Grothendieck's inequality and applications

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The famous Grothendieck inequality, which can be seen as a matrix inequality associated to certain bilinear operators and called by him "the fundamental theorem of the metric theory of tensor products of Banach spaces", is equivalent to the following assertion:

Let $\{a_{ij}\}_{i,j=1}^n$ *be a finite matrix of real numbers such that* $|\sum_{i,j=1}^n a_{ij}t_is_j| \leq 1$ *whenever* $|t_i|, |s_j| \leq 1$. *Then for every set of unit vectors* $\{x_i\}_{i=1}^n$ *and* $\{y_j\}_{j=1}^n$ *in a Hilbert space* $\| \sum_{i,j} a_{ij}(x_i, y_j)\| \leq K$, *where* K *is an absolute constant.*

This theorem has a lot of generalizations and applications in very different directions. Some of them are investigations of multilinear extensions of the inequality as well as considerations of the cases of so-called operator spaces and of non-commutative L_p-spaces (such as the Schatten spaces S_p). Let us mention just a few fields of applications:

• Theory of absolutely p-summing operators with application to the isomorphic classification of Banach spaces and to the geometry of normed spaces in general (an example: disk algebra $C_A(\mathbb{T})$ is not isomorphic to a factor space of a $C(K)$-space);

• Investigations of uniform Banach algebras and, generally, of Q-algebras (commutative Banach algebras which are isomorphic as Banach algebras to the quotients of uniform Banach algebras). An example: the answers (for $1 \leq p \leq \infty$) to the old (essentially, due to Varopoulos) problem whether S_p-spaces (with their Schur products) should be Q-algebras;

• Problems of vector measures theory and related questions in geometric theory of Banach spaces (such as constructions of counterexamples to some long standing problems. For instance, to the question of whether a separable Banach space does not contain l_1 if and only if its dual space is separable).

We shall be concerned with just some small (but hope, ones of the main) parts of the topic in connection with this beautiful Grothendieck inequality.

I. SOME CLASSICAL APPLICATIONS.

We begin with definitions of absolutely p-summing operators (for general notations, definitions and results, see, e.g., [**4**], [**6**], [**18**], [**5**]).

An operator T from a Banach space X into a Banach space Y is said to be p-*absolutely summing*, notation $T \in \Pi_p(X, Y)$, where $0 < p \leq \infty$, if there is a constant

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$C \in (0,\infty)$ such that for all finite families $(x_i)_{i=1}^n \subset X$

$$\sum_{i=1}^n ||Tx_i||^p \leq C^p \sup\{\sum_{i=1}^n |\langle x', x_i\rangle|^p : \ x' \in X, \ ||x'|| \leq 1\}.$$

The p-summing norm $\pi_p(T)$ is defined as $\inf C$.

The Grothendieck's inequality can be easily reformulated in terms of 1-absolutely summing operators: every linear continuous operator from a $L_1(\mu)$-space into a Hilbert space is 1-absolutely summing. So, the first application of the inequality is its reformulation in terms of absolutely summing operators (the proofs can be found in [**5**], [**6**], [**18**] etc.).

Theorem I.1. $\mathrm{L}(l_1, l_2) = \Pi_1(l_1, l_2)$.

Proof. Put $P(f) := (\widehat{f}(2^n))_{n=1}^\infty$ for $f \in H^1(\mathbb{T})$. By Paley's inequality [**2**], the associated operator with operator P is a projection ("Paley projection") in $H^1(\mathbb{T})$, which can be considered as an operator from $H^1(\mathbb{T})$ onto l_2. Let T be any operator from l_1 to l_2, $J : C_A(\mathbb{T}) \to H^1(\mathbb{T})$ is the identity embedding. Then the operator $PJ : C_A(\mathbb{T}) \to l_2$ is "onto" and therefore, there is a factor map $Q : l_1 \to C_A(\mathbb{T})$ such that $T = PJQ$. Since J is 1-absolutely summing, we are done.

The next application is the so-called "little Grothendieck Theorem" (see, for example, [**3**],Theorem 5.4):

Theorem I.2. *Any operator from a $C(K)$-space to a Hilbert space is 2-absolutely summing.*

Proof. It is not difficult to see that an operator $T \in \mathrm{L}(X, Y)$ is 2-absolutely summing iff for any $U \in \mathrm{L}(l_2, X)$ the operator TU is of type Π_2. In the case where T maps $X = C(K)$ into l_2, this is the same as TU (or U^*T^*) is a Hilbert-Schmidt operator. By Grothendieck, $U^* : C^*(K) \to l_2$ is of type Π_1; thus, 2-absolutely summing. Done.

