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Attainability of infima in the critical Sobolev trace embedding theorem on manifolds

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Abstract

Sufficient conditions for the existence of extremal functions in the trace Sobolev inequality and the trace Sobolev–Poincaré inequality on Riemannian manifolds are established. It is shown that some of these conditions are sharp.

1 Introduction

Let $n \geqslant 2$, and let Ω be an n-dimensional ($\mathcal{C}^3$-smooth compact Riemannian) manifold with $\mathcal{C}^3$-smooth boundary. For $1 < p < n$ we denote by $p^\star = \frac{(n-1)p}{n-p}$ the trace Sobolev exponent for p that is the critical exponent for the trace embedding $W^1_p(\Omega) \hookrightarrow L_q(\partial\Omega)$.

Since the embedding operator $W^1_p(\Omega) \hookrightarrow L_{p^\star}(\partial\Omega)$ is noncompact, the problem of attainability of the norm of this operator (i.e. the problem of existence of an extremal function in the trace embedding theorem) is nontrivial. Corresponding problem for conventional embedding $W^1_p(\Omega) \hookrightarrow L_{p^*}(\Omega)$ (here $p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p) was treated in many papers, see, e.g., the recent survey [N] and further references therein.

The problem for the trace embedding is considerably less investigated. Let us consider the inequality

$$K(n,p) \equiv \inf_{v \in \dot{\mathcal{C}}^\infty(\mathbb{R}^n_+)\setminus\{0\}} \frac{\|\nabla v\|_{p,\mathbb{R}^n_+}}{\|v(\cdot,0)\|_{p^\star,\mathbb{R}^{n-1}}} > 0, \tag{1}$$

where $\dot{\mathcal{C}}^\infty(\mathbb{R}^n_+)$ is the set of functions on $\mathbb{R}^n_+$ with bounded support.

Obviously, the functional in (1) is invariant with respect to translations and dilations of v. It was proved in the remarkable paper [Nt] that the infimum in (1) is attained on the function with *unbounded* support

$$\widetilde{w}_\varepsilon(x) = |x - x^\varepsilon|^{-\frac{n-p}{p-1}}, \tag{2}$$

with $x^\varepsilon = (0,\dots,0,-\varepsilon)$ (for the case $p = 2$ this was established earlier in [E]). The result of [Nt] implies

$$K(n,p) = \Big(\frac{n-p}{p-1}\Big)^{\frac{1}{p'}} \Big(\frac{\omega_{n-2}}{2} \cdot \mathcal{B}\Big(\frac{n-1}{2}, \frac{n-1}{2(p-1)}\Big)\Big)^{\frac{1}{(n-1)p'}}.$$

In the paper [NR] authors considered the critical trace embedding in bounded domain $\Omega \Subset \mathbb{R}^n$, i.e. the inequality

$$(I) \qquad \lambda_1(n,p,\Omega) = \inf_{v \in W^1_p(\Omega)\setminus\{0\}} \frac{\|v\|_{W^1_p(\Omega)}}{\|v\|_{p^\star,\partial\Omega}} > 0$$

(the norm of the numerator is defined as $\|v\|^p_{W^1_p(\Omega)} = \|\nabla v\|^p_{p,\Omega} + \|v\|^p_{p,\Omega}$). In [NR, Remark 4] it is noted that the main result holds true for an arbitrary manifold Ω with a smooth boundary, provided $\partial\Omega$ contains a point with positive mean curvature (with respect to the inner normal). Namely, in this case the infimum in (I) is attained for $1 < p < \frac{n+1}{2} + \beta$, where $\beta > 0$ depends on Ω.

This paper deals with more complicated case where the mean curvature is non-positive. Our first result reads as follows.

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Theorem 1. *Let $n \geqslant 5$, and let Ω be an n-dimensional manifold with $\mathcal{C}^3$-smooth boundary. Suppose that the mean curvature of $\partial\Omega$ with respect to the inner normal is non-positive everywhere. Finally, suppose that there exists a point $y^0 \in \partial\Omega$ such that $\partial\Omega$ is a totally geodesic submanifold in Ω at y^0 and the scalar curvature of Ω is positive at y^0. Then for some $\beta > 0$, for $2 < p < \frac{n+2}{3} + \beta$, the infimum in* (I) *is attained.*

Remark 1. *By standard argument it follows that under suitable normalization the extremal function in* (I)*, if it exists, is a positive solution to the nonlinear Neumann problem*

$$-\Delta_p u + u^{p-1} = 0 \quad in \;\; \Omega; \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \mathbf{n}} = u^{p^\star - 1} \quad on \;\; \partial\Omega$$

(here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$).

Observe that, in contrast to [NR], the assumptions of Theorem 1 contain non-trivial left border of the interval for p. We show that this left border is sharp.

Theorem 2. *Let $n \geqslant 2$, and let Ω be an n-dimensional hemisphere*

$$\Omega = \{(\theta, \phi_1, \dots, \phi_{n-1}) \in \mathbb{S}^n : \; 0 < \theta < \frac{\pi}{2}\}.$$

Then for any $1 < p < 2$, there exists $\varkappa^* > 0$ such that for $\varkappa > \varkappa^*$ the infimum in (I) *is not attained on $\varkappa\Omega$. For $n \geqslant 3$ this is true also for $p = 2$.*

Remark 2. *It is easy to see that the non-attainability of the infimum in* (I) *on $\varkappa\Omega$ for large values of $\varkappa$ is equivalent to the validity of the* "optimal trace Sobolev inequality"

$$\|v\|^p_{p^\star,\partial\Omega} \leqslant K^p(n,p)\cdot\|\nabla v\|^p_{p,\Omega} + C(p,\Omega)\cdot\|v\|^p_{p,\Omega}, \qquad v \in W^1_p(\Omega).$$

For the conventional optimal Sobolev inequality see [Dr1] and references therein.

Remark 3. *Note that the fact that Ω is a hemisphere is used only for symmetrization arguments. Using the techniques of [Dr1], one can obtain the estimate (42) and prove Theorem 2 in general case under assumption that $\partial\Omega$ is a totally geodesic submanifold in Ω.*

Finally, we consider the trace Sobolev–Poincaré inequality

$$(II) \qquad \lambda_2(n,p,\Omega) = \inf_{v \in W^1_p(\Omega)\setminus\{c\}} \frac{\|\nabla v\|_{p,\Omega}}{\|v - \overline{v}\|_{p^\star,\partial\Omega}} > 0$$

(here we use the notation $\overline{v} = |\partial\Omega|^{-1}\int\limits_{\partial\Omega} v\, d\Sigma$).

Theorem 3. *Let $n \geqslant 4$, and let Ω be as in Theorem 1. Then for some $\beta(\Omega) > 0$ and for $1 < p < \frac{n+2}{3} + \beta$, the infimum in* (II) *is attained.*

The structure of our paper is as follows. In §2 we establish required integral estimates and prove Theorem 1. Theorem 2 is proved in §3; inequality (II) is considered in §4.

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Let us recall some notations. $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ is a point in $\mathbb{R}^n$. Put $|x'| = \sqrt{x_1^2 + \dots + x_{n-1}^2}$ and $Q_\rho = \{x : |x'| < \rho, 0 < x_n < \rho\}$. By y we denote points in Ω.

