APPROXIMATION OF OPERATORS IN BANACH SPACES

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§0. Notations

This work is the first part of some investigations which are concerned with an approximation of ones or others classes of operators in Banach spaces (the approximation can be understood in different seances). The main objects of the study will be the spaces, possessing (or not possessing) the so-called approximation properties AP_p and the bounded approximation properties BAP_p of order p, where $p > 0$ (see, e.g., [9], [10]). In the propounded part, we consider only the properties AP_p where $p \geqslant 1$, and our considerations are bounded by some equivalent reformulations of the first-given definition from [9]. The main accent is put on the investigation of the conditions, under which the finite dimensional operators are dense in norm in the space of quasi-p-nuclear operators, as well as in the space of all absolutely p-summing operators in the topology of "π_p-compact convergence".

We will keep the standard notations and the terminology. If A is a bounded subset of a Banach space X, then $\overline{\Gamma(A)}$ is the closed absolutely convex hull of A; X_A is the Banach space with "the unit ball" $\overline{\Gamma(A)}$; $\Phi_A : X_A \to X$ is the canonical embedding. For $B \subset X$, it is denoted by $\overline{B}^{\tau}$ and $\overline{B}^{\|\cdot\|}$ the closures of the set B in the topology τ and in the norm $\|\cdot\|$ respectively. When it is necessary, we denote by $\|\cdot\|_X$ the norm in X. Other notations: Π_p, QN_p, N_p, I_p are the ideals of absolutely p-summing, quasi-p-nuclear, p-nuclear , strictly p-integral operators, respectively (see [8], [10]); $X^*\widehat{\otimes}_p Y$ is the complete tensor product associated with $\mathrm{N}_p(X,Y)$; $X^*\widehat{\widehat{\otimes}}_p Y = \overline{X^* \otimes Y}^{\pi_p}$ (i.e. the closure of the set of finite dimensional operators in $\Pi_p(X,Y)$). Finally, if $p \in [1,+\infty]$ then p' is the adjoint exponent.

Everywhere bellow, we use to suppose that $p \in [1,+\infty]$; however, the proofs are given for $p \in (1,+\infty)$ (with some non-essential changes, all proofs pass through the cases $p = 1$ and $p = +\infty$). The domain of changes for p is noted specially only in the places where it is necessary, or where a vagueness can be arised.

[R–KGU] Reinov O.I., Approximation of operators in Banach spaces, in book Primenenie funkcional'nogo analiza v teorii priblizheni, Kalinin: KGU, 1985, 128-142.

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§1. Approximation of absolutely p-summing operators

In this paragraph we will describe the topology τ_p in the space $\Pi_p(Y,X)$, for which the closures of convex sets in τ_p and in *-weak topology of the space $\Pi_p(Y,X)$ are coincident (this is an answer to a corresponding question of P. Saphar in [12], p.385). Thereafter, we will investigate some properties of the space Π_p, related to this new topology.

Thus, consider in the space $\Pi_p(Y,X)$ the *topology τ_p of π_p-compact convergence*, a local base (in zero) of which is defined by sets of type

$$\omega_{K,\varepsilon} = \{U \in \Pi_p(Y,X) :\ \pi_p(U\Phi_K) < \varepsilon\},$$

where $\varepsilon > 0, K = \overline{\Gamma(K)}$ — a compact subset of Y.

1.1. Proposition.. *Let* R *be a linear subspace in* $\Pi_p(Y,X)$, *containing* $Y^* \otimes X$. *Then* $(\mathrm{R}, \tau_p)'$ *is isomorphic to a factor space of the space* $X^*\widehat{\otimes}_{p'}Y$. *More precisely, if* $\varphi \in (\mathrm{R}, \tau_p)'$, *then there exists an element* $z = \sum_1^\infty x'_n \otimes y_n \in X^*\widehat{\otimes}_{p'}Y$ *such that*

$$(*) \qquad \varphi(U) = \text{trace } U \circ z,\ \ U \in \mathrm{R}.$$

On the other hand, for every $z \in X^*\widehat{\otimes}_{p'}Y$ *the relation* $(*)$ *defines a linear continuous functional on* (R, τ_p).

The *Proof* of this proposition will be given in detail. However, we will omit details in analogues cases hereinafter. So, let φ be a linear continuous functional on (R, τ_p). Then one can find a neighborhood of zero $\omega_K = \omega_{K,\varepsilon}$, such that φ is bounded on it: $\forall\, U \in \omega_K,\ |\varphi(U)| \leqslant 1$. We may assume that $\varepsilon = 1$. Consider the operator $U\Phi_K : Y_K \xrightarrow{\Phi_K} Y \xrightarrow{U} X$. Since the mapping Φ_K is compact, $U\Phi_K \in \mathrm{QN}_p(Y_K, X)$. Put $\varphi_k(U\Phi_K) = \varphi(U)$ for $U \in \mathrm{R}$. On the linear subspace $\mathrm{R}_K = \left\{V \in \mathrm{QN}_p(Y_K, X) :\ V = U\Phi_K\right\}$ of the space $\mathrm{QN}_p(Y_K, X)$, the linear functional φ_K is bounded: if $V = U\Phi_K \in \mathrm{R}_K$ and $\pi_p(V) \leqslant 1$, then $|\varphi_K(V)| = \varphi(U)| \leqslant 1$. Therefore, φ_K can be extended to a linear continuous functional $\widetilde{\varphi}$ on the whole $\mathrm{QN}_p(Y_K, X)$; moreover, because of the injectivity of the ideal QN_p, considering X as a subspace of some space $C(K)$, we may assume that $\widetilde{\varphi} \in \mathrm{QN}_p(Y_K, C(K))^*$. Let us mention that

$$(1) \qquad \widetilde{\varphi}(jU\Phi_K) = \varphi_K(U\Phi_K) = \varphi(U)$$

(here j is an isometric embedding of X into $C(K)$).

