Banach spaces without approximation properties of type p

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Abstract. The main purpose of this note is to show that the question posed in the paper of Sinha D.P. and Karn A.K. ("Compact operators which factor through subspaces of l_p Math. Nachr. 281, 2008, 412-423; see the very end of that paper) has a negative answer, and that the answer was known, essentially, in 1985 after the papers [2], [3] by Reinov O.I. have been appeared in 1982 and in 1985 respectively.

The main purpose of this note is to show that the question posed in [1] (see the very end of that paper) has a negative answer, and that the answer was known, essentially, in 1985 after the papers [2], [3] have been appeared in 1982 and in 1985 respectively.

For the sake of completeness (and since the article [3] is now quite difficult to get) we will reconstruct some of the facts from [3], maybe, in a somewhat changed form (a translation of [3] with some remarks can be found in arXiv:1002.3902v1 [math.FA]).

All the notation and terminology, we use, are from [2], [4], or [5] and more or less standard. For the general theory of absolutely p-summing, p-nuclear and other operator ideals, we refer to [6].

Now, we recall the main definition from [3] of the topology τ_p of the π_p-compact convergence, which is just the same as the λ_p-topology in [1][1], where it was proved, essentially, that that $T \in K_p^d(X,Y)$ if and only if $T \in QN_p(X,Y)$.

For Banach spaces X, Y, the *topology* τ_p *of* π_p*-compact convergence* in the space $\Pi_p(Y,X)$ is the topology, a local base (in zero) of which is defined by sets of type

$$\omega_{K,\varepsilon} = \{U \in \Pi_p(Y,X) : \ \pi_p(U\Phi_K) < \varepsilon\},$$

where $\varepsilon > 0,\ K = \overline{\Gamma(K)}$ — a compact subset of Y.

Proposition 1 [3]. *Let* R *be a linear subspace in* $\Pi_p(Y,X)$, *containing* $Y^* \otimes X$. *Then* $(\mathrm{R}, \tau_p)'$ *is isomorphic to a factor space of the space* $X^*\widehat{\otimes}_{p'}Y$. *More precisely,*

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[1]Definition 4.3 from [1]: "Given a compact subset $K \subset X$ we define a seminorm $||\cdot||$ on $\Pi_p(X,Y)$ given by $||T||_K = \inf\{\kappa_p^d(Ti_Z) : i_Z : Z \to X$ as above$\}$. The family of seminorms $\{||\cdot|| : K \subset X$ is compact$\}$ determines a locally convex topology λ_p on $\Pi(X,Y)$."; — for details, and for the definition of maps i_Z, see [1], Lemma 4.2.

if $\varphi \in (\mathrm{R}, \tau_p)'$, *then there exists an element* $z = \sum_1^\infty x_n' \otimes y_n \in X^* \widehat{\otimes}_{p'} Y$ *such that*

$$\varphi(U) = \text{trace } U \circ z, \ \ U \in \mathrm{R}\,. \tag{$*$}$$

On the other hand, for every $z \in X^* \widehat{\otimes}_{p'} Y$ *the relation* $(*)$ *defines a linear continuous functional on* (R, τ_p).

Proof. Let φ be a linear continuous functional on (R, τ_p). Then one can find a neighborhood of zero $\omega_K = \omega_{K,\varepsilon}$, such that φ is bounded on it: $\forall\, U \in \omega_K$, $|\varphi(U)| \leqslant 1$. We may assume that $\varepsilon = 1$. Consider the operator $U\Phi_K : Y_K \overset{\Phi_K}{\to} \longrightarrow Y \overset{U}{\to} \longrightarrow X$. Since the mapping Φ_K is compact, $U\Phi_K \in \mathrm{QN}_p(Y_K, X)$. Put $\varphi_k(U\Phi_K) = \varphi(U)$ for $U \in \mathrm{R}$. On the linear subspace $\mathrm{R}_K = \left\{V \in \mathrm{QN}_p(Y_K, X) : \ V = U\Phi_K\right\}$ of the space $\mathrm{QN}_p(Y_K, X)$, the linear functional φ_K is bounded: if $V = U\Phi_K \in \mathrm{R}_K$ and $\pi_p(V) \leqslant 1$, then $|\varphi_K(V)| = \varphi(U)| \leqslant 1$. Therefore, φ_K can be extended to a linear continuous functional $\widetilde{\varphi}$ on the whole $\mathrm{QN}_p(Y_K, X)$; moreover, because of the injectivity of the ideal QN_p, considering X as a subspace of some space $C(K)$, we may assume that $\widetilde{\varphi} \in \mathrm{QN}_p(Y_K, C(K))^*$. Let us mention that

$$\widetilde{\varphi}(jU\Phi_K) = \varphi_K(U\Phi_K) = \varphi(U) \tag{1}$$

(here j is an isometric embedding of X into $C(K)$).