Consider the exponential map at $y^0 \in \partial\Omega$ and recall (see, e.g., [BZ, §11]) that there exists $\rho_0 > 0$ such that $\exp^{-1}_{y^0}$ is a diffeomorphism from the Riemannian ball $B_{2\rho_0}(y^0) \cap \Omega$ onto the

set $B_{2\rho_0}(0) \cap \{x_n > F(x')\} \subset \mathbb{R}^n$. Moreover, one can suppose $F(0) = \nabla' F(0) = 0$. In what follows we always assume $\rho < \rho_0$. For a function f on $B_{2\rho_0}(y^0) \cap \Omega$ we define the "transplanted" function $\widetilde{f}(x) \equiv f(\exp_{y^0}(x))$.

We denote by $G = (g_{ij}(x))$ the Riemannian metric tensor on Ω and by $(g^{ij}(x))$ the inverse tensor. It is well known that for $f : \Omega \to \mathbb{R}$ the Riemannian length of ∇f is given by $|\nabla f|^2 = \sum\limits_{ij} g^{ij} \widetilde{f}_{x_i} \widetilde{f}_{x_j}$. Also $dy = \sqrt{\det(G)}dx$.

$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ stands for the area of the unit sphere in $\mathbb{R}^n$. We denote by $p' = \frac{p}{p-1}$ the Hölder conjugate exponent to p, and put $q = \frac{n-1}{2}p'$. Γ is the Euler gamma-function, $\mathcal{B}$ is the Euler beta-function.

We denote by $o_\rho(1)$ a quantity which tends to zero as $\rho \to 0$. We use letter C to denote various positive constants. To indicate that C depends on some parameters, we write $C(\dots)$.

2 Existence of the minimizer

Proof of Theorem 1. Our main tool is the concentration-compactness principle of Lions ([Ls]). It is used in various forms; for the problem (I) it can be reformulated as follows (the proof is verbatim repetition of [NR, Proposition 1]).

Proposition 1. *Let the infimum in* (I) *satisfy the inequality*

$$\lambda_1(n, p, \Omega) < K(n, p).$$

Then the infimum is attained.

Thus, to prove the attainability of the infimum in (I) it is sufficient to present a function such that the quotient in (I) is less than $K(n, p)$. Following [DN], see also [LPT], we succeed, constructing a function with a small support, simulating the behavior of $\widetilde{w}_\varepsilon$ in Riemannian normal coordinates. Namely, for sufficiently small $\varepsilon > 0$ and $\rho > 0$ we introduce the function $u(y)$ such that

$$\widetilde{u}(x) = \widetilde{\varphi}(|x'|, x_n)\widetilde{w}_\varepsilon(x). \tag{3}$$

Here $\widetilde{w}_\varepsilon$ is defined in (2), while $\widetilde{\varphi}$ is a smooth cut-off function such that

$$\widetilde{\varphi} = 1 \quad \text{in} \quad Q_{\frac{\rho}{2}}; \qquad \widetilde{\varphi} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus Q_\rho,$$

and $|\nabla\widetilde{\varphi}| \leqslant \frac{C}{\rho}$.

2.1 Auxiliary relations

It is well known (see, e.g., [BZ, §12.6]) that

$$g_{ij}(0) = \delta_{ij}, \ \frac{\partial g_{ij}}{\partial x_k}(0) = 0. \tag{4}$$

Further, from [BZ, §§14.5 and 19.5] we conclude that for $i \neq j$

$$\frac{\partial^2 g_{ii}}{\partial x_j^2}(0) = -\frac{\mathcal{R}_{ij}}{3}; \qquad \frac{\partial^2 g_{ij}}{\partial x_i \partial x_j}(0) = \frac{2\mathcal{R}_{ij}}{3}. \tag{5}$$

Here $\mathcal{R}_{ij} = K_\sigma(e_i, e_j)$ is a sectional curvature of Ω at y^0. We write $\mathcal{R}_{ii} = 0$ so the formulas (5) hold true even for $i = j$.

Also we denote by

$$\mathcal{R}_i = \sum_j \mathcal{R}_{ij} = Ric(e_i)$$

a Ricci curvature at y^0 and introduce the notation

$$\mathcal{R}'_i = \sum_{j<n} \mathcal{R}_{ij}, \qquad i < n.$$

Next, consider the function F giving the local representation of $\partial\Omega$ at y^0. Since the boundary is $\mathcal{C}^3$-smooth, we have

$$F(x') = \sum_{i,j<n} a_{ij} x_i x_i + \sum_{i,j,k<n} b_{ijk} x_i x_j x_k + o(|x'|^3) \quad \text{as} \quad x' \to 0. \tag{6}$$

Moreover, the assumption on $\partial\Omega$ to be totally geodesic at y^0 means that $a_{ij} = 0$.

2.2 Estimate of $\|\nabla u\|_{p,\Omega}$ for $1 < p \leqslant \frac{n+2}{3}$

We apply the estimate

$$|\nabla(fg)|^p \leqslant |f\,\nabla g|^p + C\,\left(|\nabla f\, g|^p + |\nabla f\, g| \cdot |f\,\nabla g|^{p-1}\right) \tag{7}$$

to ∇u. Since $\nabla\widetilde{\varphi}$ does not vanish only in $Q_\rho \setminus Q_{\frac{\rho}{2}}$, one obtains

$$\int\limits_\Omega \left(|\nabla\varphi \cdot w_\varepsilon|^p + |\nabla\varphi \cdot w_\varepsilon| \cdot |\varphi\,\nabla w_\varepsilon|^{p-1}\right)\, dy \leqslant$$

$$\leqslant C\left(\int\limits_0^{\rho/2}\int\limits_{\rho/2}^{\rho} + \int\limits_{\rho/2}^{\rho}\int\limits_0^{\rho}\right)\, \frac{\rho^{-p} r^{n-2}\, dx_n dr}{(r^2+x_n^2)^{\frac{p(n-p)}{2(p-1)}}} \cdot \left[1 + \left(\frac{\rho}{\sqrt{r^2+x_n^2}}\right)^{p-1}\right] \leqslant C\,\rho^{n-2q}. \tag{8}$$

The inequalities (7) and (8) imply

$$\int\limits_\Omega |\nabla u|^p dy \leqslant \int\limits_{\Omega\cap\exp_{y^0}(Q_\rho)} |\nabla w_\varepsilon|^p dy + C\rho^{n-2q}. \tag{9}$$

We have

$$\int\limits_{\Omega\cap\exp_{y^0}(Q_\rho)} |\nabla w_\varepsilon|^p dy = \int\limits_{|x'|<\rho}\int\limits_{F(x')}^{\rho} \left(\sum_{ij} g^{ij} \frac{\partial\widetilde{w_\varepsilon}}{\partial x_i}\frac{\partial\widetilde{w_\varepsilon}}{\partial x_j}\right)^{\frac{p}{2}} \cdot \sqrt{\det(G)}\, dx_n dx'.$$

