On the existence of an extremal function in critical Sobolev trace embedding theorem

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Abstract

Sufficient conditions for the existence of extremal functions in the trace Sobolev inequality and the trace Sobolev–Poincaré inequality are established. It is shown that some of these conditions are sharp.

1 Introduction

Let $n \geqslant 2$, and let Ω be a domain in $\mathbb{R}^n$ with piecewise smooth boundary. For $1 < p < n$ denote by $p^\star = \frac{(n-1)p}{n-p}$ the trace Sobolev exponent for p, that is the critical exponent for the trace embedding $W_p^1(\Omega) \hookrightarrow L_q(\partial\Omega)$.

Since the embedding operator $W_p^1(\Omega) \hookrightarrow L_{p^\star}(\partial\Omega)$ is noncompact, the problem of attainability of the norm of this operator (i.e. the problem of existence of an extremal function in the trace embedding theorem) is nontrivial. Corresponding problem for conventional embedding $W_p^1(\Omega) \hookrightarrow L_{p^*}(\Omega)$ (here $p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p) was treated in many papers, see, e.g., the recent survey [N] and further references therein.

The problem for the trace embedding is considerably less investigated. Let us consider the inequality

$$K(n,p) \equiv \inf_{v \in \dot{\mathcal{C}}^\infty(\mathbb{R}^n_+)\setminus\{0\}} \frac{\|\nabla v\|_{p,\mathbb{R}^n_+}}{\|v(\cdot,0)\|_{p^\star,\mathbb{R}^{n-1}}} > 0, \tag{1}$$

where $\dot{\mathcal{C}}^\infty(\mathbb{R}^n_+)$ is the set of functions on $\mathbb{R}^n_+$ with bounded support.

Obviously, the functional in (1) is invariant with respect to translations and dilations of v. It is well known that the infimum in (1) is attained on some function with a *unbounded* support. Escobar [E] conjectured that the minimizer in (1) is

$$w_\varepsilon(x) = |x - x^\varepsilon|^{-\frac{n-p}{p-1}}, \tag{2}$$

with $x^\varepsilon = (0,\dots,0,-\varepsilon)$, and proved it for $p = 2$. Later his conjecture was proved in full generality in the remarkable paper [Nt]. The result of [Nt] implies

$$K(n,p) = \Big(\frac{n-p}{p-1}\Big)^{\frac{1}{p'}} \Big(\frac{\omega_{n-2}}{2} \cdot \mathcal{B}\Big(\frac{n-1}{2}, \frac{n-1}{2(p-1)}\Big)\Big)^{\frac{1}{(n-1)p'}}.$$

In this paper we consider the critical trace embedding in bounded domains, i.e. the inequality

$$\lambda_1(n,p,\Omega) = \inf_{v \in W_p^1(\Omega)\setminus\{0\}} \frac{\|v\|_{W_p^1(\Omega)}}{\|v\|_{p^\star,\partial\Omega}} > 0 \tag{I}$$

(the norm of the numerator is defined as $\|v\|^p_{W_p^1(\Omega)} = \|\nabla v\|^p_{p,\Omega} + \|v\|^p_{p,\Omega}$).

Our first result reads as follows.

[1]This paper was partially supported by grant NSh.227.2008.1. The first author was also supported by RFBR grant 08-01-00748.

Theorem 1. *Let $n \geqslant 2$, and let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in \mathcal{C}^2$. Then for some $\beta(\Omega) > 0$ and for $1 < p < \frac{n+1}{2} + \beta$, the infimum in* (I) *is attained.*

Remark 1. *By standard argument it follows that under suitable normalization the extremal function in* (I)*, if it exists, is a positive solution to the nonlinear Neumann problem*

$$-\Delta_p u + u^{p-1} = 0 \quad \textit{in} \ \ \Omega; \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \mathbf{n}} = u^{p^\star - 1} \quad \textit{on} \ \ \partial\Omega$$

(here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$).

Our main tool is the concentration-compactness principle of Lions ([Ls]; see also [LPT]). It is used in various forms; for the problem (I) it can be reformulated as follows.

Proposition 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in \mathcal{C}^1$. Let the infimum in* (I) *satisfy the inequality*

$$\lambda_1(n, p, \Omega) < K(n, p).$$

Then the infimum is attained.

Proof. Let us consider a minimizing sequence $\{v_k\}$, normalized in $L_{p^\star}(\partial\Omega)$. Without loss of generality one can assume that $v_k \rightharpoonup v$ in $W_p^1(\Omega)$. By the Lions theorem ([Ls, Part 2]; see also [LPT, Lemma 2.2]),

$$|v_k|^{p^\star} \rightharpoonup |v|^{p^\star} + \sum_j c_j\delta(x - x^j), \quad |\nabla v_k|^p \rightharpoonup \mu \geqslant |\nabla v|^p + \sum_j C_j\delta(x - x^j)$$

(the convergence is in the sence of measures on $\partial\Omega$ and on $\overline{\Omega}$, respectively), where $\{x^j\}$ is at most countable set of distinct points in $\partial\Omega$ while c_j and C_j are positive constants.

Since $\{v_k\}$ is a minimizing sequence, by verbal repetition of the proof of Theorem 2.2 [LPT] we obtain the alternative — either v is a minimizer of the corresponding problem and the set $\{x^j\}$ is empty, or $v = 0$ and the set $\{x^j\}$ contains a single point x^0. In the second case $c_0 = 1$ and $C_0 = \lambda_1^p(n, p, \Omega)$. Note that until now all the arguments do not use the smoothness of $\partial\Omega$.

Let the second case occur. Then, similarly to the Corollary 2.1 [LPT], by multiplying v_k by a cut-off function we can assume that supports of v_k are sufficiently small. Due to the assumption $\partial\Omega \in \mathcal{C}^1$, the neighborhood of x^0 in the large scale looks like a half-space. Hence $\lambda_1(n, p, \Omega) \geqslant K(n, p)$. This contradiction proves the desired statement. $\square$

Thus, to prove the attainability of the infimum in (I) it is sufficient to present a function such that the quotient in (I) is less than $K(n, p)$. Following [DN], see also [LPT], we succeed, constructing a function with a small support, simulating the behavior of $w_\varepsilon(x)$.

Remark 2. *The attainability of the infimum in (I) was proved in [FBR] under some additional assumption on Ω. This assumption means that the quotient in (I) taken on constant function is less then $K(n,p)$. In particular, this is the case for "small" domains. Namely, for a domain with a smooth boundary define $\varkappa\Omega$ as a "dilation" of Ω with the coefficient $\varkappa$. Then for any $1<p<n$ there exists $\varkappa_*>0$ such that the infimum in (I) is attained on $\varkappa\Omega$ for $\varkappa<\varkappa_*$ (note that, in contrast with (1), the quotient in (I) is not homogeneous with respect to dilations).*

Further, we slightly strengthen the statement of [FBR] and show that the the infimum in (I) is attained for "small" domains without assumption of smoothness.

Theorem 2. *Let $n\geqslant 2$. Suppose Ω is a bounded domain in $\mathbb{R}^n$ with strictly Lipschitz boundary. Then for any $1<p<n$ there exists $\varkappa_*>0$ such that for $\varkappa<\varkappa_*$, the infimum in* (I) *is attained on $\varkappa\Omega$.*

On the another hand, for "large" domains the assumption of smoothness in Theorem 1 is essential.