Furthermore, since $\mathrm{QN}_p(Y_K, C(K))^* = \mathrm{I}_{p'}(C(K), (Y_K)^{**})$, we can find an operator $\Psi : C(K) \to (Y_K)^{**}$, for which

$$\widetilde{\varphi}(A) = \text{trace } \Psi A,\ \ A \in (Y_K)^* \otimes C(K).$$

Let $A_n \in (Y_K)^* \otimes C(K),\ \pi_p(A_n - jU\Phi_K) \to 0$. Then

$$(2) \qquad \widetilde{\varphi}(jU\Phi_K) = \lim \text{trace } \Psi A_n.$$

Consider the operator $\Phi_K^{**}\Psi : C(K) \xrightarrow{\Psi} (Y_K)^{**} \xrightarrow{\Phi_K^{**}} Y$. Since $\Psi \in \mathrm{I}_{p'}$, and Φ_K is compact, we have $\Phi_K^{**}\Psi \in \mathrm{N}_{p'}(C(K), Y) = C(K)^*\widehat{\otimes}_{p'}Y$. Let $\sum_1^\infty \mu_n \otimes y_n \in C(K)^*\widehat{\otimes}_{p'}Y$

be a representation of the operator $\Phi_K^{**}\Psi$. Put $z = \sum j^*(\mu_n) \otimes y_n$. The element z generates an operator $\Phi_K^{**}\Psi j$ from X to Y. We will show now that trace $U \circ z = \widetilde{\varphi}(jU\Phi_K)$ (note that $U \circ z$ is an element of the space $X^*\widehat{\otimes}X$, so the trace is well defined). We have:

$$\text{(3)} \quad \operatorname{trace} U \circ z = \operatorname{trace}\left(\sum j^*(\mu_n) \otimes Uy_n\right) = \sum \langle j^*(\mu_n), Uy_n\rangle = \\ = \sum \langle \mu_n, jUy_n\rangle = \operatorname{trace} jU\Phi_K^{**}\Psi = \operatorname{trace}\,(jU\Phi_K)^{**}\Psi,$$

where $(jU\Phi_K)^{**}\Psi : C(K) \xrightarrow{\Psi} (Y_K)^{**} \xrightarrow{\Phi_K^{**}} Y \xrightarrow{U} X \xrightarrow{j} C(K)$. Since $\pi_p(A_n - jU\Phi_K) \to 0$, then $\pi_p\left(A_n^{**} - (jU\Phi_K)^{**}\right) \to 0$. Moreover, if $A := A_n = \sum_1^N w_m \otimes f_m \in (Y_K)^* \otimes C(K)$, then

$$\operatorname{trace} A_n^{**}\Psi = \sum_m \langle \Psi^* w_m, f_m\rangle = \sum_m \langle w_m, \Psi f_m\rangle = \operatorname{trace} \Psi A.$$

Hence, trace $(jU\Phi_K)^{**}\Psi = \lim$ trace $A_n^{**}\Psi = \lim$ trace ΨA_n. Now, it follows from (3) and (2) that $\widetilde{\varphi}(jU\Phi_K) = $ trace $U \circ z$. Finally, we get from (1): $\varphi(U) = $ trace $U \circ z$. Thus, the functional φ is defined by an element of $X^*\widehat{\otimes}_{p'}Y$.

Inversely, if $z \in X^*\widehat{\otimes}_{p'}Y$, put $\varphi(U) = $ trace $U \circ z$ for $U \in \mathrm{R}$ (the trace is defined since $U \circ z \in X^*\widehat{\otimes}X$). We have to show that the linear functional φ is bounded on a neighborhood $\omega_{K,\varepsilon}$ of zero in τ_p. For this, we need

1.2. Lemma. *If $z \in X^*\widehat{\otimes}_qY$, then $z \in X^*\widehat{\otimes}_qY_K$, where $K = \overline{\Gamma(K)}$ is a compact in Y.*

Proof of the lemma. Let $z = \sum x_n' \otimes y_n$, $\{c_n\} \in c_0$ and $\sum \|x_n'\|^q\, c_n^{-1} < +\infty$. Consider the operator $A_1 : l_q \to Y$, $A_1\{a_n\} = \sum a_n c_n y_n$. Since this operator is compact, one can find a compact $K \subset Y$ and an operator $A_2 : l_q \to Y_K$, for which $A_1 = \Phi_K A_2$. Put $z_0 = \sum c_n^{-1}\, x_n' \otimes e_n$ (e_n are orths in l_q). Then $\overline{z} := (\mathbf{1} \otimes A_2)(z_0) \in X^*\widehat{\otimes}_qY_K$, and $\Phi_K(\overline{z}) = z$. ■