One more nice application:

Theorem I.3. *Hardy space $H^1(\mathbb{T})$ is not isomorphic to any complemented subspace of $L_1(\mu)$-space.*

Proof. There is an operator from $H^1(\mathbb{T})$ to l_2 which is not 1-absolutely summing (for one of the possible proof, see [**18**], pp. 300-301, where a Hardy inequality is used; another proof consists of considering the Paley projection: in above notation, if P is 1-absolutely summing then PJ is a nuclear (hence, compact) from $C_A(\mathbb{T})$ *onto* l_2!).

And also, we have a nice characterization of Hilbert spaces (see [**18**]):

Theorem I.4. *A Banach space is isomorphic to a Hilbert space iff it is isomorphic to a subspace of an $L_1(\nu)$-space and to a quotient of an $L_\infty(\mu)$-space.*

Proof. Every operator from $L_\infty(\mu)$ to $L_1(\nu)$ is 2-absolutely summing (see, e.g., [**18**], Theorem 5.1), hence, can be factored through a Hilbert space.

The following interesting application will be given without any proof (a proof can be found, e.g., in [**4**] or in [**18**]).

Theorem I.5. *All normalized unconditional bases in $l_1(\Gamma)$ are equivalent to the unit vector basis in $l_1(\Gamma)$. The same is true for the space $c_0(\Gamma)$.*

We conclude this section by a result on the disc algebra (see [**5**] or [**6**]).

Theorem I.6. *The disk algebra $C_A(\mathbb{T})$ is not isomorphic to a factor space of $C(K)$.*

Proof. Consider $S : C(K) \to C_A(\mathbb{T})$ and $J : C_A(\mathbb{T}) \to H^1(\mathbb{T})$, where J is the natural inclusion. If S is "onto" then JS is 1-integral, so it is nuclear (values in a separable dual). Thus, J is compact.

II. AN APPLICATION TO $(\mathrm{I}_p, \mathrm{N}_p)$-MULTIPLICATORS.

One of the simplest way to get a multi-dimensional analogue of Grothendieck's inequality (see [**7**], [**8**],)by using Grothendieck inequality itself, can be found in [**9**]. The authors define there (p. 95) a notion of the so-called p-regular norm on the tensor product $X \otimes Y$ of two Banach spaces in such a way that this definition gives them, almost immediately, the multi-dimensional inequality of A. Grothendieck (see [**9**], Theorem 3 and its Corollary 1). Namely:

Definition II.1. Let (S, μ) and (T, ν) be finite measure spaces, $p > 0$, A and B be subspaces of $L_p(\mu)$ and $L_p(\nu)$ respectively. Identify $A \otimes B$ with the set

$$\mathrm{span}_{L_p(\mu\times\nu)}\{h : \ h(s,t) = f(s)g(t),\ f \in A, g \in B\}$$

and put $A \otimes_p B = clos_{L_p(\mu\times\nu)} A \otimes B$. A norm α on $X \otimes Y$ is said to be *p-regular* if for any (S, μ), (T, ν), A and B as above and for any operators $U : A \to X$ and $V : B \to Y$ the operator $U \otimes V$ can be extended to a continuous operator (still denoted by $U \otimes V$) from $A \otimes_p B$ into $X \bar{\otimes}_\alpha Y$ (the completion with respect to α) and $||U \otimes V|| \le ||U||\,||V||$.

Theorem II.1. *Let C, D, X, Y be Banach spaces, $p > 0$, $S \in \Pi_p(C, X)$, $T \in \Pi_p(D, Y)$. If α is a p-regular norm on $X \otimes Y$, then $S \otimes T \in \Pi_p(C \hat{\hat{\otimes}} D, X \bar{\otimes}_\alpha Y)$ and $\pi_p(S \otimes T) \le \pi_p(S)\, \pi_p(T)$.*

Proof. Just apply Pietsch factorization theorem and the definition II.1.

Thus, this theorem contains essentially only a modification of the definition; but it has some nice consequences which justify that the authors called it "the theorem". In follows immediately from the theorem, e. g. :

Corollary II.1. *If $T_i \in \mathrm{L}(l_1, l_2)$, $i = 1, \dots, n$, then*

$$T_1 \otimes \dots \otimes T_n \in \Pi_1(l_1 \hat{\hat{\otimes}} \dots \hat{\hat{\otimes}} l_1, l_2(\mathbb{Z}_+^n)) \quad \textit{and} \quad \pi_1(T_1 \otimes \dots \otimes T_n) \le K_G^n\, ||T_1|| \dots ||T_n||.$$

Therefore, the multi-dimensional generalization of Grothendieck's inequality is just a "right" definition (of a tensor norm) plus an application of the classical 1-dimensional inequality of A. Grothendieck. This "right" definition let us to get a lot of other applications. As an example, we obtain also another interesting consequence (with not very difficult proof). Bellow, it is denoted by $I_{\mu,p}$ the identity imbedding from $C(K)$ into $L_p(\mu)$ (where μ is a finite Radon measure on a compact K).