Theorem 3. *Let $n\geqslant 2$. Suppose Ω is a polyhedron in $\mathbb{R}^n$, and $1<p<n$. Then there exists $\varkappa^*>0$ such that for $\varkappa>\varkappa^*$, the infimum in* (I) *is not attained on $\varkappa\Omega$.*

Finally, we consider the trace Sobolev–Poincaré inequality

$$\lambda_2(n,p,\Omega)=\inf_{v\in W_p^1(\Omega)\setminus\{c\}}\frac{\|\nabla v\|_{p,\Omega}}{\|v-\overline{v}\|_{p^\star,\partial\Omega}}>0 \qquad (II)$$

(here we use the notation $\overline{v}=|\partial\Omega|^{-1}\int\limits_{\partial\Omega}v\,d\Sigma$).

Theorem 4. *Let $n\geqslant 3$, and let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega\in\mathcal{C}^2$. Then for some $\beta(\Omega)>0$ and for $1<p<\frac{n+1}{2}+\beta$, the infimum in* (II) *is attained.*

The structure of our paper is the following. In §2 we establish required integral estimates and prove Theorem 1. Theorems 2 and 3 are proved in §3; inequality (II) is considered in §4. In §5 we generalize Theorems 1 and 4 to the case of non-euclidean norm of gradient.

Let us recall some notations. A point in $\mathbb{R}^n$ is denoted by $x=(x',x_n)$ where $x'\in\mathbb{R}^{n-1}$; r stands for $|x'|$. $\mathbb{R}^n_+=\{x\in\mathbb{R}^n:\ x_n>0\}$. $Q_\rho=\{x: r<\rho, 0<x_n<\rho\}$ is a cylinder in $\mathbb{R}^n$, $B_\rho(x^0)=\{x:|x-x^0|<\rho\}$.

$\omega_{n-1}=\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the area of the unit sphere in $\mathbb{R}^n$. $p'=\frac{p}{p-1}$ is the Hölder conjugate exponent to p. $\mathcal{B}$ is the Euler beta-function. We denote by $o_\rho(1)$ a quantity which tends to zero as $\rho\to 0$.

We use letter C to denote various positive constants. To indicate that C depends on some parameters, we write $C(\dots)$.

2 Proof of Theorem 1

Consider the least ball containing Ω. Let $x^0\in\partial\Omega$ be a point of contact of Ω with this ball. Then all the principal curvatures of $\partial\Omega$ at x^0 are positive, and therefore

$H(x^0) > 0$. We introduce a local coordinate system such that x^0 is the origin, x' lie in the tangent plane and the axis x_n is directed into Ω. Since $\partial\Omega \in \mathcal{C}^2$, in some neighborhood of the origin $\partial\Omega$ is the graph of a function $x_n = F(x')$, and $F(x') = (\mathcal{A}x', x') + o(r^2)$ as $r \to 0$, where $\mathcal{A}$ is a positive definite matrix.

For sufficiently small $\varepsilon > 0$ and $\rho > 0$ we introduce the function

$$u(x) = u(r, x_n) = \varphi(r, x_n) w_\varepsilon(r, x_n), \tag{3}$$

where w_ε is defined in (2), while φ is a smooth cut-off function such that

$$\varphi = 1 \quad \text{in} \quad Q_{\frac{\rho}{2}}; \qquad \varphi = 0 \quad \text{in} \quad \mathbb{R}^n \setminus Q_\rho,$$

and $|\nabla\varphi| \leqslant \frac{C}{\rho}$.

2.1 Estimate of $\|\nabla u\|_{p,\Omega}$ for $1 < p \leqslant \frac{n+1}{2}$

Let us apply the estimate

$$|\nabla(fg)|^p \leqslant |f\,\nabla g|^p + C\,\left(|\nabla f\, g|^p + |\nabla f\, g| \cdot |f\,\nabla g|^{p-1}\right). \tag{4}$$

to ∇u. Since $\nabla\varphi$ does not vanish only in $Q_\rho \setminus Q_{\frac{\rho}{2}}$, one obtains

$$\int\limits_\Omega \left(|\nabla\varphi \cdot w_\varepsilon|^p + |\nabla\varphi \cdot w_\varepsilon| \cdot |\varphi\,\nabla w_\varepsilon|^{p-1}\right) dx \leqslant$$

$$\leqslant C\Bigg(\int\limits_0^{\rho/2}\int\limits_{\rho/2}^{\rho} + \int\limits_{\rho/2}^{\rho}\int\limits_0^{\rho}\Bigg) \frac{\rho^{-p} r^{n-2}\, dx_n dr}{(r^2+x_n^2)^{\frac{p(n-p)}{2(p-1)}}} \cdot \left[1 + \left(\frac{\rho}{\sqrt{r^2+x_n^2}}\right)^{p-1}\right] \leqslant C\,\rho^{-\frac{n-p}{p-1}}. \tag{5}$$

Furthermore, one has

$$|\varphi\,\nabla w_\varepsilon|^p \leqslant |\nabla w_\varepsilon|^p = \left(\frac{n-p}{p-1}\right)^p |x - x^\varepsilon|^{-(n-1)p'}.$$

The inequalities (4) and (5) imply

$$\int\limits_\Omega |\nabla u|^p\, dx \leqslant \int\limits_{Q_\rho \cap \Omega} \left(\frac{n-p}{p-1}\right)^p |x - x^\varepsilon|^{-(n-1)p'} dx_n dx' + C\,\rho^{-\frac{n-p}{p-1}} = I_1 - I_2 + C\,\rho^{-\frac{n-p}{p-1}}, \tag{6}$$

where

$$I_1 = \int\limits_{|x'|<\rho} \int\limits_0^{\rho} \left(\frac{n-p}{p-1}\right)^p |x - x^\varepsilon|^{-(n-1)p'} dx_n dx',$$

$$I_2 = \int\limits_{|x'|<\rho} \int\limits_0^{F(x')} \left(\frac{n-p}{p-1}\right)^p |x - x^\varepsilon|^{-(n-1)p'} dx_n dx'.$$

It is easy to see that

$$I_1 = \omega_{n-2}\Big(\frac{n-p}{p-1}\Big)^p \int\limits_0^\rho \int\limits_0^\rho \frac{r^{n-2}\,dx_n dr}{(r^2+(x_n+\varepsilon)^2)^{\frac{p(n-1)}{2(p-1)}}} \leqslant E_1\,\varepsilon^{-\frac{n-p}{p-1}}, \tag{7}$$

where

$$E_1 = \omega_{n-2}\Big(\frac{n-p}{p-1}\Big)^p \int\limits_0^\infty \int\limits_0^\infty \frac{t^{n-2}\,dtds}{(t^2+(s+1)^2)^{\frac{p(n-1)}{2(p-1)}}} = \\ = \frac{\omega_{n-2}}{2}\Big(\frac{n-p}{p-1}\Big)^{p-1} \mathcal{B}\Big(\frac{n-1}{2}, \frac{n-1}{2(p-1)}\Big). \tag{8}$$