Let us continue the proof of the theorem. Let K be a compact subset of Y, for which $z \in X^*\widehat{\otimes}_{p'}Y_K$. If $U \in \omega_{K,1}$, then $\pi_p(U\Phi_K) < 1$ and $|\operatorname{trace} U\Phi_K \circ z| \leqslant \|z\|_{X^*\widehat{\otimes}_{p'}Y_K} \cdot \pi_p(U\Phi_K) \leqslant C$. ■

1.3. Corollary. $(\mathrm{R}, \tau_p)' = (\mathrm{R}, \sigma)'$, *where $\sigma = \sigma(\mathrm{R}, X^*\widehat{\otimes}_{p'}Y)$. Thus, the closures of convex subsets of the space $\Pi_p(Y, X)$ in τ_p and in σ are the same.* ■

Denote by $X^*\widetilde{\otimes}_{p'}Y$ the closure of the set $X^* \otimes Y$ in the space $\mathrm{I}_{p'}(X, Y^{**})$ (dual to $Y^*\widehat{\widehat{\otimes}}_pX$).

1.4. Proposition. *Let A be the intersection of the unit ball of the (dual to $X^*\widetilde{\otimes}_{p'}Y$) space $G = G(Y, X^{**})$ with the subspace $Y^* \otimes X$. $*$-weak closure of the set A in $G \cap \Pi_p(Y, X)$ coincides with the closure of A in $(\Pi_p(Y, X), \tau_p)$.*

Proof. Let us consider the canonical mappings

$$X^*\widehat{\otimes}_{p'}Y \xrightarrow{\ j\ } X^*\widetilde{\otimes}_{p'}Y,$$
$$\Pi_p(Y, X^{**}) \xleftarrow{\ j^*\ } G(Y, X^{**}).$$

Since j^* is one-to-one, then the closures of the bounded sets in $G(Y, X^{**})$, in topologies $\sigma(G, X^*\widetilde{\otimes}_{p'}Y)$ and $\sigma(G, X^*\widehat{\otimes}_{p'}Y)$ are the same. Therefore, if B is a convex bounded set in $G(Y, X) \subset \Pi_p(Y, X)$, then the closure of the set B in $\big(\Pi_p(Y, X) \cap G, \sigma(G, X^*\widetilde{\otimes}_{p'}Y)\big)$ coincides with the closure of the set B in the space $\big(\Pi_p(Y, X), \sigma(\Pi_p(Y, X), X^*\widehat{\otimes}_{p'}Y)\big)$ and, therefore, by Corollary 1,3, — with the closure of B in $(\Pi_p(Y, X), \tau_p)$. ■

1.5. Corollary. *With notations of the proposition* 1.4*, the closure of the set A in τ_p coincides with the closure of A in the space* $\mathrm{L}(Y, X)$ *in the topology of compact convergence.*

For the *proof,* it is enough to use the previous assertion, considering the canonical mapping from $X^*\widehat{\otimes}_1 Y$ into $X^*\widetilde{\otimes}_{p'}Y$ instead of the map j from the proof of the proposition 1.4 (and to apply either Proposition 1.1 for $p = +\infty$, or results on duality from [4]). ■

1.6. Corollary. *Let $C > 0$ and $T \in \Pi_p(Y, X)$. The following assertions are equivalent:*

1) *there ia a net $\{T_\alpha\}, T_\alpha \in Y^* \otimes X$, converging to T in the topology τ_p such that $\pi_p(T_\alpha) \leqslant C$;*

2) *there is a net $\{T_\alpha\}, T_\alpha \in Y^* \otimes X$, converging to T in the topology of compact convergence, such that $\pi_p(T_\alpha) \leqslant C$.* ■

1.7. Proposition. *For an operator $T \in \Pi_p(Y, X), \overline{T(Y)} = X$, the following are the same:*

1) *$T \in \overline{Y^* \otimes X}^{\tau_p}$;*

2) *there is a net of operators $R_\alpha \in Y^* \otimes Y$ such that $TR_\alpha \to T$ in the topology τ_p.*

Proof. Assuming that 2) is not valid, we (by Corollary 1.3) get:

$$\nexists\, R_\alpha \in Y^* \otimes Y : \; TR_\alpha \to T \;\text{ in }\; \big(\Pi_p(Y, X), \sigma(\Pi_p(Y, X), X^*\widehat{\otimes}_{p'}Y)\big)\,. \tag{4}$$

