$.

Theorem 2.1 (the Harnack inequality). *Let $\mathcal{L}$ be an operator of the form (**DE**) in a domain Ω. Let the condition (1) be satisfied, and*

$$\mathbf{b} \in L_{n,loc}(\Omega) \quad \textit{for} \quad n \geq 3; \qquad\qquad \mathbf{b} \ln^{\frac{1}{2}} |\mathbf{b}| \in L_{2,loc}(\Omega) \quad \textit{for} \quad n = 2 \tag{2}$$

(here $\mathbf{b} = (b_i)$). Then for arbitrary subdomain $\Omega' \Subset \Omega$ there exists a positive constant N_1 such that for any nonnegative solution of $\mathcal{L}u = 0$ in Ω the estimate

$$\sup_{\Omega'} u \leq N_1 \cdot \inf_{\Omega'} u. \tag{3}$$

holds true. Moreover, N_1 depends only on n, ν, Ω' and moduli of continuity of b_i in corresponding space.

Theorem 2.2 (strong maximum principle). *Let $\mathcal{L}$ be an operator of the form (**DE**) in a domain Ω. Let the conditions (1) and (2) be satisfied. Then any nonconstant solution of $\mathcal{L}u = f \geq 0$ in Ω cannot attain its minimum at interior point of Ω.*

Theorem 2.3 (the Liouville theorem). *Let $\mathcal{L}$ be an operator of the form (**DE**) in $\mathbb{R}^n$. Let the condition (1) be satisfied, and*

$$\mathbf{b} \in L_{n,loc}(\mathbb{R}^n) \quad \textit{for} \quad n \geq 3; \qquad\qquad \mathbf{b} \ln^{\frac{1}{2}} |\mathbf{b}| \in L_{2,loc}(\mathbb{R}^2) \quad \textit{for} \quad n = 2.$$

Suppose also that for some $\delta > 0$

$$\liminf_{R\to\infty} \sup_{|x|=R} \|\mathbf{b}\|_{n,B_{R\delta}(x)} < \varepsilon, \quad n \geq 3; \quad \liminf_{R\to\infty} \sup_{|x|=R} \|\mathbf{b} \ln^{\frac{1}{2}} |\mathbf{b}|\|_{2,B_{R\delta}(x)} < \varepsilon, \quad n = 2, \tag{4}$$

where $\varepsilon(n, \nu)$ is a sufficiently small constant. Then any semibounded solution of $\mathcal{L}u = 0$ in $\mathbb{R}^n$ is a constant.

Remark 1. If $\mathbf{b} \in L_n(\mathbb{R}^n)$ (respectively, $\mathbf{b} \ln^{\frac{1}{2}} |\mathbf{b}| \in L_2(\mathbb{R}^2)$), then (4) is obviously satisfied.

Remark 2. In some applications the lower-order coefficients satisfy the structure condition $\mathrm{div}(\mathbf{b}) \equiv 0$ in the sense of distributions. In this case the assumption (2) can be weakened in the scale of Morrey spaces

$$\mathbb{M}^{\alpha}_{p}(\Omega) = \{f \in L_p(\Omega) : \sup_{B_\rho(x) \subset \Omega} \rho^{-\alpha} \|f\|_{p,B_\rho(x)} < \infty\}$$

to

$$\mathbf{b} \in \mathbb{M}^{\frac{n}{p}-1}_{p,loc}(\Omega), \;\; n \geq 3; \quad \mathbf{b} \ln^{\frac{1}{p}} |\mathbf{b}| \in \mathbb{M}^{\frac{2}{p}-1}_{p,loc}(\Omega), \;\; n = 2, \qquad \text{for some} \quad \frac{n}{2} < p \leq n. \tag{5}$$

The constant N in (3) depends on n, ν, p and the norm of $\mathbf{b}$ in corresponding space. Note that in Lebesgue spaces the assumption (2) cannot be weakened.