Further, since $\sup_{|x'|<\rho}|F(x')| \to 0$ as $\rho \to 0$, we have

$$I_2 = \Big(\frac{n-p}{p-1}\Big)^p \int\limits_{|x'|<\rho} \frac{F(x')\,dx'}{(|x'|^2+\varepsilon^2)^{\frac{p(n-1)}{2(p-1)}}} \cdot (1+o_\rho(1)) =$$

$$= \Big(\frac{n-p}{p-1}\Big)^p \int\limits_0^\rho \int\limits_{\mathbb{S}_1} \big((\mathcal{A}\omega,\omega)+o_\rho(1)\big)d\omega \cdot \frac{r^n\,dr}{(r^2+\varepsilon^2)^{\frac{p(n-1)}{2(p-1)}}} \cdot (1+o_\rho(1)) =$$

$$= \frac{\omega_{n-2}}{n}\Big(\frac{n-p}{p-1}\Big)^p \varepsilon^{1-\frac{n-p}{p-1}} \int\limits_0^{\frac{\rho}{\varepsilon}} \frac{t^n\,dt}{(t^2+1)^{\frac{p(n-1)}{2(p-1)}}} \cdot (H(x^0)+o_\rho(1)).$$

Thus, for $p < \frac{n+1}{2}$ the following estimate holds:

$$I_2 \geqslant E_2\,\varepsilon^{1-\frac{n-p}{p-1}} \cdot (H(x^0)+o_\rho(1)) - C\,\rho^{1-\frac{n-p}{p-1}}. \tag{9}$$

Here

$$E_2 = \frac{\omega_{n-2}}{n}\,\Big(\frac{n-p}{p-1}\Big)^p \int\limits_0^\infty \frac{t^n\,dt}{(t^2+1)^{\frac{p(n-1)}{2(p-1)}}} = \frac{\omega_{n-2}}{2n}\,\Big(\frac{n-p}{p-1}\Big)^p \cdot \mathcal{B}\Big(\frac{n+1}{2}, \frac{n-1}{2(p-1)}-1\Big). \tag{10}$$

For $p = \frac{n+1}{2}$ one gets the estimate

$$I_2 \geqslant C\,\big(\ln(\rho/\varepsilon)-1\big). \tag{11}$$

2.2 Estimates of $\|u\|_{p^\star,\partial\Omega}$ for $1<p<n$

We have

$$\int\limits_{\partial\Omega} u^{p^\star}\,d\Sigma \geqslant \int\limits_{|x'|<\frac{\rho}{2}} \frac{\sqrt{1+|\nabla F(x')|^2}\,dx'}{(|x'|^2+(\varepsilon+F(x'))^2)^{\frac{p(n-1)}{2(p-1)}}} \geqslant$$

$$\geqslant \int\limits_{|x'|<\frac{\rho}{2}} \frac{dx'}{(|x'|^2+(\varepsilon+F(x'))^2)^{\frac{p(n-1)}{2(p-1)}}} = \int\limits_{|x'|<\frac{\rho}{2}} \frac{dx'}{(|x'|^2+\varepsilon^2)^{\frac{p(n-1)}{2(p-1)}}} \times$$

$$\times \Big[1-(n-1)p'\frac{\varepsilon F(x')}{|x'|^2+\varepsilon^2} + \frac{\varepsilon^2F^2(x')}{(|x'|^2+\varepsilon^2)^2}\cdot O_\rho(1)\Big] =: J_1 - J_2 + J_3. \tag{12}$$

It is easy to see that

$$J_1=\omega_{n-2}\int\limits_0^{\frac{\rho}{2}}\frac{r^{n-2}\,dr}{(r^2+\varepsilon^2)^{\frac{p(n-1)}{2(p-1)}}}\geqslant D_1\varepsilon^{-\frac{n-1}{p-1}}-C\rho^{-\frac{n-1}{p-1}}, \tag{13}$$

where

$$D_1=\omega_{n-2}\int\limits_0^{\infty}\frac{t^{n-2}\,dt}{(t^2+1)^{\frac{p(n-1)}{2(p-1)}}}=\frac{\omega_{n-2}}{2}\cdot\mathcal{B}\Big(\frac{n-1}{2},\frac{n-1}{2(p-1)}\Big). \tag{14}$$

Further, we obtain

$$J_2=(n-1)p'\int\limits_0^{\frac{\rho}{2}}\int\limits_{\mathbb{S}_1}\big((\mathcal{A}\omega,\omega)+o_\rho(1)\big)d\omega\cdot\frac{\varepsilon r^n\,dr}{(r^2+\varepsilon^2)^{\frac{p(n-1)}{2(p-1)}+1}}=$$

$$=\frac{\omega_{n-2}p'}{n'}\,\varepsilon^{1-\frac{n-1}{p-1}}\int\limits_0^{\frac{\rho}{2\varepsilon}}\frac{t^n\,dt}{(t^2+1)^{\frac{p(n-1)}{2(p-1)}+1}}\cdot(H(x^0)+o_\rho(1))\leqslant D_2\,\varepsilon^{1-\frac{n-1}{p-1}}\cdot(H(x^0)+o_\rho(1)), \tag{15}$$

where

$$D_2=\frac{\omega_{n-2}p'}{n'}\int\limits_0^{\infty}\frac{t^n\,dt}{(t^2+1)^{\frac{p(n-1)}{2(p-1)}+1}}=\frac{\omega_{n-2}p'}{2n'}\cdot\mathcal{B}\Big(\frac{n+1}{2},\frac{n-1}{2(p-1)}\Big). \tag{16}$$

Finally, since $|F(x')|\leqslant Cr^2$, we have

$$|J_3|\leqslant CJ_1\varepsilon^2. \tag{17}$$

2.3 Estimates of $\|u\|_{p,\Omega}$

We have

$$\int\limits_\Omega u^p\,dx\leqslant C\int\limits_0^\rho\int\limits_0^\rho\frac{r^{n-2}\,dx_n dr}{(r^2+(x_n+\varepsilon)^2)^{\frac{p(n-p)}{2(p-1)}}}\leqslant C\,\varepsilon^{\frac{p^2-n}{p-1}}\int\limits_0^{\frac{\rho}{\varepsilon}}\int\limits_0^{\infty}\frac{t^{n-2}\,dtds}{(t^2+(s+1)^2)^{\frac{p(n-p)}{2(p-1)}}}\leqslant$$

$$\leqslant\begin{cases}C\,\varepsilon^{\frac{p^2-n}{p-1}}, & \text{if}\quad 1<p<\sqrt{n};\\ C\,\big(1+\ln(\rho/\varepsilon)\big), & \text{if}\quad p=\sqrt{n};\\ C, & \text{if}\quad p>\sqrt{n}.\end{cases} \tag{18}$$

2.4 Completion of the proof

The estimates (12), (13), (15) and (17) yield

$$\int\limits_{\partial\Omega}u^{p^\star}\,d\Sigma\geqslant D_1\,\varepsilon^{-\frac{n-1}{p-1}}-D_2\,(H(x_0)+o_\rho(1))\,\varepsilon^{1-\frac{n-1}{p-1}}-C\,\varepsilon^{2-\frac{n-1}{p-1}}-C(\rho),$$

where D_1, D_2 are defined in (14), (16).