Consider the associated with T mappings:

$$\begin{array}{ccc} X^*\widehat{\otimes}_{p'}Y & \xrightarrow{\widetilde{T}} & Y^*\widehat{\otimes}_1 Y, \\ \Pi_p(Y, X^{**}) & \xleftarrow{\widetilde{T}^*} & \mathrm{L}(Y, Y^{**}). \end{array}$$

where $\widetilde{T}(z) = z \circ T$ for $z \in X^*\widehat{\otimes}_{p'}Y$. Let $Z = \overline{\widetilde{T}^*(Y^* \otimes Y)}^*$ (the closure is taken in the space $\Pi_p(Y, X)$ in *-weak topology). It follows from (4) that T is not zero on the subspace $Z^\perp \subset X^*\widehat{\otimes}_{p'}Y$, i.e. there exists an element $A \in Z^\perp$ such that $\langle T, A\rangle =$ trace $AT = 1$. But $AT = \widetilde{T}(A)$, and if $R \in Y^* \otimes Y$, then $\langle \widetilde{T}(A), R\rangle = \langle A, (\widetilde{T})^*(R)\rangle = 0$. Hence, the element $\widetilde{T}(A)$ of the projective tensor product $Y^*\widehat{\otimes}Y$ is not zero (since trace $\widetilde{T}(A) = 1$), but generates a null-operator in Y. For any $y' \in Y^*$ and $Ty \in T(Y)$ we have: $\langle A, Ty \otimes y'\rangle = \langle ATy, y'\rangle = 0$. Since $\overline{T(Y)} = X$, we obtain that a non-zero tensor element $A \in X^*\widehat{\otimes}_{p'}Y$ generates a zero-operator. Again, by using the equality trace $AT = 1$, we conclude that T can not be approximated in *-weak topology by finite rank operators. Now, it follows from Corollary 1.3 that 1) is not fulfilled. ■

Next two statements give us sufficient (but not necessary, as we will see below) conditions for the density of the set of all finite rank operators in the space of operators $\Pi_p(Y, X)$ in the topology τ_p of π_p-compact convergence.

1.8. Proposition[1]**.** *If* $\mathrm{QN}_p(Y,X) = \overline{Y^* \otimes X}^{\pi_p}$ *for every Banach space* Y*, then for each* Y $\Pi_p(Y,X) = \overline{Y^* \otimes X}^{\tau_p}$.

Proof. Let $U \in \Pi_p(Y,X)$, $\varepsilon > 0$, $K = \overline{\Gamma(K)}$ be a compact in Y. By the assumptions, there is an operator $V \in (Y_K)^* \otimes X$, such that $\pi_p(V - U\Phi_K) < \varepsilon$. We need to set successfully instead of V an operator $\widetilde{V}\Phi_K$, where $\widetilde{V} \in Y^* \otimes X$. Let $V = \sum_{n=1}^N z_n \otimes x_n$. Note that we can consider only the case where Φ_K^{**} is one-to-one (else, with the help of the construction of [2] we change Y_K by a space Y_{K_0}, for which the operator Φ_{K_0} is compact and the operator $\Phi_{K_0}^{**}$ is one-to-one). In this case Y^* is norm dense in $(Y_K)^*$ and, therefore, for every positive number sequence $\{\varepsilon_n\}$ there exist the elements $y'_n \in Y^*$, for which $\|y'_n - z_n\|_{Y_K^*} < \varepsilon_n$. Put $\widetilde{V} = \sum_1^N y'_n \otimes x_n \in Y^* \otimes X$. Let $\{a_n\}$ be a sequence of the elements of the space Y_K, such that $\sup\left\{\sum |\langle a_n, a'\rangle|^p : \ \|a'\|_{Y_K^*} \leqslant 1\right\} \leqslant 1$. We have:

$$\sum_{i=1}^m \|(\widetilde{V}\Phi_K - V)a_i\|^p = \sum_{i=1}^m \|\sum_{n=1}^N \langle y'_n - z_n, a_i\rangle\, x_n\|^p \leqslant$$

$$\leqslant \sum_{i=1}^m \sum_{n=1}^N |\langle y'_n - z_n, a_i\rangle|^p \left(\sum_{n=1}^N \|x_n\|^{p'}\right)^{p/p'} \leqslant$$

$$\leqslant \left(\sum_{n=1}^N \|x_n\|^{p'}\right)^{p/p'} \sum_{n=1}^N \|y'_n - z_n\|^p_{Y_K^*} \leqslant \left(\sum_{n=1}^N \|x_n\|^{p'}\right)^{p/p'} \sum_{n=1}^N \varepsilon_n^p.$$

If we take ε_n small enough then the last number is less then ε, and, from the inequality $\pi_p(V - U\Phi_K) \leqslant \varepsilon$, we get that $\pi_p(\widetilde{V}\Phi_K - U\Phi_K) \leqslant \varepsilon + \pi_p(V - \widetilde{V}\Phi_K) \leqslant 2\varepsilon$. Hence, $\widetilde{V} - U \in \omega_{K,2\varepsilon}$. Thus, we have shown that for every neighborhood $\omega_{K,\varepsilon}$ there exists an operator $\widetilde{V} \in Y^* \otimes X$, for which $\widetilde{V} - U \in \omega_{K,\varepsilon}$. Therefore $U \in \overline{Y^* \otimes X}^{\tau_p}$. ■

1.9. Proposition. *If the canonical mapping* $j : X^* \widehat{\otimes}_{p'} Y \to \mathrm{N}_{p'}(X,Y)$ *is one-to-one then* $\Pi_p(Y,X) = \overline{Y^* \otimes X}^{\tau_p}$.