Theorem 2.2 in this case also holds true under the assumption (5) instead of (2). Theorem 2.3 is fulfilled under the assumption

$$\mathbf{b} \in \mathbb{M}_p^{\frac{n}{p}-1}(\mathbb{R}^n), \quad n \geq 3; \quad \mathbf{b} \ln^{\frac{1}{p}} |\mathbf{b}| \in \mathbb{M}_p^{\frac{2}{p}-1}(\mathbb{R}^2), \quad n = 2, \qquad \text{for some} \quad \frac{n}{2} < p \leq n.$$

As for the statement **4°** (the Hopf lemma), it is not true for divergence equations. Let us describe corresponding counterexample.

Let Ω be a convex domain, and let $0 \in \partial\Omega$. Assume that at the neighborhood of the origin $\partial\Omega$ is the graph of a function $x_n = \omega(x')$. Finally, suppose that $\omega \in \mathcal{C}^1$ but $D'\omega$ is not Dini continuous at the origin.

It is well known that the Hopf lemma for the Laplacian fails in such domain. Now we rectify the boundary near the origin and obtain an operator of the form (**DE**) in smooth domain. This operator has even *continuous* leading coefficients and $b_i \equiv 0$, but the Hopf lemma fails. Considering functions depending only on spatial variables we see that this example works also for the parabolic operator (**DP**).

2.2 Parabolic case

In this subsection we assume $u \in L_{2,\infty,loc}(Q)$, $Du \in L_{2,loc}(Q)$.

Theorem 2.4 (the Harnack inequality). *Let $\mathcal{M}$ be an operator of the form (**DP**) in a cylinder Q. Let the condition (1) be satisfied, and*

$$\mathbf{b} \in L_{p,q,loc}(Q) \quad \textit{for some } p \textit{ and } q \textit{ satisfying} \quad q < \infty, \quad \frac{n}{p} + \frac{2}{q} = 1. \tag{6}$$

Then for arbitrary subcylinders $Q' = \Omega' \times]-T+\delta, -T']$ and $Q'' = \Omega' \times]-T'+\delta, 0]$ (here $\Omega' \Subset \Omega$, $0 < \delta < T' < T - \delta$) there exists a positive constant N_2 such that for any nonnegative solution of $\mathcal{M}u = 0$ in Q the estimate

$$\sup_{Q'} u \leq N_2 \cdot \inf_{Q''} u. \tag{7}$$

holds true. Moreover, N_2 depends only on n, ν, p, Ω', δ, T' and moduli of continuity of b_i in $L_{p,q,loc}(Q)$.

Theorem 2.5 (strong maximum principle). *Let $\mathcal{M}$ be an operator of the form (**DP**) in a cylinder Q. Let the conditions (1) and (6) be satisfied. Then any nonconstant solution of $\mathcal{M}u = f \geq 0$ in Q cannot attain its minimum at points of the "top" $\Omega \times \{0\}$.*

Theorem 2.6 (the Liouville theorem). *Let $\mathcal{M}$ be an operator of the form (**DP**) in $\mathbb{R}^n \times \mathbb{R}_-$. Let the condition (1) be satisfied, and $\mathbf{b} \in L_{p,q}(\mathbb{R}^n \times \mathbb{R}_-)$ for some p and q satisfying $q < \infty$, $\frac{n}{p} + \frac{2}{q} = 1$.*

Then any bounded solution of $\mathcal{M}u = 0$ in $\mathbb{R}^n \times \mathbb{R}_-$ is a constant.

Remark 3. If the lower-order coefficients satisfy $\operatorname{div}(\mathbf{b}) \equiv 0$ then, similarly to elliptic case, the assumption (6) can be weakened in the scale of Morrey spaces

$$\mathbb{M}^{\alpha}_{p,q}(Q) = \{f \in L_{p,q}(Q) : \sup_{Q_\rho(x,t) \subset Q} \rho^{-\alpha} \|f\|_{p,q,Q_\rho(x,t)} < \infty\}$$

to

$$\mathbf{b} \in \mathbb{M}^{\frac{n}{p}+\frac{2}{q}-1}_{p,q,loc}(Q) \quad \text{for some } p \text{ and } q \text{ satisfying} \quad 1 \le \frac{n}{p} + \frac{2}{q} < 2. \tag{8}$$

The constant N_2 in (7) depends on n, ν, p, q and the norm of $\mathbf{b}$ in corresponding space. In particular, in this case the estimate (7) holds for $\mathbf{b} \in L_{n,\infty,loc}(Q)$.