Furthermore, since $\mathrm{QN}_p(Y_K, C(K))^* = \mathrm{I}_{p'}(C(K), (Y_K)^{**})$, we can find an operator $\Psi : C(K) \to (Y_K)^{**}$, for which

$$\widetilde{\varphi}(A) = \text{trace } \Psi A, \ \ A \in (Y_K)^* \otimes C(K).$$

Let $A_n \in (Y_K)^* \otimes C(K)$, $\pi_p(A_n - jU\Phi_K) \to 0$. Then

$$\widetilde{\varphi}(jU\Phi_K) = \lim \text{trace } \Psi A_n. \tag{2}$$

Consider the operator $\Phi_K^{**}\Psi : C(K) \overset{\Psi}{\to} \longrightarrow (Y_K)^{**} \overset{\Phi_K^{**}}{\to} \longrightarrow Y$. Since $\Psi \in \mathrm{I}_{p'}$, and Φ_K is compact, we have $\Phi_K^{**}\Psi \in \mathrm{N}_{p'}(C(K), Y) = C(K)^* \widehat{\otimes}_{p'} Y$. Let $\sum_1^\infty \mu_n \otimes y_n \in C(K)^* \widehat{\otimes}_{p'} Y$ be a representation of the operator $\Phi_K^{**}\Psi$. Put $z = \sum j^*(\mu_n) \otimes y_n$. The element z generates an operator $\Phi_K^{**}\Psi j$ from X to Y. We will show now that trace $U \circ z = \widetilde{\varphi}(jU\Phi_K)$ (note that $U \circ z$ is an element of the space $X^* \widehat{\otimes} X$, so the trace is well defined). We have:

where $(jU\Phi_K)^{**}\Psi : C(K) \overset{\Psi}{\to} \longrightarrow (Y_K)^{**} \overset{\Phi_K^{**}}{\to} \longrightarrow Y \overset{U}{\to} \longrightarrow X \overset{j}{\to} \longrightarrow C(K)$. Since $\pi_p(A_n - jU\Phi_K) \to 0$, then $\pi_p\left(A_n^{**} - (jU\Phi_K)^{**}\right) \to 0$. Moreover, if $A := A_n = \sum_1^N w_m \otimes f_m \in (Y_K)^* \otimes C(K)$, then

$$\text{trace } A_n^{**}\Psi = \sum_m \langle \Psi^* w_m, f_m\rangle = \sum_m \langle w_m, \Psi f_m\rangle = \text{trace } \Psi A.$$

$$(1) \quad \text{trace } U \circ z = \text{trace } \left(\sum j^*(\mu_n) \otimes Uy_n\right) = \sum \langle j^*(\mu_n), Uy_n\rangle =$$
$$= \sum \langle \mu_n, jUy_n\rangle = \text{trace } jU\Phi_K^{**}\Psi = \text{trace}\,(jU\Phi_K)^{**}\Psi, 3)$$

Hence, trace$\,(jU\Phi_K)^{**}\Psi = \lim$ trace $A_n^{**}\Psi = \lim$ trace ΨA_n. Now, it follows from (3) and (2) that $\widetilde{\varphi}(jU\Phi_K) =$ trace $U \circ z$. Finally, we get from (1): $\varphi(U) =$ trace $U \circ z$. Thus, the functional φ is defined by an element of $X^* \widehat{\otimes}_{p'} Y$.