Proof. If the map j is one-to-one then the annihilator $j^{-1}(0)^\perp$ of its kernel in the space, dual to $X^* \widehat{\otimes}_{p'} Y$, coincides with $\Pi_p(Y, X^{**})$. On the other hand, in any case $j^{-1}(0)^\perp = \overline{Y^* \otimes X}^*$ (the closure in *-weak topology of the space $\Pi_p(Y, X^{**})$); by Corollary 1.3,

$$\Pi_p(Y,X) \cap \overline{Y^* \otimes X}^* = \overline{Y^* \otimes X}^{\tau_p}.$$

Therefore, $\Pi_p(Y,X) = \overline{Y^* \otimes X}^{\tau_p}$. ■

For a reflexive space X, the dual space to $X^* \widehat{\otimes}_{p'} Y$ is equal to $\Pi_p(Y,X)$. Consequently, it follows from 1.3 and 1.9

1.10. Corollary. *For a reflexive space* X *the canonical mapping* $j : X^* \widehat{\otimes}_{p'} Y \to \mathrm{N}_{p'}(X,Y)$ *is one-to-one iff the set of finite rank operators is dense in the space* $\Pi_p(Y,X)$ *in the topology* τ_p *of* π_p*-compact convergence.* ■

To conclude this part of our considerations, let us give an assertion which shows the following. It follows from the existing of an absolutely p-summing operator, which is non-approximated in the topology τ_p, that there esits a non-approximated (in the same topology) quasi-p-nuclear operator (with values in the same space).

[1] In the original paper [R–KGU] this Proposition was sounded as follows: "*If* $\mathrm{QN}_p(Y,X) = \overline{Y^* \otimes X}^{\pi_p}$*, then* $\Pi_p(Y,X) = \overline{Y^* \otimes X}^{\tau_p}$."

1.11. Proposition. *If there exists an operator* $T \in \Pi_p(Y,X) \setminus \overline{Y^* \otimes X}^{\tau_p}$, *then there exist a reflexive space* Z *and an operator* $U \in \mathrm{QN}_p(Z,X)$, *which is not in the closure* $\overline{Y^* \otimes X}^{\tau_p}$.

For the *proof* it is enough to remember the definition of the topology τ_p and to use the following two facts: a) if V is a compact operator then it can be represented as a composition of two compact operators; b) the product AB of a compact operator B and absolutely p-summing operator A is a quasi-p-nuclear map (see [5], [8]). $\blacksquare$

1.12. Remark. [2]

§2. Approximation properties of Banach spaces

In this paragraph, we fix the space of images of operators (or the spaces where operators are defined), and investigate the conditions under which it is possible to approximate (by finite rank operators) *all* absolutely p-summing mappings with values in a given space (or acting from a given space).[3]

2.1. Proposition. *For a Banach space* X *the following are equivalent:*

1) *for every Banach space* Y *the equality* $\mathrm{QN}_p(Y,X) = \overline{Y^* \otimes X}^{\pi_p}$ *holds;*
2) *for every Banach space* Y *the equality* $\Pi_p(Y,X) = \overline{Y^* \otimes X}^{\tau_p}$ *holds;*
3) *for every Banach space* Y *one has* $\mathrm{QN}_p(Y,X) \subset \overline{Y^* \otimes X}^{\tau_p}$;[4]
4) *for every reflexive Banach space* Y *the equality* $\mathrm{QN}_p(Y,X) = \overline{Y^* \otimes X}^{\tau_p}$ *holds.*

Proof. Implications 2) $\implies$ 3) $\implies$ 4) are evident; 1) $\implies$ 2) by Proposition 1.8. For the proof of the implication 4) $\implies$ 1), consider an operator $V \in \mathrm{QN}_p(Y,X)$. By using the results of [2], we can factorize the operator V by the following way: $V = UA$, where $A \in \mathrm{L}_c(Y,Z)$, $U \in \mathrm{QN}_p(Z,X)$, and, moreover, the space Z is reflexive. Put $K = \overline{A(\mathrm{Ball}_Y)}$. From 4), it follows that

$$\forall \varepsilon > 0 \; \exists \widetilde{V} \in Z^* \otimes X : \quad \widetilde{V} - U \in \omega_{K,\varepsilon},$$

i.e. $\pi_p(\widetilde{V}\Phi_K - U\Phi_K) < \varepsilon$. Hence, $\pi_p(V - \widetilde{V}A) \leqslant C_A\, \pi_p(U\Phi_K - \widetilde{V}\Phi_K) \leqslant C_A\varepsilon$. $\blacksquare$

2.2. Corollary. *If for every reflexive space* Y *the canonical mapping* $X^* \widehat{\otimes}_{p'} Y \to \mathrm{N}_{p'}(X,Y)$ *is one-to-one then for each Banach space* Y *the equality* $\mathrm{QN}_p(Y,X) = \overline{Y^* \otimes X}^{\pi_p}$ *holds.* $\blacksquare$

For the *proof,* Proposition 1.9 may be applied, and then use the implication 2) $\implies$ 1) of the previous fact. $\blacksquare$

It follows from Corollaries 1.10 and 2.2

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[2] In the original paper [R–KGU] this non-essential remark is sounded exactly as "At the moment, I do not know whether the inverse of Proposition 1.8 is true in the case where the space X is reflexive.