Theorem 2.5 also holds true under the assumption (8) instead of (6). Theorem 2.6 is fulfilled if $\mathbf{b} \in \mathbb{M}^{\frac{n}{p}+\frac{2}{q}-1}_{p,q}(\mathbb{R}^n \times \mathbb{R}_-)$ for some p and q satisfying $1 \le \frac{n}{p} + \frac{2}{q} < 2$.

3 Operators in nondivergence form

3.1 Elliptic case

In this subsection we assume $D(Du) \in L_{n,loc}(\Omega)$.

Strong maximum principle for (**NDE**) was established by A.D. Aleksandrov in 1961 under natural assumption $b_i \in L_{n,loc}(\Omega)$.

Theorem 3.1 (the Harnack inequality). *Let $\mathcal{L}$ be an operator of the form (*__NDE__*) in a domain Ω. Let the condition (1) be satisfied, and $\mathbf{b} \in L_{n,loc}(\Omega)$. Then for arbitrary subdomain $\Omega' \Subset \Omega$ there exists a positive constant N_1 such that for any nonnegative solution of $\mathcal{L}u = 0$ in Ω the estimate (3) holds true. Moreover, N_1 depends only on n, ν, Ω' and moduli of continuity of b_i in $L_{n,loc}(\Omega)$.*

Theorem 3.2 (the Liouville theorem). *Let $\mathcal{L}$ be an operator of the form (*__NDE__*) in $\mathbb{R}^n$. Let the condition (1) be satisfied, and let $\mathbf{b} \in L_{n,loc}(\mathbb{R}^n)$. Suppose also that for some $\delta > 0$ $\liminf\limits_{R\to\infty} \sup\limits_{|x|=R} \|\mathbf{b}\|_{n,B_{R\delta}(x)}$ is sufficiently small (we recall that this assumption is obviously satisfied for $\mathbf{b} \in L_n(\mathbb{R}^n)$). Then any semibounded solution of $\mathcal{L}u = 0$ in $\mathbb{R}^n$ is a constant.*

Theorem 3.3 (the Hopf lemma). *Let $\mathcal{L}$ be an operator of the form (*__NDE__*) in Ω, and let $\partial\Omega \in \mathcal{C}^2$. Suppose that the condition (1) be satisfied, and $\mathbf{b} \in L_p(\Omega)$ with $p > n$. Let u be a solution of the equation $\mathcal{L}u = f \ge 0$ in Ω, and $u > u(x^0)$ in Ω for some $x^0 \in \partial\Omega$. Then*

$$\limsup_{\delta\to 0+} \frac{u(x^0) - u(x^0 - \delta\mathbf{n}(x^0))}{\delta} < 0.$$

In particular, if there exists $\frac{\partial u}{\partial \mathbf{n}}(x^0)$ then it is negative.

Note that the restriction on $\mathbf{b}$ cannot be weakened to $\mathbf{b} \in L_n(\Omega)$. Indeed, let $\Omega = B^+_{1/2}$. Put $u(x) = -\frac{x_n}{\ln|x|}$. Then direct calculation shows that u satisfies an equation

$$-\Delta u + b_n(x) D_n u = 0 \qquad \text{with} \quad b_n \in L_n(\Omega),$$

and $u > 0 = u(0)$ in Ω. However, it is easy to see that $D_n u(0) = 0$.

3.2 Parabolic case

Let $1 < p, q < \infty$ be some numbers such that $\frac{n}{p} + \frac{2}{q} = 1$. In this subsection we assume that

$$\partial_t u, D(Du) \in \left[\begin{array}{ll} L_{p,q,loc}(Q), & \text{if} \quad p \ge n+1, \\ \widetilde{L}_{p,q,loc}(Q), & \text{if} \quad q \ge n+1 \end{array} \right.$$

(if as p as q satisfy these assumptions then both scales are admissible).