By making use of (6), (7), (9), (11) and (18) we obtain

$$\|u\|^p_{W^1_p(\Omega)} \leqslant E_1\,\varepsilon^{-\frac{n-p}{p-1}} +$$
$$+\begin{cases} -E_2\,(H(x^0)+o_\rho(1))\,\varepsilon^{1-\frac{n-p}{p-1}} + C\,\varepsilon^{\frac{p^2-n}{p-1}} + C(\rho), & \text{if} \quad 1<p<\sqrt{n};\\ -E_2\,(H(x^0)+o_\rho(1))\,\varepsilon^{1-\frac{n-p}{p-1}} + C\ln\varepsilon^{-1} + C(\rho), & \text{if} \quad p=\sqrt{n};\\ -E_2\,(H(x^0)+o_\rho(1))\,\varepsilon^{1-\frac{n-p}{p-1}} + C(\rho), & \text{if} \quad \sqrt{n}<p<\frac{n+1}{2};\\ -C\ln\varepsilon^{-1} + C(\rho), & \text{if} \quad p=\frac{n+1}{2}, \end{cases}$$

where E_1, E_2 are defined in (8) and (10).

Since $1-\frac{n-p}{p-1} < \frac{p^2-n}{p-1}$ for $p>1$, for $1<p<\frac{n+1}{2}$ we get

$$\frac{\|u\|^p_{W^1_p(\Omega)}}{\|u\|^p_{p^\star,\partial\Omega}} \leqslant \frac{E_1}{D_1^{p/p^\star}}\cdot\Big[1+\Big(\frac{p}{p^\star}\,\frac{D_2}{D_1}-\frac{E_2}{E_1}\Big)\,(H(x^0)+o_\rho(1))\,\varepsilon + o(\varepsilon)\Big]. \tag{19}$$

Direct computation shows that for $1<p<\frac{n+1}{2}$

$$\frac{p}{p^\star}\,\frac{D_2}{D_1}-\frac{E_2}{E_1} = -\frac{2(n-p)(p-1)}{n(n-2p+1)} < 0. \tag{20}$$

Hence, for ε and ρ sufficiently small, one has

$$\frac{\|u\|_{W^1_p(\Omega)}}{\|u\|_{p^\star,\partial\Omega}} < \frac{E_1^{1/p}}{D_1^{1/p^\star}}. \tag{21}$$

For $p=\frac{n+1}{2}$ the relation (21) also holds true for ε and ρ sufficiently small. By continuity, for some $\beta>0$ and $p<\frac{n+1}{2}+\beta$

$$\lambda_1(n,p,\Omega) < \frac{E_1^{1/p}}{D_1^{1/p^\star}} = K(n,p).$$

The application of Proposition 1 completes the proof. $\square$

Remark 3. *One can see from the proof that the requirement $\partial\Omega\in\mathcal{C}^2$ can be relaxed. For example, we can assume that $\partial\Omega\in\mathcal{C}^1$, and the normal map of $\partial\Omega$ is absolutely continuous.*

Remark 4. *Theorem 1 is also valid for an arbitrary manifold Ω with a smooth boundary, if $\partial\Omega$ contains a point with the positive mean curvature (with respect to the inner normal). The proof is carried out in the same way.*

3 On "small" and "large" domains

Proof of Theorem 2. Assume that the infimum in (I) is not attained on $\varkappa\Omega$. Then the arguments of Proposition 1 show that a minimizing sequence v_k, normalized in $L_{p^\star}(\partial\Omega)$, satisfies

$$|v_k|^{p^\star} \rightharpoonup \delta(x-x^0), \qquad |\nabla v_k|^p \rightharpoonup \lambda_1^p(n,p,\varkappa\Omega)\cdot\delta(x-x^0)$$

(the convergence is in the sence of measures on $\partial\Omega$ and on $\overline{\Omega}$, respectively).

By multiplying v_k by a cut-off function we can assume that supports of v_k are arbitrarily small. This implies that $\lambda_1^p(n,p,\varkappa\Omega)$ does not depend on $\varkappa$.

Since $\partial\Omega$ is strictly Lipschitz, $\lambda_1^p(n,p,\varkappa\Omega)$ cannot be zero. However, the quotient (I) on constants tends to zero as $\varkappa \to 0$. This contradiction proves the Theorem. $\square$

Proof of Theorem 3. The proof is similar to [DN, Theorem 7.2], see also [Dr, Theorem 2].

Suppose there exists an unbounded sequence of $\varkappa$'s such that the infimum in (I) is attained on $\varkappa\Omega$. Then, by a standard argument, the corresponding minimizers $u_\varkappa$ are positive in $\varkappa\Omega$.

Consider the functions $v_\varkappa(x) = C(\varkappa)u_\varkappa(\frac{x}{\varkappa})$ in Ω, with $C(\varkappa)$ given by the normalization condition $\|v_\varkappa\|_{p^\star,\partial\Omega} = 1$. Then the weak Euler equation for the $v_\varkappa$ reads as follows:

$$\int\limits_\Omega |\nabla v_\varkappa|^{p-2}\langle\nabla v_\varkappa, \nabla h\rangle\, dx + \varkappa^p \int\limits_\Omega v_\varkappa^{p-1} h\, dx = \lambda_\varkappa^p \int\limits_{\partial\Omega} v_\varkappa^{p^\star-1} h\, d\Sigma, \tag{22}$$

where $\lambda_\varkappa = \lambda_1(n,p,\varkappa\Omega)$.

We claim that $\lambda_\varkappa$ are bounded. Indeed, let us consider a point $\widehat{x} \in \partial\Omega$. It is easy to see that for sufficiently small ρ the set $\partial\Omega \cap B_\rho(\widehat{x})$ is a conic surface. Thus, for functions v supported in $B_\rho(\widehat{x})$, the quotient (I) in $\varkappa\Omega$ does not depend on $\varkappa$. We denote by $\gamma(\widehat{x})$ the infimum of the quotient (I) over the set of such functions v and arrive at

$$\lambda_\varkappa \leqslant \gamma := \inf_{\widehat{x}\in\partial\Omega} \gamma(\widehat{x}). \tag{23}$$

Put $h = v_\varkappa$ in (22). Then

$$\|\nabla v_\varkappa\|_{p,\Omega}^p + \varkappa^p \|v_\varkappa\|_{p,\Omega}^p = \lambda_\varkappa^p. \tag{24}$$

Without loss of generality, $v_\varkappa \rightharpoonup v$ in $W_p^1(\Omega)$. From (24) and (23) we conclude that $v = 0$. Arguing as in the proof of Theorem 2.2 [LPT] one deduces that there exists a point $x^0 \in \partial\Omega$ such that $|v_\varkappa|^{p^\star} \rightharpoonup \delta(x-x^0)$ in the sense of measures on $\partial\Omega$. Therefore, for a given $\rho > 0$,

$$\|v_\varkappa\|_{p^\star,\partial\Omega\setminus B_{\frac{\rho}{2}}(x^0)} \to 0. \tag{25}$$