[3] In the original paper [R–KGU], there was here the phrase "The next statement, among other things, gives us a partial inversion of Proposition 1.8."

[4] In the original paper [R–KGU], there was here an evident misprint: "for every Banach space Y the equality $\mathrm{QN}_p(Y,X) = \overline{Y^* \otimes X}^{\tau_p}$ holds;"

2.3. Corollary. *For a reflexive Banach space X the following are equivalent:*

1) *for each space Y the canonical mapping $X^*\widehat{\otimes}_{p'}Y \to \mathrm{N}_{p'}(X,Y)$ is one-to-one;*

2) *for each space Y the set of finite rank operators is dense in the Banach space $\mathrm{QN}_p(Y,X)$.* ■

The next statement, among other things, gives us a partial inversion of Proposition 1.9.

2.4. Corollary. *For every Banach space X the following are equivalent:*

1) *for each space Y the canonical mapping $Y^*\widehat{\otimes}_pX \to \mathrm{N}_p(Y,X)$ is one-to-one;*

2) *for each reflexive Banach space Y the canonical mapping $Y^*\widehat{\otimes}_pX \to \mathrm{N}_p(Y,X)$ is one-to-one;*

3) *for each (reflexive) Banach space Y $\overline{X^* \otimes_{p'} Y}^{\tau_{p'}} = \Pi_{p'}(X,Y)$.*

Proof. Concerning the equivalence 1) $\iff$ 2), see [1]; the implications 1) $\implies$ 3) and 3) $\implies$ 2) follow from 1.9 and 1.10, respectively. ■

The previous statement yields the following well known result:

2.5. Corollary. *If a Banach space X has the approximation property then for any $p > 1$ and any space Y the canonical mapping $Y^*\widehat{\otimes}_pX \to \mathrm{N}_p(Y,X)$ is one-to-one.* ■

The above results lead us to the following definition which is equivalent to corresponding definition in [9], [10], [11].

2.6. Definition. Let $p \geqslant 1$. A Banach space X has the property AP_p (the approximation property of order p), if every absolutely p'-summing operator, acting from the space X, can be approximated in the topology of π_p'-compact convergence $\tau_{p'}$ by operators of finite rank.

It follows from Proposition 1.8 and Corollary 2.4 that

2.7. Corollary. *If for each Banach space Y the equality $\mathrm{QN}_{p'}(X,Y) = \overline{X^* \otimes Y}^{\pi_{p'}}$ holds then the space X has the property AP_p.* ■

Recall for the sake of completeness the following assertion on a characterization of the spaces with the property AP_p (a proof can be found in [1]).

2.8. Proposition. *A Banach space X has the property AP_p iff for every Banach space Y, for every operator $T \in \Pi_{p'}(X,Y)$, for each weakly p'-summable sequence $\{x_k\}$ of elements of the space X and for every $\varepsilon > 0$ there is a finite rank operator $R : X \to Y$, such that $\sum \|Ux_k - Rx_k\|^{p'} < \varepsilon$.*

As the property AP_1 (the usual approximation property og Grothendieck), the properties AP_p are very useful in investigations of the questions of different kinds in the geometrical theory of operatos (for instance, when describing the dual spaces of some spaces of operators; so, if X possesses the property AP_p, then $\mathrm{N}_p(Y,X)^* = \Pi_{p'}(X,Y^{**})$ for every space Y). However, we would like (concluding this paragrap) to adduce a simple fact, which is valid without any assumptions on approximation (see [3] for $p = 1$ and [12] for $p > 1$, where an analogues assertion was proved with a supposition of approximation property; see also [7], Theorem 4.6, where it is obtained a little bit less general result).

2.9. Proposition. *Let $p \in [1, +\infty)$. The following are equivalent*

1) *Banach spaces X and Y are reflexive;*
2) *the space $\mathrm{QN}_p(X, Y)$ is reflexive;*
3) *the space $\Pi_p(X, Y)$ is reflexive.*

Proof. Since, for reflexive spaces X and Y, the equality $\mathrm{QN}_p(X, Y) = \Pi_p(X, Y)$ holds, it is sufficient to prove only the implication 1) $\implies$ 2). For this, imbedd the space Y into some space $C(K)$ isometrically, and let us consider the space $\mathrm{QN}_p(X, Y)$ as a subspace of $\mathrm{QN}_p(X, C(K))$. Then any continuous functional Φ on $\mathrm{QN}_p(X, Y)$ has an extension with the same norm to a linear functional $\widetilde{\Phi}$ on $\mathrm{QN}_p(X, C(K))$. In turn, the functional $\widetilde{\Phi}$ is generated by an operator $U \in \mathrm{I}_{p'}(C(K), X^{**})$, so $\langle \Phi, T\rangle = \operatorname{trace} UjT$ for $T \in \mathrm{QN}_p(X, Y)$ (where j is the imbedding of Y into $C(K)$). Since the space Y is reflexive, one has $Uj \in \mathrm{N}_{p'}(Y, X)$ [7]. Therefore, as a functional, Φ is generated by an element of the space $Y^*\widehat{\otimes}_{p'} X$. Thus, the natural mapping $Y^*\widehat{\otimes}_{p'} X \to \mathrm{QN}_p(X, Y)^*$ is an "onto" map. Since $\left(Y^*\widehat{\otimes}_{p'} X\right)^* = \mathrm{QN}_p(X, Y)$, we get that the space $\mathrm{QN}_p(X, Y)$ is reflexive. ■