Theorem 3.4 (the Harnack inequality). *Let $\mathcal{M}$ be an operator of the form (**NDP**) in a cylinder Q. Let the condition (1) be satisfied, and*

$$\mathbf{b} \in \left[\begin{array}{ll} L_{p,q,loc}(Q), & \textit{if} \quad p \geq n+1, \\ \widetilde{L}_{p,q,loc}(Q), & \textit{if} \quad q \geq n+1 \end{array} \right. \tag{9}$$

(if as p as q satisfy these assumptions then both scales are admissible).

Then for arbitrary subcylinders $Q' = \Omega' \times]-T+\delta, -T']$ and $Q'' = \Omega' \times]-T'+\delta, 0]$ (here $\Omega' \Subset \Omega$, $0 < \delta < T' < T - \delta$) there exists a positive constant N_2 such that for any nonnegative solution of $\mathcal{M}u = 0$ in Q the estimate (7) holds true. Moreover, N_2 depends only on n, ν, p, Ω', δ, T' and moduli of continuity of b_i in corresponding spaces.

Theorem 3.5 (strong maximum principle). *Let $\mathcal{M}$ be an operator of the form (**NDP**) in a cylinder Q. Let the conditions (1) and (9) be satisfied. Then any nonconstant solution of $\mathcal{M}u = f \geq 0$ in Q cannot attain its minimum at points of the "top" $\Omega \times \{0\}$.*

Theorem 3.6 (the Liouville theorem). *Let $\mathcal{M}$ be an operator of the form (**NDP**) in $\mathbb{R}^n \times \mathbb{R}_-$. Let the condition (1) be satisfied, and let*

$$\mathbf{b} \in \left[\begin{array}{ll} L_{p,q}(\mathbb{R}^n \times \mathbb{R}_-), & \textit{if} \quad p \geq n+1, \\ \widetilde{L}_{p,q}(\mathbb{R}^n \times \mathbb{R}_-), & \textit{if} \quad q \geq n+1 \end{array} \right.$$

(if as p as q satisfy these assumptions then both scales are admissible).

Then any bounded solution of $\mathcal{M}u = 0$ in $\mathbb{R}^n \times \mathbb{R}_-$ is a constant.

Theorem 3.7 (the Hopf lemma). *Let $\mathcal{M}$ be an operator of the form (**NDP**) in Q, and let $\partial\Omega \in \mathcal{C}^2$. Suppose that the condition (1) be satisfied, and for some $1 < r, s < \infty$ such that $\frac{n}{r} + \frac{2}{s} < 1$,*

$$\mathbf{b} \in \left[\begin{array}{ll} L_{r,s,loc}(Q), & \textit{if} \quad r \geq n+1, \\ \widetilde{L}_{r,s,loc}(Q), & \textit{if} \quad s \geq n+1 \end{array} \right.$$

(if as r as s satisfy these assumptions then both scales are admissible).

Let u be a solution of the equation $\mathcal{M}u = f \geq 0$ in Q, and $u > u(x^0; 0)$ in Q for some $x^0 \in \partial\Omega$. Then

$$\limsup_{\delta \to 0+} \frac{u(x^0; 0) - u(x^0 - \delta \mathbf{n}(x^0); 0)}{\delta} < 0.$$

In particular, if there exists $\frac{\partial u}{\partial \mathbf{n}}(x^0; 0)$ then it is negative.

Note that the restriction on r, s cannot be weakened to $\frac{n}{r} + \frac{2}{s} = 1$. Indeed, let $Q = Q^+_{1/2}$. Put $u(x,t) = -\frac{x_n}{\ln(|x|^2 - t)}$. Then direct calculation shows that u satisfies an equation

$$\partial_t u - \Delta u + b_n(x,t) D_n u = 0 \qquad \text{with} \quad b_n \in L_{r,s}(Q) \cap \widetilde{L}_{r,s}(Q),$$

where $1 < r, s < \infty$ are arbitrary numbers such that $\frac{n}{r} + \frac{2}{s} = 1$. Moreover, $u > 0 = u(0,0)$ in Q. However, it is easy to see that $D_n u(0;0) = 0$.