CHARACTERIZATION OF WEIGHTED INTEGRAL MEANS

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Introduction

In recent paper we will continue investigation of means

$$K_f(u,v) = \frac{f(f'^{-1}(u)) - f(f'^{-1}(v))}{f'^{-1}(u) - f'^{-1}(v)}, \tag{1}$$

introduced in the previous preprint [1], where $f \in C^1$ and f' strictly monotone. If for some mean K there is the function f such that $K = K_f$ we will call it *solvable mean.* Our aim is to proof a simple criteria of solvability of the given mean: the non-degenerate mean $K \in C^1([\alpha,\beta]\times[\alpha,\beta])$ is solvable if and only if for any four numbers $a,b,c,d \in [\alpha,\beta]$

$$\begin{vmatrix} K(a,b) & K(a,b) & K(c,d) \\ K(c,a) & K(a,d) & K(c,d) \\ K(c,b) & K(b,d) & K(c,d) \end{vmatrix} = \begin{vmatrix} K(a,d) & K(b,d) & K(b,d) \\ K(a,d) & K(a,b) & K(c,b) \\ K(c,a) & K(c,a) & K(c,b) \end{vmatrix}.$$

This is absolutely intrinsic characterization of the above class. Usually the problem of intrinsic or axiomatic characterization of concrete class of functions is open. We know a few examples of positive solutions of such type: for difference quotent [2], for cross-ratio [3], for (α,ϕ)-means [4], for some other classes [5]. Many classes of means are functionally conjugate. But even we have the characterization of one of them we cannot apply the result to another one. For this one should solve the Cauchy type equation, which is not solvable in general. For example we cannot apply our result to class of integral means which is conjugate with described above means.

Making the change of variables $p(s) = f'^{-1}(s)$ we obtain another form of this mean which coincides with the special case of weighted integral mean described by G. Toader and J. Sandor in [6]

$$K_f(u,v) = M_p(u,v) = \frac{\int_{p(u)}^{p(v)} p^{-1}(s)ds}{p(v)-p(u)}. \tag{1'}$$

Remark that the f'-conjugation $f'(K_f(f'^{-1}, f'^{-1}))$ of K_f gives ordinary integral mean generated by function f'.

In [1] was proven that for solvable means holds $K_f = K_{\tilde{f}}$ where $\tilde{f}(t) = \frac{1}{k}f(kt+a)+b$, for all k, a, b. Thus we can assume without lose of generality that

$$f(t) : [0,1] \to I\!R, \quad f(0) = 0, \quad f'(1) > f'(0),$$

and K_f is defined on $[f'(0), f'(1)] \times [f'(0), f'(1)]$.

§1. Sections of mean

Analyzing (1) we see that for $f \in C^1$ and f' strictly monotone the mean K_f is continuous, and for all admissible C the section $\theta_C(u) = K_f(u, C)$ is strictly monotone as well. More over the equation $\theta_C(u) = u$ has the only solution $u = C$.

Let solvable mean K_f be defined on $[C, C'] \times [C, C']$. We can write (1) for $u = C = f'(0)$

$$\theta_C(f'(v)) = K_f(f'(0), f'(v)) = \frac{f(v) - f(0)}{v - 0} = \frac{f(v)}{v}.$$

Therefore

$$f'(v) = \theta_C^{-1}\left(\frac{f(v)}{v}\right). \tag{2}$$

There is the only stationary point for this homogeneous equation $f'(v) = C$ which is corresponding the linear function $f(v) = Cv$, hence boundary conditions $C < f(1) = K(C, C') < C'$ gives C^1-solution, extended by continuity in 0. This solution is unique while the mean is solvable, i.e. it is generalized by convex function which has not linear segments, therefore this solution is separated from stationary one. This ODE can be explicitly solved by substitution $f(v) = vz(v)$ and separation of variables. Thus the explicit form of generating function $f(t)$ is

$$f(t) = tF^{-1}(\ln(t)), \tag{3}$$

where

$$F(z) = \int_{K(C,C')}^{z} \frac{d\mu}{\theta_C^{-1}(\mu) - \mu}, \tag{4}$$

$$F(C) = -\infty.$$

Remark 1. The solutions of (2) for different initial data coincide up to homothety. Thus every section generates the same mean K_f.