§3. Counterexamples

A lot of counterexamples concerning AP's can be found in [9], [10], [11]. We will use them partially. Recall that for any $p \geqslant 1$, $p \neq 2$, there exists a separable reflexive Banach space without the property AP_p. Moreover, for every $p \geqslant 1$, $p \neq 2$, there exist a separable reflexive E, an operator $R \in \Pi_{p'}(E, E)$ and a tensor element $t \in E^*\widehat{\otimes}_p E$, so that $\operatorname{trace} t \circ R = 1$ and $\operatorname{trace} t \circ A = 0$ for each finite rank operator $A \in E^* \otimes E$ (for details, we refer the reader to the papers [9], [10], [11]).

It is often very useful to apply the following fact when constructing some counterexamples (see [6]):

3.1. Lemma. *For every separable Banach space E there exist a separable conjugate Banach space $H = Y^*$ with a basis and operators $Q : H \to E$ and $U : H^* \to E^*$ so that $Q(H) = E$, $U(H^*) = E^*$, $\|U\| \leqslant 1$, $\|Q\| \leqslant 1$, $UQ^* = \mathrm{id}_{E^*}$ and the space $(\mathrm{id}_{E^*} - Q^*U)(H^*)$ is isomorphic to the space Y.*

Now we are ready for constructions of our counterexamples. Firstly, we will show that the inversion of Proposition 1.9 and Corollary 2.2 are not valid.

3.2. Proposition. *For every $p \in [1, +\infty]$, $p \neq 2$, there exist a separable reflexive space E, a separable conjugate space H with a basis such that the canonical mapping $j : H^*\widehat{\otimes}_{p'} E \to \mathrm{N}_{p'}(H, E)$ is not one-to-one. On the other hand, since H has the AP, $\mathrm{QN}_p(E, H) = \overline{E^* \otimes H}^{\pi_p}$ and $\Pi_p(E, H) = \overline{E^* \otimes H}^{\tau_p}$.*

Proof. Let E, $t \in E^*\widehat{\otimes}_{p'} E$ and $R \in \Pi_p(E, E)$ be the mentioned above spaces, tensor element and operator so that $\operatorname{trace} t \circ R = 1$ and $t = 0$ as an operator. Set $g = t \circ Q \in H^*\widehat{\otimes}_{p'} E$ and $V = U^*R \in \mathrm{QN}_p(E, H^{**})$. Then $\operatorname{trace} V \circ g = 1$ (consequently, the tensor element g is not equal to zero) and $g = 0$ as an operator. ■

3.3. Corollary. *For every $p \in [1, +\infty]$, $p \neq 2$, there exist a Banach space Z, an element $z \in Z^*\widehat{\otimes}_{p'} Z$ and an operator $\Psi \in \Pi_p(Z, Z^{**})$ such that* $\operatorname{trace} \Psi \circ z = 1$, *but* $\operatorname{trace} \Phi \circ z = 0$ *for all* $\Phi \in \Pi_p(Z, Z)$.

Proof. Let us use notation introduced in the proof of Proposition 3.2. Let $\sum y'_n \otimes y_n$ be any representation of a tensor element $g \in H^* \widehat{\otimes}_{p'} E$. Note that trace $A \circ g = 0$ for every operator $A \in \Pi_p(E, H)$ (because of the space H has approximation property). Put $Z = H \oplus E$, $z = \sum (y'_n, 0) \otimes (0, y_n)$ and define the operator $\Psi \in \mathrm{QN}_p(Z, Z^{**})$ by $\Psi(h, y) = (Vy, 0)$. We have:

$$\operatorname{trace} \Psi \circ z = \sum \langle (y'_n, 0), (Vy_n, 0) \rangle = \sum \langle y'_n, Vy_n \rangle = \operatorname{trace} V \circ g = 1.$$

On the other hand, denoting by P_H and P_E the natural projectors from Z onto H and E respectively, we have, for arbitrary operator $\Phi \in \Pi_p(Z, Z)$,

$$\Phi(h, y) = \Phi|_H(h) + \Phi|_E(y) = (P_H \Phi|_H(h) + P|_E \Phi|_H(h)) + (P_H \Phi|_E(y) + P|_E \Phi|_E(y)),$$

whence,

$$\operatorname{trace} \Phi \circ z = \sum \langle (y'_n, 0), [P_H \Phi(0, y_n)] \rangle = \sum \langle (y'_n, 0), P_H \Phi|_E(y_n) \rangle.$$

Denoting by A the operator $P_H \Phi|_E$, we get:

$$A \in \Pi_p(E, H); \quad \operatorname{trace} \Phi \circ z = \operatorname{trace} A \circ g = 0. \quad \blacksquare$$

3.4. Remark. For $p = +\infty$ we get nonzero tensor element $z \in Z^* \widehat{\otimes}_1 Z$, vanishing on the subspace $\mathrm{L}(Z, Z)$ of the space $\mathrm{L}(Z, Z^{**})$. This is an answer to a question of Swedish mathematician Sten Kaijser, who was one who is directed my attention for a possibility of the existemce of such an element z.