Expanding the metric tensor by the Taylor formula and using (4), we obtain

$$\int\limits_{\Omega\cap\exp_{y^0}(Q_\rho)} |\nabla w_\varepsilon|^p dy =$$

$$= \mathcal{I}_1 + \frac{1}{4}\sum_{ijk} \frac{\partial^2 g_{kk}}{\partial x_i \partial x_j}(0) \cdot \mathcal{I}_2^{(ij)} - \frac{p}{4}\sum_{ijkl} \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(0) \cdot \mathcal{I}_3^{(ijkl)} + \mathcal{I}_4 - \mathcal{I}_5, \tag{10}$$

where

$$\mathcal{I}_1 = \int\limits_{|x'|<\rho} \int\limits_0^\rho |\nabla \widetilde{w}_\varepsilon|^p dx_n dx';$$

$$\mathcal{I}_2^{(ij)} = \int\limits_{|x'|<\rho} \int\limits_0^\rho x_i x_j |\nabla \widetilde{w}_\varepsilon|^p dx_n dx';$$

$$\mathcal{I}_3^{(ijkl)} = \int\limits_{|x'|<\rho} \int\limits_0^\rho x_k x_l \frac{\partial \widetilde{w}_\varepsilon}{\partial x_i} \frac{\partial \widetilde{w}_\varepsilon}{\partial x_j} |\nabla \widetilde{w}_\varepsilon|^{p-2} dx_n dx';$$

$$\mathcal{I}_4 = \int\limits_{|x'|<\rho} \int\limits_0^\rho |x|^2 |\nabla \widetilde{w}_\varepsilon|^p dx_n dx' \cdot o_\rho(1);$$

$$\mathcal{I}_5 = \int\limits_{|x'|<\rho} \int\limits_0^{F(x')} |\nabla \widetilde{w}_\varepsilon|^p \cdot \big(1 + o(\rho^2)\big)\, dx_n dx'.$$

It is easy to see that

$$\mathcal{I}_1 = \Big(\frac{n-p}{p-1}\Big)^p \cdot \big(\varepsilon^{n-2q} \mathcal{E}_1 - O(\rho^{n-2q})\big), \tag{11}$$

where

$$\mathcal{E}_1 = \omega_{n-2} \int\limits_0^\infty \int\limits_0^\infty \frac{r^{n-2} dr dx_n}{(r^2 + (x_n+1)^2)^q} = \frac{\omega_{n-2}}{2(2q-n)} \cdot \mathcal{B}\big(\tfrac{n-1}{2}, \tfrac{n-1}{2(p-1)}\big). \tag{12}$$

In a similar way we get

$$\mathcal{I}_2^{(ij)} = \Big(\frac{n-p}{p-1}\Big)^p \delta_{ij} \cdot \left\{ \begin{array}{rcll} \frac{\varepsilon^{n-2q+2}}{2q-n-2} \cdot \mathcal{E}_2^{(j)} & - & O(\rho^{n-2q+2}), & p < \frac{n+2}{3}; \\ \ln(\frac{1}{\varepsilon}) \cdot \mathcal{E}_2^{(j)} & - & O(\ln(\frac{1}{\rho})), & p = \frac{n+2}{3}, \end{array} \right. \tag{13}$$

where

$$\mathcal{E}_2^{(j)} = \frac{\omega_{n-2}}{2q-n-1} \cdot \mathcal{B}\big(\tfrac{n-1}{2}, \tfrac{n-1}{2(p-1)}\big) \cdot \left\{ \begin{array}{ll} \frac{1}{2}, & j < n; \\ \frac{1}{2q-n}, & j = n. \end{array} \right. \tag{14}$$

To calculate $\mathcal{I}_3$, we consider three cases.

1. $i < n$, $j < n$. Then

$$\mathcal{I}_3^{(ijkl)} = \Big(\frac{n-p}{p-1}\Big)^p \int\limits_{|x'|<\rho} \int\limits_0^\rho \frac{x_i x_j x_k x_l dx}{(|x'|^2 + (x_n+\varepsilon)^2)^{q+1}} dx =$$

$$= (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \mathcal{F}^{(ij)} + \delta_{ij}\delta_{kl} \mathcal{F}^{(ik)}, \tag{15}$$

where, for $i \neq k$,

$$\mathcal{F}^{(ik)} = \Big(\frac{n-p}{p-1}\Big)^p \int\limits_{|x'|<\rho} \int\limits_0^\rho \frac{x_i^2 x_k^2 dx}{(|x'|^2 + (x_n+\varepsilon)^2)^{q+1}} dx =$$

$$= \Big(\frac{n-p}{p-1}\Big)^p \cdot \frac{1}{2q} \cdot \left\{ \begin{array}{rcll} \frac{\varepsilon^{n-2q+2}}{2q-n-2} \cdot \mathcal{E}_2^{(k)} & + & O(\rho^{n-2q+2}), & p < \frac{n+2}{3}; \\ \ln(\frac{1}{\varepsilon}) \cdot \mathcal{E}_2^{(k)} & + & O(\ln(\frac{1}{\rho})), & p = \frac{n+2}{3} \end{array} \right. \tag{16}$$

(note that the case $i = k$ is not interesting for us due to $\mathcal{R}_{ii} = 0$).

2. $i < n$, $j = n$. Then

$$\mathcal{I}_3^{(ijkl)} = \Big(\frac{n-p}{p-1}\Big)^p \int\limits_{|x'|<\rho} \int\limits_0^\rho \frac{x_i(x_n+\varepsilon)x_k x_l dx}{(|x'|^2+(x_n+\varepsilon)^2)^{q+1}} dx = (\delta_{ik}\delta_{ln} + \delta_{il}\delta_{kn})\mathcal{F}_1^{(i)}, \tag{17}$$

where

$$\mathcal{F}_1^{(i)} = \Big(\frac{n-p}{p-1}\Big)^p \int\limits_{|x'|<\rho} \int\limits_0^\rho \frac{x_i^2(x_n+\varepsilon)x_n dx}{(|x'|^2+(x_n+\varepsilon)^2)^{q+1}} dx =$$

$$= \Big(\frac{n-p}{p-1}\Big)^p \cdot \frac{1}{2q} \cdot \begin{cases} \frac{\varepsilon^{n-2q+2}}{2q-n-2} \cdot \mathcal{E}_2^{(i)} & + \quad O(\rho^{n-2q+2}), & p < \frac{n+2}{3}; \\ \ln(\frac{1}{\varepsilon}) \cdot \mathcal{E}_2^{(i)} & + \quad O(\ln(\frac{1}{\rho})), & p = \frac{n+2}{3}. \end{cases} \tag{18}$$

Obviously, similar result is true for $i = n$, $j < n$.

2. $i = j = n$. Then

$$\mathcal{I}_3^{(ijkl)} = \Big(\frac{n-p}{p-1}\Big)^p \int\limits_{|x'|<\rho} \int\limits_0^\rho \frac{(x_n+\varepsilon)^2 x_k x_l dx}{(|x'|^2+(x_n+\varepsilon)^2)^{q+1}} dx = \delta_{kl}\mathcal{F}_2^{(k)}, \tag{19}$$

where, for $k \neq n$,

$$\mathcal{F}_2^{(k)} = \Big(\frac{n-p}{p-1}\Big)^p \int\limits_{|x'|<\rho} \int\limits_0^\rho \frac{x_k^2(x_n+\varepsilon)^2 dx}{(|x'|^2+(x_n+\varepsilon)^2)^{q+1}} dx =$$

$$= \Big(\frac{n-p}{p-1}\Big)^p \cdot \frac{2q-n-1}{2q} \cdot \begin{cases} \frac{\varepsilon^{n-2q+2}}{2q-n-2} \cdot \mathcal{E}_2^{(k)} & + \quad O(\rho^{n-2q+2}), & p < \frac{n+2}{3}; \\ \ln(\frac{1}{\varepsilon}) \cdot \mathcal{E}_2^{(k)} & + \quad O(\ln(\frac{1}{\rho})), & p = \frac{n+2}{3}. \end{cases} \tag{20}$$