§2. Necessary conditions of solvability

We see that if the mean K is solvable and for some C the section $\theta_C \in \mathit{Diff}^{\,k}$ for some $k \geqslant 0$, then $K \in C^k$ and therefore all sections have the same smoothness $\mathit{Diff}^{\,k}$. Hence it is the necessary condition of solvability. In the case of continuous mean K the uniqueness of solution of (2) for every section is also necessary condition of solvability. The mean K_f generated by the solution f of (2) for arbitrary mean K satisfying above conditions may not coincide with initial one $K_f \neq K$. It means that

$$K$$

is not solvable.

Let K_f be solvable mean defined on $[C, C'] \times [C, C']$. Rewrite (2) in the form

$$\theta_C(u) = \frac{f(f'^{-1}(u))}{f'^{-1}(u)}. \tag{5}$$

Put now (5) in (1)

$$K_f(u, v) = \frac{\theta_C(u) f'^{-1}(u) - \theta_C(v) f'^{-1}(v)}{f'^{-1}(u) - f'^{-1}(v)}. \tag{6}$$

In particular for $v = C' = f'(1)$ we have

$$K_f(u, C') = \frac{\theta_C(u) f'^{-1}(u) - \theta_C(C')}{f'^{-1}(u) - 1}. \tag{7}$$

Let us now express $f'^{-1}(u)$ from the last

$$f'^{-1}(u) = \frac{K_f(u, C') - \theta_C(C')}{K_f(u, C') - \theta_C(u)}. \tag{8}$$

Substituting now (2) in obtained expression we get

$$t = \frac{K_f(\theta_C{}^{-1}\left(\frac{f(t)}{t}\right), C') - \theta_C(C')}{K_f(\theta_C{}^{-1}\left(\frac{f(t)}{t}\right), C') - \frac{f(t)}{t}} \tag{9}$$

This equation has the simplier form

$$f(t) - (t-1) K_f(\theta_C{}^{-1}\left(\frac{f(t)}{t}\right), C') - \theta_C(C') = 0, \tag{10}$$

where $f(0)=0, f'(0)=C, f(1)=\theta_C(C'), f'(1)=C'$. If we introduce again the function $z(t)=\frac{f(t)}{t}$, where $z(0)=C, z(1)=\theta_C(C'), z'(1)=C'-\theta_C(C')$, we get

$$tz(t)-(t-1)\theta_{C'}\left(\theta_C{}^{-1}\left(z(t)\right)\right)-\theta_C(C')=0.$$

Substituting $t=z^{-1}(s)$ we get

$$z^{-1}(t)=\frac{\theta_C(C')-\theta_{C'}\left(\theta_C^{-1}(t)\right)}{t-\theta_{C'}\left(\theta_C^{-1}(t)\right)}. \tag{11}$$

Theorem 1. *The mean K defined on $[C,C']\times[C,C']$ is solvable if it is generated by C^1-function $tz(t)$ where*

1)

$$z^{-1}(t)=e^{\int_{K(C,C')}^{t}\frac{d\mu}{\theta_C^{-1}(\mu)-\mu}},$$

or

2)

$$z^{-1}(t)=\frac{\theta_C(C')-\theta_{C'}\left(\theta_C^{-1}(t)\right)}{t-\theta_{C'}\left(\theta_C^{-1}(t)\right)}. \quad \square$$