Remark 5. *Really, the more strong property* $\sup_{\overline{\Omega}\setminus B_\rho(x^0)} v_\varkappa \to 0$ *can be proved, but we do not need it.*

By η we denote a smooth cut-off function such that

$$\eta = 0 \quad \text{in} \quad B_{\frac{\rho}{2}}(x^0); \qquad \eta = 1 \quad \text{in} \quad \mathbb{R}^n \setminus B_\rho(x^0).$$

We substitute $h = \eta^p v_\varkappa$ into the equation (22):

$$\int\limits_\Omega \eta^p |\nabla v_\varkappa|^p\, dx + p\int\limits_\Omega \eta^{p-1}|\nabla v_\varkappa|^{p-2}\langle\nabla v_\varkappa, \nabla\eta\rangle\, dx + \varkappa^p \int\limits_\Omega v_\varkappa^p\, dx \leqslant \lambda_\varkappa^p \int\limits_{\partial\Omega} \eta^p v_\varkappa^{p^\star} d\Sigma.$$

Recalling (23), we arrive at

$$\|\eta\nabla v_\varkappa\|_{p,\Omega}^p + \varkappa^p\|\eta v_\varkappa\|_{p,\Omega}^p \leqslant \gamma^p \int\limits_{\partial\Omega} \eta^p v_\varkappa^{p^\star} d\Sigma + C\|v_\varkappa\|_{p,\Omega}\cdot\|\eta\nabla v_\varkappa\|_{p,\Omega}^{p-1}. \qquad (26)$$

The Hölder inequality and (25) give us

$$\int\limits_{\partial\Omega} \eta^p v_\varkappa^{p^\star} d\Sigma = \int\limits_{\partial\Omega\setminus B_{\frac{\rho}{2}}(x^0)} \eta^p v_\varkappa^{p^\star} d\Sigma \leqslant \|\eta v_\varkappa\|_{p^\star,\partial\Omega}^p \cdot \|v_\varkappa\|_{p^\star,\partial\Omega\setminus B_{\frac{\rho}{2}}(x^0)}^{p^\star-p} = \|\eta v_\varkappa\|_{p^\star,\partial\Omega}^p \cdot o_\varkappa(1).$$

We apply the trace embedding theorem to the function $\eta v_\varkappa$:

$$\|\eta v_\varkappa\|_{p^\star,\partial\Omega}^p \leqslant C(\|\nabla(\eta v_\varkappa)\|_{p,\Omega}^p + \|\eta v_\varkappa\|_{p,\Omega}^p) \leqslant C(\|\eta\nabla v_\varkappa\|_{p,\Omega}^p + \|v_\varkappa\|_{p,\Omega}^p). \qquad (27)$$

Substituting these relations to (26), we obtain

$$\left(\frac{\|\eta\nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}}\right)^p \leqslant C\left(\frac{\|\eta\nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}}\right)^{p-1} + o_\varkappa(1)\cdot\left(1+\left(\frac{\|\eta\nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}}\right)^p\right).$$

Therefore,

$$\frac{\|\eta\nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}} \leqslant C. \qquad (28)$$

Now we consider a smooth cut-off function η_1 such that

$$\eta_1 = 1 \quad \text{in} \quad B_\rho(x^0); \qquad \eta_1 = 0 \quad \text{in} \quad \mathbb{R}^n \setminus B_{2\rho}(x^0).$$

The normalization of $v_\varkappa$ and the inequality (23) yield

$$\|v_\varkappa\|_{p^\star,\partial\Omega\cap B_\rho(x^0)}^{p^\star} \leqslant \|v_\varkappa\|_{p^\star,\partial\Omega\cap B_\rho(x^0)}^p \leqslant \|\eta_1 v_\varkappa\|_{p^\star,\partial\Omega}^p \leqslant$$
$$\leqslant \frac{1}{\gamma^p(x^0)}\cdot\|\eta_1 v_\varkappa\|_{W_p^1(\Omega)}^p \leqslant \frac{1}{\gamma^p}\cdot(\|\nabla(\eta_1 v_\varkappa)\|_{p,\Omega}^p + \|\eta_1 v_\varkappa\|_{p,\Omega}^p).$$

By (4),

$$\|v_\varkappa\|_{p^\star,\partial\Omega\cap B_\rho(x^0)}^{p^\star} \leqslant \frac{1}{\gamma^p}\cdot\|\nabla v_\varkappa\|_{p,\Omega}^p + C\Big[\|v_\varkappa\|_{p,\Omega}^p + \|v_\varkappa\|_{p,\Omega}\cdot\|\nabla v_\varkappa\|_{p,\Omega\setminus B_\rho(x^0)}^{p-1}\Big].$$

Substituting $\|\nabla v_\varkappa\|_{p,\Omega}^p$ from (24) and recalling (23), we get

$$\frac{\varkappa^p}{\gamma^p}\cdot\|v_\varkappa\|_{p,\Omega}^p \leqslant 1 - \|v_\varkappa\|_{p^\star,\partial\Omega\cap B_\rho(x^0)}^{p^\star} + C\Big[\|v_\varkappa\|_{p,\Omega}^p + \|v_\varkappa\|_{p,\Omega}\cdot\|\nabla v_\varkappa\|_{p,\Omega\setminus B_\rho(x^0)}^{p-1}\Big].$$

Thus,

$$\frac{\varkappa^p}{\gamma^p} \leqslant \frac{\|v_\varkappa\|_{p^\star,\partial\Omega\setminus B_\rho(x^0)}^{p^\star}}{\|v_\varkappa\|_{p,\Omega}^p} + C\left(1+\left(\frac{\|\nabla v_\varkappa\|_{p,\Omega\setminus B_\rho(x^0)}}{\|v_\varkappa\|_{p,\Omega}}\right)^{p-1}\right). \qquad (29)$$

Using normalization of $v_\varkappa$, (27) and (28), we obtain

$$\frac{\|v_\varkappa\|_{p^\star,\partial\Omega\setminus B_\rho(x^0)}^{p^\star}}{\|v_\varkappa\|_{p,\Omega}^p} \leqslant \frac{\|v_\varkappa\|_{p^\star,\partial\Omega\setminus B_\rho(x^0)}^p}{\|v_\varkappa\|_{p,\Omega}^p} \leqslant \frac{\|\eta v_\varkappa\|_{p^\star,\partial\Omega}^p}{\|v_\varkappa\|_{p,\Omega}^p} \leqslant C\left(1+\left(\frac{\|\eta\nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}}\right)^p\right) \leqslant C,$$

while the second term in (29) is bounded by (28):

$$\frac{\|\nabla v_\varkappa\|_{p,\Omega\setminus B_\rho(x^0)}}{\|v_\varkappa\|_{p,\Omega}} \leqslant \frac{\|\eta\nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}} \leqslant C.$$

Therefore, the right-hand side in (29) is bounded. This gives a contradiction for $\varkappa$ large enough. □

4 Proof of Theorem 4

The following statement is proved in the same way as Proposition 1.