Further, it is obvious that

$$\mathcal{I}_4 = \mathcal{I}_2^{(nn)} \cdot o_\rho(1). \tag{21}$$

Finally, the last term in (10) can be estimated using (6):

$$\mathcal{I}_5 = \Big(\frac{n-p}{p-1}\Big)^p \int\limits_{|x'|<\rho} \sum_{i,j,k<n} \frac{b_{ijk}x_i x_j x_k}{(|x'|^2+\varepsilon^2)^q}\, dx' +$$

$$+ \int\limits_{|x'|<\rho} \frac{|x'|^3\, dx'}{(|x'|^2+\varepsilon^2)^q} \cdot o_\rho(1) = \mathcal{I}_2^{(nn)} \cdot o_\rho(1). \tag{22}$$

Substituting relations (10)-(22) into (9), we obtain with regards to (5),

$$\int\limits_\Omega |\nabla u|^p dy \leqslant \Big(\frac{n-p}{p-1}\Big)^p \times \begin{cases} \varepsilon^{n-2q}\mathcal{E}_1 - \mathcal{E}_2\frac{\varepsilon^{n-2q+2}}{2q-n-2} + O(\rho^{n-2q}), & p < \frac{n+2}{3}; \\ \varepsilon^{n-2q}\mathcal{E}_1 - \mathcal{E}_2\ln(\frac{1}{\varepsilon}) + O(\rho^{n-2q}), & p = \frac{n+2}{3}, \end{cases} \tag{23}$$

where

$$\mathcal{E}_2 = \frac{\mathcal{E}_2^{(1)}}{12(2q-n+1)} \cdot \big(12\mathcal{R}_n + (2q-n+4)\sum_{i<n}\mathcal{R}_i' + o_\rho(1)\big). \tag{24}$$

2.3 Estimate of $\|u\|_{p^\star,\partial\Omega}$ for $1 < p < \frac{n+1}{2}$

It is obvious that the area on $\partial\Omega$ satisfies the relation

$$d\Sigma = \sqrt{1+|\nabla F(x')|^2}\cdot\sqrt{\det(G')}\,dx' \geqslant \sqrt{\det(G')}\,dx',$$

where $G' = (g_{ij})_{i,j<n}$. Thus, we have

$$\int\limits_{\partial\Omega} u^{p^\star}\,d\Sigma \geqslant \int\limits_{|x'|<\frac{\rho}{2}} \frac{\sqrt{\det(G')}\,dx'}{(|x'|^2+(\varepsilon+F(x'))^2)^q} =$$

$$= \int\limits_{|x'|<\frac{\rho}{2}} \frac{\sqrt{\det(G')}\,dx'}{(|x'|^2+\varepsilon^2)^q} \times \left[1 - 2q\frac{\varepsilon F(x')}{|x'|^2+\varepsilon^2} - \frac{F^2(x')}{|x'|^2+\varepsilon^2}\cdot O(1)\right]. \quad (25)$$

Expanding the metric tensor by the Taylor formula and using (4), we obtain

$$\int\limits_{\partial\Omega} u^{p^\star}\,d\Sigma \geqslant \mathcal{J}_1 + \frac{1}{4}\sum_{i,j,k<n}\frac{\partial^2 g_{kk}}{\partial x_i\partial x_j}(0)\cdot\mathcal{J}_2^{(ij)} - \mathcal{J}_3 - 2q\mathcal{J}_4 - \mathcal{J}_5, \quad (26)$$

where

$$\mathcal{J}_1 = \int\limits_{|x'|<\frac{\rho}{2}} \frac{dx'}{(|x'|^2+\varepsilon^2)^q};$$

$$\mathcal{J}_2^{(ij)} = \int\limits_{|x'|<\frac{\rho}{2}} \frac{x_i x_j dx'}{(|x'|^2+\varepsilon^2)^q};$$

$$\mathcal{J}_3 = \int\limits_{|x'|<\frac{\rho}{2}} \frac{|x'|^2\,dx'}{(|x'|^2+\varepsilon^2)^q}\cdot o_\rho(1);$$

$$\mathcal{J}_4 = \int\limits_{|x'|<\frac{\rho}{2}} \frac{\varepsilon F(x')\big(1+o(\rho^2)\big)\,dx'}{(|x'|^2+\varepsilon^2)^{q+1}};$$

$$\mathcal{J}_5 = \int\limits_{|x'|<\frac{\rho}{2}} \frac{F^2(x')\,dx'}{(|x'|^2+\varepsilon^2)^{q+1}}\cdot O(1).$$

It is easy to see that

$$\mathcal{J}_1 = \varepsilon^{n-2q-1}\mathcal{E}_1' - O(\rho^{n-2q-1}), \quad (27)$$

where

$$\mathcal{E}_1' = \omega_{n-2}\int\limits_0^\infty \frac{r^{n-2}dr}{(r^2+1)^q} = \frac{\omega_{n-2}}{2}\cdot\mathcal{B}\big(\tfrac{n-1}{2},\tfrac{n-1}{2(p-1)}\big). \quad (28)$$

Similarly,

$$\mathcal{J}_2^{(ij)} = \delta_{ij}\varepsilon^{n-2q+1}\mathcal{E}_2'^{(i)} - O(\rho^{n-2q+1}), \quad (29)$$

where

$$\mathcal{E}_2'^{(i)} = \int\limits_{\mathbb{R}^{n-1}} \frac{x_i^2\,dx'}{(|x'|^2+1)^q} = \frac{\omega_{n-2}}{2(2q-n-1)}\cdot\mathcal{B}\big(\tfrac{n-1}{2},\tfrac{n-1}{2(p-1)}\big). \quad (30)$$

Further, using (6) we obtain

$$\mathcal{J}_4 = \int\limits_{|x'|<\rho} \sum_{i,j,k<n} \frac{\varepsilon b_{ijk} x_i x_j x_k}{(|x'|^2+\varepsilon^2)^{q+1}}\, dx' + \\ + \int\limits_{|x'|<\rho} \frac{\varepsilon |x'|^3\, dx'}{(|x'|^2+\varepsilon^2)^{q+1}} \cdot o_\rho(1) = \mathcal{J}_2^{(11)} \cdot o_\rho(1). \quad (31)$$

Finally, it is obvious that

$$\mathcal{J}_3 = \mathcal{J}_2^{(11)} \cdot o_\rho(1); \qquad \mathcal{J}_5 = \mathcal{J}_2^{(11)} \cdot o_\rho(1). \quad (32)$$

Substituting relations (27)-(32) into (26), we obtain with regards to (5),

$$\int\limits_{\partial\Omega} u^{p^\star} d\Sigma \geqslant \varepsilon^{n-2q-1}\mathcal{E}_1' - \varepsilon^{n-2q+1}\mathcal{E}_2' - O(\rho^{n-2q-1}), \quad (33)$$

where

$$\mathcal{E}_2' = \frac{\mathcal{E}_2'^{(1)}}{12} \cdot \big(\sum_{i<n} \mathcal{R}_i' + o_\rho(1)\big). \quad (34)$$

2.4 Estimate of $\|u\|_{p,\Omega}$

Since $dy \leqslant C dx$ and $F(x') = O(|x'|^3)$ for $|x'| < \rho_0$, we have

$$\int\limits_\Omega u^p dy \leqslant C \int\limits_{|x'|<\rho} dx' \int\limits_{-C|x'|^3}^{\infty} \frac{dx_n}{(|x'|^2+(x_n+\varepsilon)^2)^{\frac{p(n-p)}{2(p-1)}}} \leqslant \\ \leqslant \begin{cases} C\varepsilon^{\frac{p^2-n}{p-1}}, & p<\sqrt{n}; \\ C(1+\ln\frac{\rho}{\varepsilon}), & p=\sqrt{n}; \\ C, & p>\sqrt{n}. \end{cases} \quad (35)$$