Now we will show that the inversion of Proposition 1.8[5] and Corollary 2.7 are invalid too.

3.5. Proposition. *For every $p \in [1, +\infty]$, $p \neq 2$, there exist a separable reflexive space E, a separable conjugate space H with a basis (so, with the property $\mathrm{AP}_{p'}$) such that $\mathrm{QN}_p(H, E) \neq \overline{H^* \otimes E}^{\pi_p}$; on the other hand, $\Pi_p(H, E) = \overline{H^* \otimes E}^{\tau_p}$* (see Corollary 2.4).

Proof. With the notation of the proof of Proposition 3.2, set $L = RQ$. Since trace $(t \circ RQ^{**}Y^*) = 1$ and $t = 0$ as an operator, the map RQ^{**} can not be approximated by finite rank operators in the space $\mathrm{QN}_p(H^{**}, E)$. Moreover, $L \not\subset \overline{H^* \otimes E}^{\pi_p}$. $\blacksquare$

In conclusion, let us bring the following, at first sight somewhat surprising, statement which shows, roughly speaking, that there are spaces X and Y, for which the closures in $(\Pi_p(X, Y^{**}), w^*)$ of the set of all finite dimensional operators is minimal: coincides with $X^* \widehat{\widehat{\otimes}}_p Y$.

[5]In [R–KGU] I meant that, for fixed X and Y, in Proposition 1.8
$\Pi_p(Y, X) = \overline{Y^* \otimes X}^{\tau_p} \not\Rightarrow \mathrm{QN}_p(Y, X) = \overline{Y^* \otimes X}^{\pi_p}$

3.6. Proposition. *For every $p \in [1, +\infty)$ there exist (separable and reflexive) Banach spaces X and Y such that*

$$\overline{X^* \otimes Y}^{\tau_p} = X^* \widehat{\widehat{\otimes}}_p Y.$$

Proof. Let X and Y be th separable and reflexive spaces such that the natural mapping $Y^* \widehat{\otimes}_{p'} X \to \mathrm{N}_{p'}(Y, X)$ is not one-to-one. By Proposition 1.1, the dual space to $\mathrm{N}_{p'}(Y, X)$ can be identified with a subspace $\overline{X^* \otimes Y}^{\tau_p}$ of the spaces $\Pi_p(X, Y)$ (see also the proof of Proposition 1.9). Since this subspace is reflexive (Proposition 2.9) and $(X^* \widehat{\widehat{\otimes}}_p Y)^* = \mathrm{N}_{p'}(Y, X)$, we have: $X^* \widehat{\widehat{\otimes}}_p Y = \overline{X^* \otimes Y}^{\tau_p}$. ∎

3.7. Remark. Seemingly, it is unknown whether Proposition 3.6 is true in the case where $p = +\infty$.

References

1. Bourgain J., Reinov O.I., *On the approximation properties for the space H^∞*, Math. Nachr. **122** (1985), 19-27.
2. Davis W.J, Figiel T., Johnson W.B., Pelczynski A., *Factoring weakly compact operators*, J. Functional Analysis **17** (1974), 311-327, MR50#8010..
3. Y.Gordon, D.R.Lewis, H.R.Retherford, *Banach ideals of operators with applications*, J. Funct. Anal. **14** (1973), no. 1, 85-129.
4. Grothendieck A., *Produits tensoriels topologiques et espases nucléaires*, Mem. Amer. Math. Soc. **16** (1955), pp. 196 + 140.
5. Johnson W.B., *Factoring compact operators*, Israel J. Math. **9** (1971), 337-345.
6. Lindenstrauss J., *On James' paper "Separable Conjugate Spaces"*, Israel J. Math. **9** (1971), 279-284.
7. Makarov B.M., Samarskij V.G., *Weak sequencial completeness and close properties of he spaces of operators*, Theory of operators and theory of functions, 1, LGU, Leningrad, 1983, 122-144
8. Pietsch A., *Operator ideals*, North-Holland, Deutscher Verlag der Wiss., Berlin, 1978. 451 p..
9. Reinov O.I., *Approximation properties of order p and the existence of non-p-nuclear operators with p-nuclear second adjoints*, Doklady AN SSSR **256** (1981), no. 1, 43-47.
10. Reinov O.I., *Approximation properties of order p and the existence of non-p-nuclear operators with p-nuclear second adjoints*, Math. Nachr. **109** (1982), 125-134.
11. Reinov O.I., *Disappering tensor elements in the scale of p-nuclear operators*, Theory of operators and theory of functions (LGU) (1983), 145-165.
12. Saphar P., *Produits tensoriels d'espaces de Banach et classes d'applications lineaires*, Studia Math. **38** (1970), 71-100 MR43#878..

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