The first point follows directly from (3). This theorem allows us to find the generating function via integration of the inverse of the section θ_C or directly using both sections θ_C and $\theta_{C'}$. Comparison of these two formulae gives some kind of "the operational research". For example, as a sequence of above theorem we have the formula for every solvable mean and its sections

$$\int_{\theta_C(C')}^{\theta_C(t)}\frac{d\mu}{\theta_C^{-1}(\mu)-\mu}=\ln\left(\frac{\theta_C(C')-\theta_{C'}(t)}{\theta_C(t)-\theta_{C'}(t)}\right). \tag{12}$$

§3. Reconstruction of the solvable mean by section

Combining now (2) and (3) we get the explicit expression for f'

$$f'(t)=\theta_C^{-1}\left(F^{-1}(\ln t)\right). \tag{13}$$

Thus

$$f'^{-1}(u)=e^{F(\theta_C(u))}, \tag{14}$$

and with (6) it gives

$$K_f(u,v) = \frac{\theta_C(u) e^{\int_{\theta_C(C')}^{\theta_C(u)} \frac{d\mu}{\theta_C^{-1}(\mu)-\mu}} - \theta_C(v) e^{\int_{\theta_C(C')}^{\theta_C(v)} \frac{d\mu}{\theta_C^{-1}(\mu)-\mu}}}{e^{\int_{\theta_C(C')}^{\theta_C(u)} \frac{d\mu}{\theta_C^{-1}(\mu)-\mu}} - e^{\int_{\theta_C(C')}^{\theta_C(v)} \frac{d\mu}{\theta_C^{-1}(\mu)-\mu}}}. \tag{15}$$

This formula reconstructs the mean by its section. Considered above equations (15), (12) and Theorem are necessary conditions for solvability of mean.

Let us continue to get the sequences. Put (8) in (6)

$$K_f(u,v) = \frac{\theta_C(u) \frac{K_f(u,C')-\theta_C(C')}{K_f(u,C')-\theta_C(u)} - \theta_C(v) \frac{K_f(v,C')-\theta_C(C')}{K_f(v,C')-\theta_C(v)}}{\frac{K_f(u,C')-\theta_C(C')}{K_f(u,C')-\theta_C(u)} - \frac{K_f(v,C')-\theta_C(C')}{K_f(v,C')-\theta_C(v)}}. \tag{16}$$

This formula can be rewritten in the compact form

$$K_f(u,v) = \frac{\begin{vmatrix} K_f(C,v) & K_f(C',v) & K_f(C',v) \\ K_f(C,v) & K_f(C,C') & K_f(u,C') \\ K_f(u,C) & K_f(u,C) & K_f(u,C') \end{vmatrix}}{\begin{vmatrix} K_f(C,C') & K_f(C,C') & 1 \\ K_f(u,C) & K_f(C,v) & 1 \\ K_f(u,C') & K_f(C',v) & 1 \end{vmatrix}} \tag{17}$$

Right side of equation contains only values of sections and can be interpreted as reconstruction of mean's graph defined on the square with help of five points on the boarder. Necessarily the fraction tends to v while u tends to v.

Note that in symmetrical form (17) can be written as follows

$$\begin{vmatrix} K_f(C,C') & K_f(C,C') & K_f(u,v) \\ K_f(u,C) & K_f(C,v) & K_f(u,v) \\ K_f(u,C') & K_f(C',v) & K_f(u,v) \end{vmatrix} = \begin{vmatrix} K_f(C,v) & K_f(C',v) & K_f(C',v) \\ K_f(C,v) & K_f(C,C') & K_f(u,C') \\ K_f(u,C) & K_f(u,C) & K_f(u,C') \end{vmatrix}.$$

§4. Geometry of barycenter

Suppose that in (1') $p(s)$ is smooth and $p'(s) = g(s)$, than we have the formula of barycenter, i.e the x-coordinate of g-ordinate set,

$$K_f(u,v) = M_p(u,v) = B_g(u,v) = \frac{\int_u^v \mu g(\mu) d\mu}{\int_u^v g(\mu) d\mu}.$$