2.5 Completion of the proof

Estimates (23) and (35) imply

$$\|u\|^p_{W^1_p(\Omega)} \leqslant \begin{cases} \big(\frac{n-p}{p-1}\big)^p \varepsilon^{n-2q}\mathcal{E}_1 - \varepsilon^{n-2q+2}\frac{\mathcal{E}_2}{2q-n-2} + C\,\varepsilon^{\frac{p^2-n}{p-1}} + C(\rho), & 1<p<\sqrt{n}; \\ \big(\frac{n-p}{p-1}\big)^p \varepsilon^{n-2q}\mathcal{E}_1 - \varepsilon^{n-2q+2}\frac{\mathcal{E}_2}{2q-n-2} + C\ln(\frac{1}{\varepsilon}) + C(\rho), & p=\sqrt{n}; \\ \big(\frac{n-p}{p-1}\big)^p \varepsilon^{n-2q}\mathcal{E}_1 - \varepsilon^{n-2q+2}\frac{\mathcal{E}_2}{2q-n-2} + C(\rho), & \sqrt{n}<p<\frac{n+2}{3}; \\ \big(\frac{n-p}{p-1}\big)^p \varepsilon^{n-2q}\mathcal{E}_1 - \mathcal{E}_2\ln(\frac{1}{\varepsilon}) + C\,\rho^{\frac{p-n}{p-1}}, & p=\frac{n+2}{3}, \end{cases}$$

where $\mathcal{E}_1$, $\mathcal{E}_2$ are defined in (12) and (24).

Since $n-2q+2 < \frac{p^2-n}{p-1}$ for $p>2$, we get for $2<p<\frac{n+2}{3}$, with regard to (33),

$$\frac{\|u\|^p_{W^1_p(\Omega)}}{\|u\|^p_{p^\star,\partial\Omega}} \leqslant \left(\frac{n-p}{p-1}\right)^p \frac{\mathcal{E}_1}{(\mathcal{E}_1')^{\frac{p}{p^\star}}} \cdot \left(1+\varepsilon^2\left(\frac{p}{p^\star}\frac{\mathcal{E}_2'}{\mathcal{E}_1'} - \frac{\mathcal{E}_2}{(2q-n-2)\mathcal{E}_1}\right) + o(\varepsilon^2)\right).$$

Direct computation shows that for $2 < p < \frac{n+2}{3}$

$$\frac{p}{p^\star}\frac{\mathcal{E}_2'}{\mathcal{E}_1'} - \frac{\mathcal{E}_2}{(2q-n-2)\mathcal{E}_1} = -\frac{(2q-n)\big(2\mathcal{R}_n + \sum_{i<n}\mathcal{R}_i' + o_\rho(1)\big)}{2(2q-n+1)(2q-n-1)(2q-n-2)}. \tag{36}$$

The last quantity is negative for sufficiently small ρ since

$$2\mathcal{R}_n + \sum_{i<n}\mathcal{R}_i' = \sum_i \mathcal{R}_i = \mathcal{R}(y^0)$$

is the scalar curvature of Ω at y^0 which is positive by assumption.

Hence, for ε and ρ sufficiently small, one has

$$\frac{\|u\|_{W_p^1(\Omega)}}{\|u\|_{p^\star,\partial\Omega}} < \frac{n-p}{p-1}\frac{\mathcal{E}_1^{\frac{1}{p}}}{(\mathcal{E}_1')^{\frac{1}{p^\star}}}. \tag{37}$$

Similarly, in the case $p = \frac{n+2}{3}$ the relation (37) holds true for ε and ρ sufficiently small. By continuity, for some $\beta > 0$ and $p < \frac{n+2}{3} + \beta$

$$\lambda_1(n,p,\Omega) < \frac{n-p}{p-1}\frac{\mathcal{E}_1^{\frac{1}{p}}}{(\mathcal{E}_1')^{\frac{1}{p^\star}}} = K(n,p).$$

The application of Proposition 1 completes the proof. □

3 Nonexistence of the minimizer

Proof of theorem 2. We proceed by contradiction. Suppose there exists an unbounded sequence of $\varkappa$'s such that the infimum in (I) is attained on $\varkappa\Omega$. Then, by a standard argument, the corresponding minimizers $u_\varkappa$ are positive in $\varkappa\Omega$.

Consider the functions $v_\varkappa(y) = C(\varkappa)u_\varkappa(\frac{y}{\varkappa})$ in Ω, with $C(\varkappa)$ given by the normalization condition $\|v_\varkappa\|_{p^\star,\partial\Omega} = 1$. Then the weak Euler equation for the $v_\varkappa$ reads as follows:

$$\int_\Omega |\nabla v_\varkappa|^{p-2}\langle \nabla v_\varkappa, \nabla h\rangle\, dy + \varkappa^p \int_\Omega v_\varkappa^{p-1} h\, dy = \lambda_\varkappa^p \int_{\partial\Omega} v_\varkappa^{p^\star-1} h\, d\Sigma, \tag{38}$$

where $\lambda_\varkappa = \lambda_1(n,p,\varkappa\Omega)$. Since, given $\varkappa$, for the sequence considered in the proof of Theorem 1 the quotient (I) tends to $K(n,p)$, one has

$$\lambda_\varkappa \leqslant K(n,p). \tag{39}$$

Put $h = v_\varkappa$ in (38). Then

$$\|\nabla v_\varkappa\|_{p,\Omega}^p + \varkappa^p \|v_\varkappa\|_{p,\Omega}^p = \lambda_\varkappa^p. \tag{40}$$

Without loss of generality, $v_\varkappa \rightharpoonup v$ in $W_p^1(\Omega)$. From (40) we conclude that $v = 0$. Arguing as in the proof of Theorem 2.2 [LPT] one deduces that there exists a point $y^0 \in \partial\Omega$ such that $|v_\varkappa|^{p^\star} \rightharpoonup \delta(y - y^0)$ in the sense of measures on $\partial\Omega$. Therefore, for a given $\rho > 0$,

$$\|v_\varkappa\|_{p^\star,\partial\Omega\setminus B_{\frac{\rho}{2}}(y^0)} \to 0. \tag{41}$$

Remark 4. *In fact, the stronger property* $\sup_{\widetilde{\Omega}\setminus B_\rho(y^0)} v_\varkappa \to 0$ *can be proved, but we do not need it.*

Now we use spherical symmetrization (see, for instance, [PS]) in "planes" $\theta = const$. Since this operation retains the denominator in (I) and does not increase the numerator, we can assume that $v_\varkappa\big|_{\partial\Omega}$ is symmetrically decreasing around the point y^0. Since the normalization of $v_\varkappa$ implies $\|v_\varkappa\|_{p^\star,weak,\partial\Omega} \leqslant C$, this symmetry provides the estimate

$$\operatorname{dist}(y, y^0)^{\frac{n}{p}-1} \cdot v_\varkappa(y) \leqslant C, \qquad y \in \partial\Omega. \tag{42}$$