Inversely, if $z\in X^*\widehat{\otimes}_{p'}Y$, put $\varphi(U)=\text{trace }U\circ z$ for $U\in \mathrm{R}$ (the trace is defined since $U\circ z\in X^*\widehat{\otimes}X$). We have to show that the linear functional φ is bounded on a neighborhood $\omega_{K,\varepsilon}$ of zero in τ_p. For this, we need the following fact, which proof is rather standard (see [3], Lemma 1.2): *If $z\in X^*\widehat{\otimes}_qY$, then $z\in X^*\widehat{\otimes}_qY_K$, where $K=\overline{\Gamma(K)}$ is a compact in Y.* Now, let K be a compact subset of Y, for which $z\in X^*\widehat{\otimes}_{p'}Y_K$. If $U\in\omega_{K,1}$, then $\pi_p(U\Phi_K)<1$ and $|\text{trace }U\Phi_K\circ z|\leqslant \|z\|_{X^*\widehat{\otimes}_{p'}Y_K}\cdot \pi_p(U\Phi_K)\leqslant C$.

Corollary 1. $(\mathrm{R},\tau_p)'=(\mathrm{R},\sigma)'$, *where $\sigma=\sigma(\mathrm{R},X^*\widehat{\otimes}_{p'}Y)$. Thus, the closures of convex subsets of the space $\Pi_p(Y,X)$ in τ_p and in σ are the same.*

Proposition 2 [3]. *If the canonical mapping $j: X^*\widehat{\otimes}_{p'}Y\to \mathrm{N}_{p'}(X,Y)$ is one-to-one then $\Pi_p(Y,X)=\overline{Y^*\otimes X}^{\tau_p}$.*

Proof If the map j is one-to-one then the annihilator $j^{-1}(0)^\perp$ of its kernel in the space, dual to $X^*\widehat{\otimes}_{p'}Y$, coincides with $\Pi_p(Y,X^{**})$. On the other hand, in any case $j^{-1}(0)^\perp=\overline{Y^*\otimes X}^{*}$ (the closure in *-weak topology of the space $\Pi_p(Y,X^{**})$); by Corollary from Proposition 1,

$$\Pi_p(Y,X)\cap\overline{Y^*\otimes X}^{*}=\overline{Y^*\otimes X}^{\tau_p}.$$

Therefore, $\Pi_p(Y,X)=\overline{Y^*\otimes X}^{\tau_p}$.

For a reflexive space X, the dual space to $X^*\widehat{\otimes}_{p'}Y$ is equal to $\Pi_p(Y,X)$. Consequently, it follows from the last two statements

Corollary 2. *For a reflexive space X the canonical mapping $j: X^*\widehat{\otimes}_{p'}Y\to \mathrm{N}_{p'}(X,Y)$ is one-to-one iff the set of finite rank operators is dense in the space $\Pi_p(Y,X)$ in the topology τ_p of π_p-compact convergence.*

Recall a definition of the *approximation property AP_p of order $p, p\in(0,\infty]$* (see, e.g., [2] and [7]): a Banach space X has the AP_p if for every Banach space Y (equivalently, for every reflexive Banach space Y, see [2] or [5]) one has the equality $Y^*\widehat{\otimes}_pX=\mathrm{N}_p(Y,X)$. It follows now from Proposition 1 and Corollary 2:

Corollary 3. *For $p\in[1,\infty]$ and for every Banach space X the following are equivalent:*

1) *X has the AP_p;*

2) *for each Banach space Y $\overline{X^*\otimes_{p'}Y}^{\tau_{p'}}=\Pi_{p'}(X,Y)$.*

3) *for each reflexive Banach space Y $\overline{X^*\otimes_{p'}Y}^{\tau_{p'}}=\Pi_{p'}(X,Y)$.*

3′) *for each reflexive Banach space Y $\overline{X^*\otimes_{p'}Y}^{\lambda_{p'}}=\Pi_{p'}(X,Y)$.*

As shown in [2], [4] and [5] , for each $p, p\in[1,\infty], p\neq 2$, there exist reflexive Banach spaces without the AP_p. Thus, we get from the last consequence

Corollary 4. *For $1\le p\le\infty, p\neq 2$, there are two reflexive Banach spaces X and Y such that the natural map $Y^*\widehat{\otimes}_pX\to\mathrm{N}_p(Y,X)$ is not one-to-one and $\overline{X^*\otimes_{p'}Y}^{\tau_{p'}}\neq\Pi_{p'}(X,Y)$.*

Since, as we said, τ_q equals to the topology λ_q from [1], we get firstly Theorem 4.11 [1] (the case $p>2$ below) and secondly the negative answer to the question in the end of the paper [1]. Or, generally, we have:

Theorem. *Let $1 \leq q \neq 2 \leq \infty$. Then there is a (reflexive) Banach space that fails the approximation property of type q of* [1].

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