All above analysis is applicable for this situation. For example, we can answer the question how to find the barycenter of the part of ordinate set between u and v knowing barycenters between C and u, C and v, u and C', v and C', and between C and C' together. This problem has the direct solution. Denote as S_1, S_2, S_3 the areas of ordinate sets between points C, u, v, C' respectively. We can write three obvious conditions on barycenters

$$\begin{cases} \dfrac{B_g(C,u)S_1 + B_g(u,v)S_2 + B_g(v,C')S_3}{S_1+S_2+S_3} = B_g(C,C') \\ \dfrac{B_g(C,u)S_1 + B_g(u,v)S_2}{S_1+S_2} = B_g(C,v) \\ \dfrac{B_g(u,v)S_2 + B_g(v,C')S_3}{S_2+S_3} = B_g(u,C') \end{cases}$$

This is the homogeneous linear system with respect to S_1, S_2, S_3. As for this system has the solution the determinant of it is equal to zero

$$\begin{vmatrix} B_g(C,u) - B_g(C,C') & B_g(u,v) - B_g(C,C') & B_g(v,C') - B_g(C,C') \\ B_g(C,u) - B_g(C,v) & B_g(u,v) - B_g(C,v) & 0 \\ 0 & B_g(u,v) - B_g(u,C') & B_g(v,C') - B_g(u,C') \end{vmatrix} = 0.$$

Expression of $B_g(u,v)$ from this equation gives exactly the formula (17).

Now we can apply the results of previous paragraphs. For given function of barycenter we can reconstruct the shape of graph up to scalar via sections. By notation $g(u) = p'(u) = (f'^{-1})'(u)$. From (14)

$$g(u) = \left(e^{F(\theta_C(u))}\right)' = e^{F(\theta_C(u))} F'(\theta_C(u))\theta_C'(u).$$

Differentiating (4) we get the expression

$$g(u) = \frac{\theta_C'(u)}{u - \theta_C(u)} \exp\left(\int_{C'}^{u} \frac{\theta_C'(\mu)d\mu}{\mu - \theta_C(\mu)}\right),$$

or by (12)

$$g(u) = \frac{\theta_C'(u)}{u - \theta_C(u)} \cdot \frac{C_0 - \theta_{C'}(u)}{\theta_C(u) - \theta_{C'}(u)}.$$

This function corresponds to generating $f(t)$ which satisfies the boundary conditions $f'(0) = C, f'(1) = C'$, all other possible functions $\lambda g(u)$ correspond to homothety $\lambda f(\frac{t}{\lambda})$.

The question if the given mean is the barycenter of some ordinate set is equivalent to solvability of this mean and we shell give the answer on it in further paragraphs.

§5. The Bernulli equation of special type

Consider the case $K_f \in C^1$. In this situation fixing v we can use L'Hopital's rule in the point $u = v$ for (17). This gives the equation

$$\frac{\theta'_{C'}(t)}{(\theta_{C'}(t) - \theta_{C'}(C))(\theta_{C'}(t) - t)} = \frac{\theta'_C(t)}{(\theta_C(t) - \theta_C(C'))(\theta_C(t) - t)}. \tag{18}$$

So, we see that both sections of solvable smooth mean should satisfy to one Riccati ODE of Bernulli type

$$y' = \xi(t)(y - t)(y - C_0), \tag{19}$$

where

$$\xi(t) = \frac{\theta'_{C'}(t)}{(\theta_{C'}(t) - \theta_{C'}(C))(\theta_{C'}(t) - t)} \in C^0(C, C'),$$

and both solutions determined by boundary conditions $\theta_C(C') = C, \theta_C(C') = C_0, \theta_{C'}(C) = C_0, \theta_{C'} = C'$. Note that the equation (19) is not defined on the ends of interval (C, C'). As we will see later the function $\xi(t)$ is unique characteristic of mean.

Let investigate the space of solutions of the equation (19) for section θ_C of some solvable mean. This equation has two obvious solutions $y(t) = \theta_C(t)$ and $y(t) = C_0$.