Proposition 2. *Let Ω be as in Proposition 1. Let the infimum in* (II) *satisfy the inequality $\lambda_2(n,p,\Omega) < K(n,p)$. Then the infimum is attained.*

To construct a function with zero trace mean-value having the quotient (II) less then $K(n,p)$ we modify the function (3). Following ([DN, Theorem 5.1]), for large p we proceed by subtracting a suitable function with a small support while for small p we will subtract a constant.

We need the follownig estimate for the function u defined in (3):

$$\begin{aligned}\int\limits_{\partial\Omega} u\,d\Sigma &\leqslant \int\limits_{|x'|<\rho} \frac{\sqrt{1+|\nabla F(x')|^2}\,dx'}{(|x'|^2+(F(x')+\varepsilon)^2)^{\frac{n-p}{2(p-1)}}} \leqslant \\ &\leqslant C\int\limits_{|x'|<\rho} \frac{dx'}{(|x'|^2+(F(x')+\varepsilon)^2)^{\frac{n-p}{2(p-1)}}} \leqslant \\ &\leqslant C\int\limits_{|x'|<\rho} \frac{dx'}{(|x'|^2+\varepsilon^2)^{\frac{n-p}{2(p-1)}}} \leqslant \begin{cases} C\varepsilon^{n-1-\frac{n-p}{p-1}}, & \text{if} \quad p<\frac{2n-1}{n} \\ C(1+\log(\rho/\varepsilon)), & \text{if} \quad p=\frac{2n-1}{n} \\ C, & \text{if} \quad p>\frac{2n-1}{n}\end{cases}.\end{aligned} \tag{30}$$

1. First, consider the case $p > \frac{2n-1}{n}$. Let u_1 be a smooth non-negative function such that $u_1\big|_{\partial\Omega} \not\equiv 0$ and $\operatorname{supp} u_1 \cap \operatorname{supp} u = \emptyset$. We define the function with zero trace mean-value

$$\widehat{u}(x) = u(x) - u_1(x)\,\frac{\int\limits_{\partial\Omega} u\,d\Sigma}{\int\limits_{\partial\Omega} u_1 d\Sigma}.$$

The estimates of subsection 2.4 and (30) yield for $p < \frac{n+1}{2}$

$$\frac{\|\nabla\widehat{u}\|^p_{p,\Omega}}{\|\widehat{u}\|^p_{p^\star,\partial\Omega}} \leqslant \frac{\|\nabla u\|^p_{p,\Omega}+C(\rho)}{\|u\|^p_{p^\star,\partial\Omega}} \leqslant \frac{E_1}{D_1^{p/p^\star}}\cdot\Big[1+\Big(\frac{p}{p^\star}\,\frac{D_2}{D_1}-\frac{E_2}{E_1}\Big)\,(H(x^0)+o_\rho(1))\,\varepsilon+o(\varepsilon)\Big].$$

By (20), for ε and ρ sufficiently small, one has

$$\frac{\|\nabla\widehat{u}\|_{p,\Omega}}{\|\widehat{u}\|_{p^\star,\partial\Omega}} < \frac{E_1^{1/p}}{D_1^{1/p^\star}}. \tag{31}$$

For $p = \frac{n+1}{2}$ this relation also holds true for ε and ρ sufficiently small. By continuity, for some $\beta > 0$ and $p < \frac{n+1}{2} + \beta$

$$\lambda_2(n,p,\Omega) < \frac{E_1^{1/p}}{D_1^{1/p^\star}} = K(n,p). \tag{32}$$

2. Now we consider the case $1 < p \leqslant \frac{2n-1}{n}$. Define u by (3) and put $\rho = \varepsilon^\gamma$ with some $\gamma \in (0, \gamma^*)$, where $\gamma^* = \frac{n+1-2p}{n-p} \in (0,1)$.

The estimates of subsection 2.2 yield

$$\int\limits_{\partial\Omega} u^{p^\star} d\Sigma \leqslant D_1 \varepsilon^{-\frac{n-1}{p-1}} - D_2(R(x_0) + o_\varepsilon(1))\varepsilon^{-\frac{n-p}{p-1}} - C\varepsilon^{-\gamma\frac{n-1}{p-1}}.$$

Immediate computation shows that

$$\gamma^* \frac{n-1}{p-1} < \frac{n-p}{p-1}.$$

Therefore, for $\gamma \in (0, \gamma^*)$

$$\int\limits_{\partial\Omega} u^{p^\star} d\Sigma \leqslant D_1 \varepsilon^{-\frac{n-1}{p-1}} - D_2(R(x_0) + o_\varepsilon(1))\varepsilon^{-\frac{n-p}{p-1}}.$$

Furthermore, the estimates of subsection 2.1 imply

$$\int |\nabla u|^p dx \leqslant E_1 \varepsilon^{-\frac{n-p}{p-1}} - E_2(R(x_0) + o_\varepsilon(1))\varepsilon^{1-\frac{n-p}{p-1}} + C\varepsilon^{-\gamma\frac{n-p}{p-1}}.$$

Direct calculations result in

$$\gamma^* \frac{n-p}{p-1} = \frac{n-p}{p-1} - 1.$$

Hence, for $\gamma \in (0, \gamma^*)$

$$\int |\nabla u|^p dx \leqslant E_1 \varepsilon^{-\frac{n-p}{p-1}} - E_2(R(x_0) + o_\varepsilon(1))\varepsilon^{1-\frac{n-p}{p-1}}. \tag{33}$$

Now we define the function with zero trace mean-value

$$\widehat{u}(x) = u(x) - \frac{1}{|\partial\Omega|} \int\limits_{\partial\Omega} u\, d\Sigma.$$

Using (30) and the Minkowski inequality we conclude that

$$\|\widehat{u}\|_{p^\star,\partial\Omega} \geqslant \Bigg(\int\limits_{\operatorname{supp} u \cap \partial\Omega} |\widehat{u}|^{p^\star} d\Sigma \Bigg)^{\frac{1}{p^\star}} \geqslant \|u\|_{p^\star,\partial\Omega} - \frac{|\operatorname{supp} u \cap \partial\Omega|^{\frac{1}{p^\star}}}{|\partial\Omega|} \int\limits_{\partial\Omega} u d\Sigma \geqslant$$
$$\geqslant D_1^{\frac{1}{p^\star}} \varepsilon^{-\frac{n-p}{p(p-1)}} \left(1 - \frac{1}{p^\star} \frac{D_2}{D_1} (R(x_0) + o_\varepsilon(1))\varepsilon - C\varepsilon^{\gamma\frac{n-1}{p^\star} + \frac{n-p}{p(p-1)}} \Phi(\varepsilon) \right), \tag{34}$$

where

$$\Phi(\varepsilon) = \begin{cases} \varepsilon^{n-1-\frac{n-p}{p-1}}, & \text{if} \quad 1 < p < \frac{2n-1}{n}; \\ 1 + \ln \varepsilon^{\gamma-1} & \text{if} \quad p = \frac{2n-1}{n}. \end{cases}$$