By η we denote a smooth cut-off function such that

$$\eta = 0 \quad \text{in} \quad B_{\frac{\varrho}{2}}(y^0); \qquad \eta = 1 \quad \text{in} \quad \mathbb{R}^n \setminus B_\rho(y^0).$$

We substitute $h = \eta^p v_\varkappa$ into the equation (38):

$$\int\limits_\Omega \eta^p |\nabla v_\varkappa|^p \, dy + p \int\limits_\Omega \eta^{p-1} |\nabla v_\varkappa|^{p-2} \langle \nabla v_\varkappa, \nabla \eta \rangle \, dy + \varkappa^p \int\limits_\Omega v_\varkappa^p \, dy \leqslant \lambda_\varkappa^p \int\limits_{\partial\Omega} \eta^p v_\varkappa^{p^\star} \, d\Sigma.$$

Recalling (39), we arrive at

$$\|\eta \nabla v_\varkappa\|_{p,\Omega}^p + \varkappa^p \|\eta v_\varkappa\|_{p,\Omega}^p \leqslant K^p(n,p) \int\limits_{\partial\Omega} \eta^p v_\varkappa^{p^\star} \, d\Sigma + C \|v_\varkappa\|_{p,\Omega} \cdot \|\eta \nabla v_\varkappa\|_{p,\Omega}^{p-1}. \tag{43}$$

The Hölder inequality and (41) give us

$$\int\limits_{\partial\Omega} \eta^p v_\varkappa^{p^\star} \, d\Sigma \leqslant \|\eta v_\varkappa\|_{p^\star,\partial\Omega}^p \cdot \|v_\varkappa\|_{p^\star,\partial\Omega\setminus B_{\frac{\varrho}{2}}(x^0)}^{p^\star - p} = \|\eta v_\varkappa\|_{p^\star,\partial\Omega}^p \cdot o_\varkappa(1).$$

We apply the trace embedding theorem to the function $\eta v_\varkappa$:

$$\|\eta v_\varkappa\|_{p^\star,\partial\Omega}^p \leqslant C(\|\nabla(\eta v_\varkappa)\|_{p,\Omega}^p + \|\eta v_\varkappa\|_{p,\Omega}^p) \leqslant C(\|\eta \nabla v_\varkappa\|_{p,\Omega}^p + \|v_\varkappa\|_{p,\Omega}^p). \tag{44}$$

Substituting these relations into (43), we obtain

$$\left(\frac{\|\eta \nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}}\right)^p \leqslant C \left(\frac{\|\eta \nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}}\right)^{p-1} + o_\varkappa(1) \cdot \left(1 + \left(\frac{\|\eta \nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}}\right)^p\right).$$

Therefore, for $\varkappa$ sufficiently large one has

$$\frac{\|\eta \nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}} \leqslant C. \tag{45}$$

To obtain accurate estimates of $v_\varkappa$ at the neighborhood of y^0, we use the idea of [Dr1]. Recall that $\exp_{y^0}$ is a diffeomorphism from the half-ball $B_{2\rho_0}^+ \subset \mathbb{R}^n$ onto Riemannian half-ball $B_{2\rho_0}^+(y^0)$. Without loss of generality, one can assume that $\rho < \rho_0$.

Denote by η_1 a smooth cut-off function such that

$$\eta_1 = 1 \quad \text{in} \quad B_\rho(y^0); \qquad \eta_1 = 0 \quad \text{in} \quad \mathbb{R}^n \setminus B_{2\rho}(y^0).$$

By (1),

$$\|\widetilde{\eta}_1 \widetilde{v}_\varkappa\|_{p^\star,\mathbb{R}^{n-1}}^p \leqslant \frac{1}{K^p(n,p)} \cdot \|\nabla(\widetilde{\eta}_1 \widetilde{v}_\varkappa)\|_{p,\mathbb{R}^n_+}^p. \tag{46}$$

By [BZ, §19.5],

$$|\nabla(\widetilde{\eta}_1\widetilde{v}_\varkappa)|(x) = |\nabla(\eta_1 v_\varkappa)|(y)\cdot\big(1+O(\mathrm{dist}^2(y,y^0))\big)$$

and

$$dx = dy\cdot\big(1+O(\mathrm{dist}^2(y,y^0))\big).$$

This gives

$$\|\nabla(\widetilde{\eta}_1\widetilde{v}_\varkappa)\|^p_{p,\mathbb{R}^n_+} = \int\limits_{B_{2\rho}(0)} |\nabla(\widetilde{\eta}_1\widetilde{v}_\varkappa)|^p(x)\,dx \leqslant$$

$$\leqslant \int\limits_{B_{2\rho}(y^0)} |\nabla(\eta_1 v_\varkappa)|^p(y)\cdot\big(1+C\mathrm{dist}^2(y,y^0)\big)\,dy \quad (47)$$

and

$$\|\widetilde{\eta}_1\widetilde{v}_\varkappa\|^{p^\star}_{p^\star,\mathbb{R}^{n-1}} = \int\limits_{B_{2\rho}(0)\cap\mathbb{R}^{n-1}} (\widetilde{\eta}_1\widetilde{v}_\varkappa)^{p^\star}(x')\,dx' \geqslant$$

$$\geqslant \int\limits_{B_{2\rho}(y^0)\cap\partial\Omega} (\eta_1 v_\varkappa)^{p^\star}(y)\cdot\big(1-C\mathrm{dist}^2(y,y^0)\big)\,dy \geqslant$$

$$\geqslant \|v_\varkappa\|^{p^\star}_{p^\star,B_\rho(y^0)\cap\partial\Omega} - C\int\limits_{\partial\Omega} v^{p^\star}_\varkappa(y)\cdot\mathrm{dist}^2(y,y^0)\,dy.$$

Moreover, with regard to the normalization of $v_\varkappa$,

$$\|\widetilde{\eta}_1\widetilde{v}_\varkappa\|^p_{p^\star,\mathbb{R}^{n-1}} \geqslant \|v_\varkappa\|^{p^\star}_{p^\star,B_\rho(y^0)\cap\partial\Omega} - C\int\limits_{\partial\Omega} v^{p^\star}_\varkappa(y)\cdot\mathrm{dist}^2(y,y^0)\,dy. \quad (48)$$

Substituting (47) and (48) into (46), we obtain by (7)

$$\|v_\varkappa\|^{p^\star}_{p^\star,\partial\Omega\cap B_\rho(y^0)} \leqslant$$

$$\leqslant \frac{1}{K^p(n,p)}\cdot\|\nabla v_\varkappa\|^p_{p,\Omega} + C\Big[\|v_\varkappa\|^p_{p,\Omega} + \|v_\varkappa\|_{p,\Omega}\cdot\|\nabla v_\varkappa\|^{p-1}_{p,\Omega\setminus B_\rho(y^0)}\Big] +$$

$$+ C\int\limits_{\Omega} |\nabla v_\varkappa|^p\,\mathrm{dist}^2(y,y^0))\,dy + C\int\limits_{\partial\Omega} v^{p^\star}_\varkappa\,\mathrm{dist}^2(y,y^0)\,dy. \quad (49)$$

We estimate the last term in the right-hand side of (49) by (42):

$$\int\limits_{\partial\Omega} v^{p^\star}_\varkappa\,\mathrm{dist}^2(y,y^0)\,dy =$$

$$= \int\limits_{\partial\Omega} \big(v^{\frac{p}{n-p}}_\varkappa\mathrm{dist}(y,y^0)\big)^{p-1}\cdot v^p_\varkappa\,\mathrm{dist}^{3-p}(y,y^0)\,dy \leqslant$$

$$\leqslant C\int\limits_{\partial\Omega} v^p_\varkappa\,\mathrm{dist}^{3-p}(y,y^0)\,dy.$$