Using the change of variables $y = \frac{1}{w} + C_0$ we get the linear ODE

$$w' + (C_0 - t)\xi(t)w = -\xi(t), \tag{20}$$

which has the particular solution $w_0 = \frac{1}{\theta_C - C_0}$. For any other solution $w_1 = \frac{1}{y_1 - C_0}$ the difference $w = w_1 - w_0$ satisfies the ODE

$$w' + (C_0 - t)\xi(t)w = 0, \tag{21}$$

which is explicit integrable

$$w(t) = w(C_0)e^{\int_{C_0}^{t}(\mu - C_0)\xi(t)d\mu}. \tag{22}$$

The integral in the right part can directly calculated. Let us make the change of variables $s=\theta_C(\mu)$ in the integral where we changed ξ on the original expression

$$\int_{C_0}^{t}(\mu-C_0)\frac{\theta_C'(\mu)d\mu}{(\theta_C(\mu)-\mu)(\theta_C(\mu)-C_0)}=\int_{\theta_C(C_0)}^{\theta_C(t)}\frac{(\theta_C^{-1}(s)-C_0)ds}{(s-\theta_C^{-1}(s))(s-C_0)}=$$

$$\int_{\theta_C(C_0)}^{\theta_C(t)}\frac{ds}{s-\theta_C^{-1}(s)}+\int_{\theta_C(C_0)}^{\theta_C(t)}\frac{ds}{C_0-s}=\int_{\theta_C(C_0)}^{\theta_C(t)}\frac{ds}{s-\theta_C^{-1}(s)}+\ln\left(\frac{1}{C_0-s}\right)\Big|_{\theta_C(C_0)}^{\theta_C(t)}.$$

So, we have

$$w(t)=w(C_0)\frac{\theta_C(C_0)-C_0}{\theta_C(t)-C_0}e^{-\int_{\theta_C(C_0)}^{\theta_C(t)}\frac{ds}{\theta_C^{-1}(s)-s}}.$$

Thus any solution with initial data $y_1(C_0)\neq C_0$ and $y_1(C_0)\neq\theta_C(C_0)$ has the form

$$y_1(t)=C_0+\frac{\theta_C(t)-C_0}{1-\frac{y_1(C_0)-\theta_C(C_0)}{y_1(C_0)-C_0}\exp\left(-\int_{\theta_C(C_0)}^{\theta_C(t)}\frac{ds}{\theta_C^{-1}(s)-s}\right)}. \tag{23}$$

For any solution $y_1(C)=C_0$ while

$$\int_{\theta_C(C_0)}^{\theta_C(C)}\frac{ds}{\theta_C^{-1}(s)-s}=\int_{\theta_C(C')}^{\theta_C(C)}\frac{ds}{\theta_C^{-1}(s)-s}=F(C)=-\infty$$

For every initial data when the denominator is not zero $y_1(C')=C_0$. In the case of zero denominator the indeterminacy appears. Let us calculate such initial data. We can find them from the condition

$$1-\frac{y_1(C_0)-\theta_C(C_0)}{y_1(C_0)-C_0}\exp\left(\int_{\theta_C(C')}^{\theta_C(C_0)}\frac{ds}{\theta_C^{-1}(s)-s}\right)=0,$$

The formula (12) gives

$$1-\frac{y_1(C_0)-\theta_C(C_0)}{y_1(C_0)-C_0}\left(\frac{C_0-\theta_{C'}(C_0)}{\theta_C(C_0)-\theta_{C'}(C_0)}\right)=0,$$

i.e

$$y_1(C_0)=\theta_{C'}(C_0).$$

Due to the uniqueness theorem there is the only solution of the Bernulli equation with such special condition $y_1 = \theta_{C'}$. In particular the section $\theta_{C'}$ is the only solution which satisfies the boundary condition $\theta_{C'}(C') = C'$.