Immediate calculations show that for $n \geqslant 3$ and $1 < p \leqslant \frac{2n-1}{n}$, the inequality

$$\gamma^* \frac{n-1}{p^\star} + \frac{n-p}{p(p-1)} + n - 1 - \frac{n-p}{p-1} > 1$$

holds true for $\gamma = \gamma^*$ and, therefore, for some $\gamma \in (0, \gamma^*)$. Choose such a γ, then (34) implies

$$\|\widehat{u}\|^p_{p^\star,\partial\Omega} \geqslant D_1^{\frac{1}{p^\star}} \varepsilon^{-\frac{n-p}{p-1}} \left(1 - \frac{p}{p^\star} \frac{D_2}{D_1} (R(x_0) + o_\varepsilon(1))\varepsilon\right). \tag{35}$$

The relations (33), (35) and (20) imply (31) for ε small enough. Thus, the inequality (32) is valid in this case. The application of Proposition 2 completes the proof. □

5 Some generalizations

We can consider the quotients (I)–(II) with a more general definition of $\|\nabla v\|_{p,\Omega}$ in the numerator. Namely, let us consider an arbitrary norm $\mathcal{M}(x)$ in $\mathbb{R}^n$. Assume, for simplicity, that the function $\mathcal{M}$ is strictly convex in non-radial directions, and $\mathcal{M} \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\})$. Denote by

$$\mathcal{M}_o(x) = \sup_{\mathcal{M}(\xi) \leqslant 1} \langle x, \xi \rangle = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{\mathcal{M}(\xi)}$$

the dual norm. In particular, if

$$\mathcal{M}(x) = |x|_q \equiv \left(\sum_{k=1}^n |x_k|^q \right)^{\frac{1}{q}}, \quad 1 < q < \infty,$$

then $\mathcal{M}_o(x) = |x|_{q'}$.

Let $\omega_{n-1,\mathcal{M}}$ stand for the area of a unit sphere $\mathbb{S}_1^{\mathcal{M}} = \{x \in \mathbb{R}^n : \ \mathcal{M}(x) = 1\}$. In particular, for $\mathcal{M}(x) = |x|_q$ we have $\omega_{n-1,\mathcal{M}} = \omega_{n-1,q} = q[2\Gamma(\frac{1}{q}+1)]^n / \Gamma(\frac{n}{q})$.

We introduce an equivalent seminorm $\|\nabla v\|_{p,\Omega,\mathcal{M}}$ in $W^1_p(\Omega)$ by

$$\|\nabla v\|^p_{p,\Omega,\mathcal{M}} = \int\limits_\Omega \mathcal{M}^p(\nabla v) \, dx$$

and replace $\|\nabla v\|_{p,\Omega}$ in the numerators of (I) and (II) by $\|\nabla v\|_{p,\Omega,\mathcal{M}}$. Denote by $(\mathrm{I}_\mathcal{M})$ (respectively, $(\mathrm{II}_\mathcal{M})$) these quotients and by $\lambda_k(n, p, \Omega, \mathcal{M})$, $k = 1, 2$, the corresponding infima.

It is proved in [Nt] that

$$K(n, p, \mathcal{M}) \equiv \inf_{v \in \dot{\mathcal{C}}^\infty(\mathbb{R}^n_+) \setminus \{0\}} \frac{\|\nabla v\|_{p,\mathbb{R}^n_+,\mathcal{M}}}{\|v(\cdot, 0)\|_{p^\star,\mathbb{R}^{n-1}}} = \frac{\|\nabla w_{\varepsilon,\mathcal{M}}\|_{p,\mathbb{R}^n_+,\mathcal{M}}}{\|w_{\varepsilon,\mathcal{M}}(\cdot, 0)\|_{p^\star,\mathbb{R}^{n-1}}},$$

where

$$w_{\varepsilon,\mathcal{M}}(x) = \big[\mathcal{M}_o(x - x^{\varepsilon})\big]^{-\frac{n-p}{p-1}}, \tag{36}$$

and $x^{\varepsilon} = (0, \dots, 0, -\varepsilon)$.

Verbatim repetition of the proof of Proposition 1 gives us

Proposition 3. *Let Ω be as in Proposition 1. Let the infimum in $(\mathrm{I}_{\mathcal{M}})$ satisfy the inequality $\lambda_1(n, p, \Omega, \mathcal{M}) < K(n, p, \mathcal{M})$. Then the infimum is attained.*

Remark 6. *Under suitable normalization the extremal function in $(\mathrm{I}_{\mathcal{M}})$, if it exists, is a positive solution to the Neumann-type problem*

$$-\Delta_{p,\mathcal{M}} u + u^{p-1} = 0 \quad in \ \ \Omega; \qquad \langle \mathbf{n}, \nabla\mathcal{M}(\xi)\rangle\Big|_{\substack{\xi=\nabla u\\ x\in\partial\Omega}} = u^{p^\star-1} \quad on \ \ \partial\Omega.$$

Here

$$\Delta_{p,\mathcal{M}} u = \mathrm{div}\Big(\mathcal{M}^{p-1}(\xi)\nabla\mathcal{M}(\xi)\Big|_{\xi=\nabla u}\Big)$$

is the generalized p-Laplacian, generated by $\mathcal{M}$. In particular, for $\mathcal{M}(x) = |x|_q$

$$\Delta_{p,\mathcal{M}} u = \Delta_{p,q} u \equiv \sum_{j=1}^{n} \big(|\nabla u|_q^{p-q} |u_{x_j}|^{q-2} u_{x_j}\big)_{x_j}.$$

Notice that $\Delta_{p,2} = \Delta_p$ is the conventional p-Laplacian, while $\Delta_{p,p}$ is the so-called pseudo-p-Laplacian (see, e.g., [BK]).

Theorem 5. *Let $n \geqslant 2$, and let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in \mathcal{C}^2$. Then for some $\beta(\Omega, \mathcal{M}) > 0$ and for $1 < p < \frac{n+1}{2} + \beta$, the infimum in $(\mathrm{I}_{\mathcal{M}})$ is attained.*

Proof. We introduce a local coordinate system just as in the proof of Theorem 1 and consider the function $u(x) = \varphi(x) w_{\varepsilon,\mathcal{M}}(x)$, where $w_{\varepsilon,\mathcal{M}}$ is defined in (36), while φ is a smooth cut-off function equal to 1 for $\mathcal{M}_o(x) < \rho/2$ and to 0 for $\mathcal{M}_o(x) > \rho$.