Using the trace embedding theorem and the Hölder inequality, we obtain

$$\int\limits_{\partial\Omega} v_{\varkappa}^{p^\star}\,\mathrm{dist}^2(y,y^0)\,dy \leqslant$$

$$\leqslant C\biggl(\int\limits_{\Omega}|\nabla(v_{\varkappa}^p\,\mathrm{dist}^{3-p}(y,y^0))|\,dy+\int\limits_{\Omega}v_{\varkappa}^p\,\mathrm{dist}^{3-p}(y,y^0)\,dy\biggr)\leqslant$$

$$\leqslant C\biggl(\int\limits_{\Omega}|\nabla v_{\varkappa}|^p\,\mathrm{dist}^{(3-p)p}(y,y^0)\,dy\biggr)^{\frac{1}{p}}\cdot\|v_{\varkappa}\|_{p,\Omega}^{p-1}+C\int\limits_{\Omega}v_{\varkappa}^p\,\mathrm{dist}^{2-p}(y,y^0)\,dy\leqslant$$

$$\leqslant C\biggl(\int\limits_{\Omega}|\nabla v_{\varkappa}|^p\,\mathrm{dist}^{2}(y,y^0)\,dy\biggr)^{\frac{1}{p}}\cdot\|v_{\varkappa}\|_{p,\Omega}^{p-1}+C\|v_{\varkappa}\|_{p,\Omega}^{p}\quad(50)$$

(the last inequality is due to $1<p\leqslant 2$).

Further, we substitute $h=v_{\varkappa}\mathrm{dist}^2(y,y^0)$ into (2). Canceling a positive term results in

$$\int\limits_{\partial\Omega}|\nabla v_{\varkappa}|^p\,\mathrm{dist}^2(y,y^0)\,dy\leqslant K^p(n,p)\int\limits_{\partial\Omega}v_{\varkappa}^{p^\star}\,\mathrm{dist}^2(y,y^0)\,dy.$$

Substituting this relation into (50), we arrive at

$$\int\limits_{\partial\Omega}|\nabla v_{\varkappa}|^p\,\mathrm{dist}^2(y,y^0)\,dy\leqslant C\biggl(\int\limits_{\partial\Omega}|\nabla v_{\varkappa}|^p\,\mathrm{dist}^2(y,y^0)\,dy\biggr)^{\frac{1}{p}}\|v_{\varkappa}\|_{p,\Omega}^{p-1}+C\|v_{\varkappa}\|_{p,\Omega}^{p},$$

which gives

$$\int\limits_{\partial\Omega}|\nabla v_{\varkappa}|^p\,\mathrm{dist}^2(y,y^0)\,dy\leqslant C\|v_{\varkappa}\|_{p,\Omega}^{p};\qquad\int\limits_{\partial\Omega}v_{\varkappa}^{p^\star}\,\mathrm{dist}^2(y,y^0)\,dy\leqslant C\|v_{\varkappa}\|_{p,\Omega}^{p}.$$

We substitute these relation into (49). Expressing $\|\nabla v_{\varkappa}\|_{p,\Omega}^p$ from (40) and recalling (39), we get

$$\frac{\varkappa^p}{\gamma^p}\cdot\|v_{\varkappa}\|_{p,\Omega}^p\leqslant 1-\|v_{\varkappa}\|_{p^\star,\partial\Omega\cap B_\rho(y^0)}^{p^\star}+C\Bigl[\|v_{\varkappa}\|_{p,\Omega}^p+\|v_{\varkappa}\|_{p,\Omega}\cdot\|\nabla v_{\varkappa}\|_{p,\Omega\setminus B_\rho(y^0)}^{p-1}\Bigr].$$

Thus,

$$\frac{\varkappa^p}{\gamma^p}\leqslant\frac{\|v_{\varkappa}\|_{p^\star,\partial\Omega\setminus B_\rho(y^0)}^{p^\star}}{\|v_{\varkappa}\|_{p,\Omega}^p}+C\left(1+\left(\frac{\|\nabla v_{\varkappa}\|_{p,\Omega\setminus B_\rho(y^0)}}{\|v_{\varkappa}\|_{p,\Omega}}\right)^{p-1}\right).\qquad(51)$$

Using normalization of $v_{\varkappa}$, (44) and (45), we obtain

$$\frac{\|v_{\varkappa}\|_{p^\star,\partial\Omega\setminus B_\rho(y^0)}^{p^\star}}{\|v_{\varkappa}\|_{p,\Omega}^p}\leqslant\frac{\|v_{\varkappa}\|_{p^\star,\partial\Omega\setminus B_\rho(y^0)}^{p}}{\|v_{\varkappa}\|_{p,\Omega}^p}\leqslant\frac{\|\eta v_{\varkappa}\|_{p^\star,\partial\Omega}^{p}}{\|v_{\varkappa}\|_{p,\Omega}^p}\leqslant C\biggl(1+\frac{\|\eta\nabla v_{\varkappa}\|_{p,\Omega}^p}{\|v_{\varkappa}\|_{p,\Omega}^p}\biggr)\leqslant C,$$

while the second term in (51) is bounded by (45):

$$\frac{\|\nabla v_{\varkappa}\|_{p,\Omega\setminus B_\rho(y^0)}}{\|v_{\varkappa}\|_{p,\Omega}}\leqslant\frac{\|\eta\nabla v_{\varkappa}\|_{p,\Omega}}{\|v_{\varkappa}\|_{p,\Omega}}\leqslant C.$$

Therefore, the right-hand side in (51) is bounded. This gives a contradiction for $\varkappa$ large enough. □

4 The trace Sobolev–Poincaré inequality

Proof of Theorem 3. We use the following analog of Proposition 1.

Proposition 2. *Let the infimum in* (II) *satisfy the inequality* $\lambda_2(n,p,\Omega) < K(n,p)$. *Then the infimum is attained.*

To construct a function with zero trace mean-value having the quotient (II) less then $K(n,p)$ we modify the function (3). Similarly to [DN, Theorem 5.1], for large p we proceed by subtracting a suitable function with a small support while for small p we will subtract a constant.

We need the followning estimate for the function u defined in (3):

$$\int\limits_{\partial\Omega} u(y)d\Sigma \leqslant C \int\limits_{|x'|<\rho} \frac{dx'}{(|x'|^2+\varepsilon^2)^{\frac{n-p}{2(p-1)}}} \leqslant \begin{cases} C\varepsilon^{n-1-\frac{n-p}{p-1}}, & p<\frac{2n-1}{n}; \\ C\ln(\frac{1}{\varepsilon})+C(\rho), & p=\frac{2n-1}{n}; \\ C, & p>\frac{2n-1}{n}. \end{cases} \tag{52}$$

1. First, consider the case $p > \frac{2n-1}{n}$. Let u_1 be a smooth non-negative function such that $u_1\big|_{\partial\Omega} \not\equiv 0$ and $\operatorname{supp} u_1 \cap \operatorname{supp} u = \emptyset$. We define the function with zero trace mean-value

$$\widehat{u}(y) = u(y) - u_1(y)\,\frac{\int\limits_{\partial\Omega} u\,d\Sigma}{\int\limits_{\partial\Omega} u_1 d\Sigma}.$$