Remark. Using (7) and (14) we can write explicit expression of $\theta_{C'}$ via θ_C

$$\theta_{C'}(u) = \frac{\theta_C(u)e^{F(\theta_C(u))} - \theta_C(C')}{e^{F(\theta_C(u))} - 1}.$$

It is true for every $C' > C$, but the Bernulli equation generated by monotone C^1-function θ_C is changed while the function ξ depends on $\theta_C(C')$. Any way it is the complete description of all mean sections.

§6. Sufficient condition of solvability

Now we are ready to proof the main result of the article.

Theorem 2. *Let $K(u,v)$ be the C^1-mean defined on $[C,C'] \times [C,C']$ which has monotone section $K(u,C)$ and satisfy the relation*

$$\begin{vmatrix} K_f(C,C') & K_f(C,C') & K_f(u,v) \\ K_f(u,C) & K_f(C,v) & K_f(u,v) \\ K_f(u,C') & K_f(C',v) & K_f(u,v) \end{vmatrix} = \begin{vmatrix} K_f(C,v) & K_f(C',v) & K_f(C',v) \\ K_f(C,v) & K_f(C,C') & K_f(u,C') \\ K_f(u,C) & K_f(u,C) & K_f(u,C') \end{vmatrix}$$

then K is solvable.

Proof. Rewrite the equality in the form of (17). Both boundary sections of the mean satisfy the Riccati ODE (19) with $\xi(t)$ generated by θ_C and boundary condition $\theta_C(C') = \theta_{C'}(C) = C_0, \theta_{C'}(C') = C'$. Let us introduce the function $f(t)$ as the solution of (2) and consider the mean K_f. In view of (5) θ_C is the section of K_f too. The mean K_f also satisfy (17), hence its section $K_f(u,C')$ satisfies (19) with the same ξ for it is generated by θ_C. Thus both sections $K(u,C')$ and $K_f(u,C')$ are the solutions of (19) with the same value in $u = C'$. Therefore these sections coincide and hence $K = K_f$. $\square$

Corollary. *If the smooth function θ_C is C^1-diffeomorphism on $[C,C']$, $\theta_C(C) = C$, $\theta_C(u) < u$ for $u > C$, and both functions θ_C and $\theta_{C'}$, where $\theta_{C'} \in C^1([C,C'])$, $\theta_{C'}(C) = \theta_C(C')$ and $\theta_{C'}(C') = C'$, generate the same function (18) then they define by (17) the solvable mean.*

Indeed, the function θ_C define C^2-function f and C^1-mean K_f. Necessarily K_f satisfies (17). Therefore its C'-section generates the same function (18) and hence coincides with $\theta_{C'}$.

Example. The Hölder mean $\sqrt{\frac{u^2+v^2}{2}}$ is not solvable over any square $[C, C'] \times [C, C']$ (see [7]). Let us calculate (18) for the sections

$$\theta_C(u) = \sqrt{\frac{C^2+u^2}{2}}, \quad \theta_{C'}(u) = \sqrt{\frac{C'^2+u^2}{2}}.$$

It is easy to see that $\xi_C(u) \neq \xi_{C'}(u)$, while

$$\frac{\xi_C(C_0)}{\xi_{C'}(C_0)} = \sqrt{\frac{3-t}{t+1}} \left(\frac{\sqrt{3-t}-\sqrt{2}}{\sqrt{t+1}-\sqrt{2}} \right)^2 \neq 1,$$

where $t = (C/C_0)^2$, $t \in (0, 1)$. The last can be proven in the following way. Let us make the change of variables $a = \sqrt{\frac{3-t}{2}}, b = \sqrt{\frac{t+1}{2}}$, then the symmetrical system

$$\begin{cases} \frac{a}{b}\left(\frac{a-1}{b-1}\right)^2 = 1 \\ a^2 + b^2 = 2 \end{cases}$$

has the only positive solution $a = b = 1$, but it is out of domain.

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