As in subsection 2.1, we obtain for $p < \frac{n+1}{2}$

$$\int\limits_{\Omega} \mathcal{M}^p(\nabla u)\, dx \leqslant \int\limits_{Q_\rho\cap\Omega} \frac{\big(\frac{n-p}{p-1}\big)^p dx}{\big[\mathcal{M}_o(x - x^{\varepsilon})\big]^{p'(n-1)}} \leqslant$$

$$\leqslant \Big(\frac{n-p}{p-1}\Big)^{p-1} E_1^{\mathcal{M}} \varepsilon^{-\frac{n-p}{p-1}} - \Big(\frac{n-p}{p-1}\Big)^{p} E_2^{\mathcal{M}} \varepsilon^{-\frac{n-1}{p-1}-1} + C(\rho).$$

Here

$$E_1^{\mathcal{M}} = \int\limits_{\mathbb{R}^{n-1}} \int\limits_{0}^{\infty} \frac{\frac{n-p}{p-1}\, dx_n dx'}{\big[\mathcal{M}_o(x', x_n + 1)\big]^{p'(n-1)}} =$$

$$= \int\limits_{\mathbb{R}^{n-1}} \frac{dy'}{\big[\mathcal{M}_o(y', 1)\big]^{p'(n-1)}} = \int\limits_{\mathbb{S}_1^{\mathcal{M}}} \int\limits_{0}^{\infty} \frac{r^{-\frac{n-1}{p-1}-1}\, dr\, d\omega}{f^{p'(n-1)}(r^{-1})};$$

$$E_2^{\mathcal{M}}=\int\limits_{\mathbb{R}^{n-1}}\frac{(\mathcal{A}x',x')\,dx'}{\big[\mathcal{M}_o(x',1)\big]^{p'(n-1)}}\cdot(1+o_\rho(1))=\int\limits_{\mathbb{S}_1^{\mathcal{M}}}(\mathcal{A}\omega,\omega)\,d\omega\int\limits_0^\infty\frac{r^{-\frac{n-p}{p-1}}\,dr}{f^{p'(n-1)}(r^{-1})}\cdot(1+o_\rho(1)),$$

and $f(t)=\mathcal{M}_o(\omega,t)$.

Further,

$$\int\limits_{\partial\Omega}u^{p^*}\,d\Sigma\geqslant\int\limits_{\mathcal{M}_o(x')\leqslant\frac{\rho}{2}}\frac{dx'}{\big[\mathcal{M}_o(x',\varepsilon+F(x'))\big]^{p'(n-1)}}=$$

$$=\varepsilon^{-\frac{n-1}{p-1}}\int\limits_{\mathbb{S}_1^{\mathcal{M}}}\int\limits_0^{\frac{\rho}{\varepsilon}}\frac{r^{-\frac{n-1}{p-1}-1}\,dr\,d\omega}{f^{p'(n-1)}\big(r^{-1}+(\varepsilon r)^{-1}F(\varepsilon r\omega)\big)}.$$

By the Taylor formula,

$$\frac{1}{f^\alpha\big(r^{-1}+(\varepsilon r)^{-1}F(\varepsilon r\omega)\big)}=\frac{1}{f^\alpha(r^{-1})}\cdot\Big[1-\alpha\frac{f'(r^{-1})}{f(r^{-1})}\,\frac{F(\varepsilon r\omega)}{\varepsilon r}+\frac{F(\varepsilon r\omega)}{\varepsilon r f(r^{-1})}\cdot o_\rho(1)\Big],$$

and hence

$$\int\limits_{\partial\Omega}u^{p^*}\,d\Sigma\geqslant E_1^{\mathcal{M}}\varepsilon^{-\frac{n-1}{p-1}}-D_2^{\mathcal{M}}\varepsilon^{1-\frac{n-1}{p-1}}-C(\rho),$$

where

$$D_2^{\mathcal{M}}=p'(n-1)\int\limits_{\mathbb{S}_1^{\mathcal{M}}}(\mathcal{A}\omega,\omega)\,d\omega\int\limits_0^\infty\frac{r^{-\frac{n-1}{p-1}}f'(r^{-1})\,dr}{f^{p'(n-1)+1}(r^{-1})}\cdot(1+o_\rho(1)),$$

Finally, the estimate (18) remains valid.

Substituting these estimates into $(\mathrm{I}_{\mathcal{M}})$, similarly to (19), we arrive at

$$\frac{\|\nabla u\|^p_{p,\Omega,\mathcal{M}}+\|u\|^p_{p,\Omega}}{\|u\|^p_{p^\star,\partial\Omega}}\leqslant K^p(n,p,\,\mathcal{M})\cdot\Big[1+\Big(\frac{p}{p^\star}\,\frac{D_2^{\mathcal{M}}}{E_1^{\mathcal{M}}}-\frac{n-p}{p-1}\,\frac{E_2^{\mathcal{M}}}{E_1^{\mathcal{M}}}\Big)\,\varepsilon+o(\varepsilon)\Big].$$

Integrating by parts, we obtain $D_2^{\mathcal{M}}=\frac{n-2p+1}{p-1}\,E_2^{\mathcal{M}}\cdot(1+o_\rho(1))$. Therefore,

$$\lambda_1^p(n,p,\Omega,\mathcal{M})\leqslant K^p(n,p,\,\mathcal{M})\cdot\Big[1-2\frac{n-p}{n-1}\,\frac{E_2^{\mathcal{M}}}{E_1^{\mathcal{M}}}\,(1+o_\rho(1))\varepsilon+o(\varepsilon)\Big]<K^p(n,p,\,\mathcal{M}).\tag{37}$$

If $p=\frac{n+1}{2}$ then $\int_\Omega\mathcal{M}^p(\nabla u)\,dx\leqslant\big(\frac{n-p}{p-1}\big)^{p-1}E_1^{\mathcal{M}}\varepsilon^{-1}-C\ln(\varepsilon)+C(\rho)$. This gives (37) for $p=\frac{n+1}{2}$ and, by continuity, for some $\beta>0$ and $p<\frac{n+1}{2}+\beta$. The application of Proposition 3 completes the proof. □

In a similar way one can prove analog of Theorem 4.

Theorem 6. *Let $n\geqslant 3$, and let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega\in\mathcal{C}^2$. Then for some $\beta(\Omega,\mathcal{M})>0$ and for $1<p<\frac{n+1}{2}+\beta$, the infimum in $(\mathrm{II}_{\mathcal{M}})$ is attained.*

References

[BK] M. Belloni and B. Kawohl, *The pseudo-p-Laplace eigenvalue problem and viscosity solutions as* $p \to \infty$, ESAIM Contrôle Optim. Calc. Var. **10** (2004), 28–52.

[DN] A. V. Demyanov and A. I. Nazarov, *On the existence of an extremal function in Sobolev embedding theorems with critical exponents*, Algebra Anal. **17** (2005), N5, 105–140 (Russian). English transl. in: St.Petersburg Math. J. **17** (2006), N5, 108-142.

[Dr] O. Druet, *Optimal Sobolev inequalities of arbitrary order on compact Riemannian manifolds*, J. Funct. Anal. **159** (1998), 217–242.

[E] J. F. Escobar, *Sharp constant in a Sobolev trace inequality*, Indiana Univ. Math. J. **37** (1988), 687–698.

[FBR] J. Fernandez Bonder and J. D. Rossi, *On the existence of extremals for the Sobolev trace embedding theorem with critical exponent*, Bull. London Math. Soc. **37** (2005), N1, 119–125.

[Ls] P.-L. Lions, *The concentration-compactness principle in the Calculus of Variations. The limit case*, Parts 1 and 2, Rev. Mat. Iberoamericana **1** (1985), 45–121 (1), P.145–201 (2).

[LPT] P.-L. Lions, F. Pacella, and M. Tricarico, *Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions*, Indiana Univ. Math. J. **37** (1988), 301–324.

[Nt] B. Nazaret, *Best constant in Sobolev trace inequalities on the half-space*, Nonlin. Analysis, **65** (2006), N10, 1977–1985.

[N] A. I. Nazarov, *Dirichlet and Neumann problems to critical Emden–Fowler type equations*, J. Global Optim. **40** (2008), 289–303.