The estimates (23), (33) and (36) yield for $p < \frac{n+2}{3}$

$$\frac{\|\nabla u\|^p_{p,\Omega}}{\|u\|^p_{p^\star,\partial\Omega}} \leqslant K^p(n,p)\cdot\left(1-\varepsilon^2\frac{(2q-n)\mathcal{R}(y^0)}{2(2q-n+1)(2q-n-1)(2q-n-2)}+o(\varepsilon^2)\right)$$

and therefore, for small ρ and ε

$$\frac{\|\nabla\widehat{u}\|_{p,\Omega}}{\|\widehat{u}\|_{p^\star,\partial\Omega}} < K(n,p). \tag{53}$$

For $p = \frac{n+2}{3}$ this relation also holds true for ε and ρ sufficiently small. By continuity, for some $\beta > 0$ and $p < \frac{n+2}{3} + \beta$

$$\lambda_2(n,p,\Omega) < K(n,p). \tag{54}$$

2. Now we consider the case $1 < p \leqslant \frac{2n-1}{n}$. Define u by (3) and put $\rho = \varepsilon^\gamma$ with some $\gamma \in (0,\gamma^*)$, where $\gamma^* = \frac{n+2-3p}{n-p} \in (0,1)$.

The estimate (33) yields

$$\int\limits_{\partial\Omega} u^{p^\star} d\Sigma \geqslant \varepsilon^{n-2q-1}\mathcal{E}_1' - \varepsilon^{n-2q+1}\mathcal{E}_2' - C\varepsilon^{\gamma(n-2q-1)}.$$

Immediate computation shows that

$$\gamma^*(n-2q-1) > n-2q+1.$$

Therefore, for $\gamma \in (0,\gamma^*)$

$$\int\limits_{\partial\Omega} u^{p^\star} d\Sigma \geqslant \varepsilon^{n-2q-1}\mathcal{E}_1' - \varepsilon^{n-2q+1}(\mathcal{E}_2' + o_\varepsilon(1)).$$

Furthermore, the estimate (23) implies

$$\int\limits_{\Omega} |\nabla u|^p \, dy \leqslant \Big(\frac{n-p}{p-1}\Big)^p \cdot \big(\varepsilon^{n-2q}\mathcal{E}_1 - \varepsilon^{n-2q+2}\frac{\mathcal{E}_2}{2q-n-2} + C\varepsilon^{\gamma(n-2q)}\big).$$

Direct calculations result in

$$\gamma^*(2q-n) = 2q-n-2.$$

Hence, for $\gamma \in (0, \gamma^*)$

$$\int\limits_{\Omega} |\nabla u|^p \, dy \leqslant \Big(\frac{n-p}{p-1}\Big)^p \cdot \Big(\varepsilon^{n-2q}\mathcal{E}_1 - \varepsilon^{n-2q+2}\frac{\mathcal{E}_2 - o_\varepsilon(1)}{2q-n-2}\Big). \tag{55}$$

Now we define the function with zero trace mean-value

$$\widehat{u}(y) = u(y) - \frac{1}{|\partial\Omega|}\int\limits_{\partial\Omega} u \, d\Sigma.$$

Using (52) and the Minkowski inequality we conclude that

$$\begin{aligned}\|\widehat{u}\|_{p^\star,\partial\Omega} &\geqslant \Big(\int\limits_{\operatorname{supp} u\cap\partial\Omega} |\widehat{u}|^{p^\star} d\Sigma\Big)^{\frac{1}{p^\star}} \geqslant \\ &\geqslant \|u\|_{p^\star,\partial\Omega} - \frac{|\operatorname{supp} u\cap\partial\Omega|^{\frac{1}{p^\star}}}{|\partial\Omega|}\int\limits_{\partial\Omega} u d\Sigma \geqslant \varepsilon^{\frac{n-2q-1}{p^\star}}(\mathcal{E}_1')^{\frac{1}{p^\star}}\times \\ &\qquad \times\Big(1-\varepsilon^2\,\frac{\mathcal{E}_2'+o_\varepsilon(1)}{p^\star\mathcal{E}_1'} - C\varepsilon^{\gamma\frac{n-1}{p^\star}-\frac{n-2q-1}{p^\star}}\Phi(\varepsilon)\Big), \end{aligned}\tag{56}$$

where

$$\Phi(\varepsilon) = \begin{cases}\varepsilon^{n-1-\frac{n-p}{p-1}}, & \text{if} \quad 1<p<\frac{2n-1}{n};\\ 1+\ln(\frac{1}{\varepsilon}), & \text{if} \quad p=\frac{2n-1}{n}.\end{cases}$$

Immediate calculations show that for $n \geqslant 4$ and $1 < p \leqslant \frac{2n-1}{n}$, the inequality

$$\gamma^*\frac{n-1}{p^\star} + \frac{n-2q-1}{p^\star} + n - 1 - \frac{n-p}{p-1} > 2$$

holds true for $\gamma = \gamma^*$ and, therefore, for some $\gamma \in (0, \gamma^*)$. Choose such a γ, then (56) implies

$$\|\widehat{u}\|^p_{p^\star,\partial\Omega} \geqslant \varepsilon^{n-2q}(\mathcal{E}_1')^{\frac{p}{p^\star}}\left(1-\varepsilon^2\frac{p}{p^\star}\frac{\mathcal{E}_2'+o_\varepsilon(1)}{\mathcal{E}_1'}\right). \tag{57}$$

The relations (55), (57) and (36) imply (53) for ε small enough. Thus, the inequality (54) is valid in this case. The application of Proposition 2 completes the proof. □

References

[BZ] Yu. D. Burago and V. A. Zalgaller, *Introduction to Riemannian geometry*, St.Petersburg, Nauka, 1995 (Russian).

[DN] A. V. Demyanov and A. I. Nazarov, *On the existence of an extremal function in Sobolev embedding theorems with critical exponents*, Algebra Anal. **17** (2005), N5, 105–140 (Russian). English transl. in: St.Petersburg Math. J. **17** (2006), N5, 108-142.

[Dr] O. Druet, *Optimal Sobolev inequalities of arbitrary order on compact Riemannian manifolds*, J. Funct. Anal. **159** (1998), 217–242.

[Dr1] O. Druet, *The best constants problem in Sobolev inequalities*, Math. Annalen **314** (1999), 327–346.

[E] J. F. Escobar, *Sharp constant in a Sobolev trace inequality*, Indiana Univ. Math. J. **37** (1988), 687–698.

[Ls] P.-L. Lions, *The concentration-compactness principle in the Calculus of Variations. The limit case*, Parts 1 and 2, Rev. Mat. Iberoamericana **1** (1985), 45–121 (1), P.145–201 (2).

[LPT] P.-L. Lions, F. Pacella, and M. Tricarico, *Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions*, Indiana Univ. Math. J. **37** (1988), 301–324.

[Nt] B. Nazaret, *Best constant in Sobolev trace inequalities on the half-space*, Nonlin. Analysis, **65** (2006), N10, 1977–1985.

[N] A. I. Nazarov, *Dirichlet and Neumann problems to critical Emden–Fowler type equations*, J. Global Optim. **40** (2008), 289–303.

[NR] A. I. Nazarov, A. B. Reznikov, *On the existence of an extremal function in critical Sobolev trace embedding theorem*, St.Petersburg Math. Soc. Preprint 2009-06, 15p. Submitted to AIHP Anal. non lin.

[PS] G. Pólya & G. Szegö, *Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, 1951.