Corollary II.2. *Let X be a Banach spaces, T is a linear continuous operator from X to $C(K)$ and $p > 0$. Suppose that there is a sequence of finite dimensional projectors $\{P_n\}$ in X with the following properties:*

1) $\sup_n ||P_n|| < +\infty$;

2) $(id - P_n)X = (X_1^n \oplus \ldots X_{k_n}^n)_p$ *for some subspaces* $X_1^n, \ldots, X_{k_n}^n$ *of the space* $(id - P_n)X$;

3) *for every n there exists a family $I_1^n, \ldots, I_{k_n}^n$ of pairwise disjoint Borel subsets of the compact K such that all the functions from $T(X_j^n)$ vanish out of the set I_j^n* $(j = 1, \ldots, k_n)$.

Let μ be a measure on K with $\lim_n \sup_{1 \le j \le k_n} \mu I_j^n = 0$. Then: (a) the operator $I_{\mu,p}T$ is compact for all $p, p > 0$; (b) if $1 \le r < 2$ and $1 < p < r'$ then $I_{\mu,p}T \in \mathrm{N}_p(X, L_p(\mu))$; (c) if $1 \le r < 2$ and $0 < p < r'$ then $I_{\mu,p}T \in \mathrm{QN}_p(X, L_p(\mu))$.

Now, we will consider one more of the applications. Namely, we will show (following [**9**]) how to obtain James-tree-like spaces JT_r with some unusual properties (applying Corollary II.2 for checking these properties). All the difficulties in the construction of such a space is knowing the corollary II.2 and applying just some beliefs in its existence and some mathematical thinking.

We will sketch a construction of the spaces JT_r for $r \in [1, \infty)$ with the properties, mentioned in the following theorem.

Theorem II.2 A. 1) *If $1 \le r < \infty$, and $p > 0$ then every p-absolutely summing operator from JT_r is compact.* 2) *If $1 \le r < 2$ and $1 < p < r'$ then every p-integral operator from JT_r is p-nuclear.* 3) *If $1 \le r < 2$ and $1 \le p < r'$ then every p-absolutely summing operator from JT_r is quasi-p-nuclear.*

B. *If $1 < r < 2$ and $p \ge r'$ or $r \ge 2$ and $p \ge 1$ then there exists an operator from JT_r which is p-integral but not quasi-p-nuclear.*

As a simple consequence of this theorem we get some more examples [cf. [**10**]] of separable Banach spaces having non-separable duals and not containing l_1.

Note that the proof of the part A of the theorem (when the space is constructed) is a simple application of the corollary II.2 (and thus, of the theorem II.1).

Let us describe shortly the James-tree-like spaces from [**9**]. The separable Banach space JT_r consists of functions on a dyadic tree. The norm in JT_r is defined in such a way that every trace of JT_r on each branch of the tree gives us the classical James's space J (of codimension 1 in its second dual); for every level of the tree, say n-th, consisting of 2^n vertexes, the corresponding restriction of JT_r onto 2^n natural subtrees of the tree (growing from those vertexes) gives the direct $l_r^{2^n}$-sum of 2^n 1-complemented subspaces of JT_r (which are isometric to JT_r itself).

More precisely, dyadic tree is a partially ordered set $\mathcal{T}$ which is uniquely (up to isomorphism) determined by the following requirements: I) there is the smallest element in $\mathcal{T}$ ("the root of the tree"), 2) if $t \in \mathcal{T}$ then $\{s \in \mathcal{T} : \ s > t\} = A \cup B$, where each of the sets A and B has the lowest element and any two elements a and b, $a \in A, b \in B$ are incomparable, and 3) no infinite chain in $\mathcal{T}$ has an upper bound. Elements of $\mathcal{T}$ are referred to as vertexes. The root of the tree is the vertex of the zero level, two vertexes (directly following it) are the vertexes of level 1, the next incomparable four vertexes are called the vertexes of level 2. In general, by induction, we can naturally define the vertexes of the n-th level (there are exactly 2^n ones).

If $s \in \mathcal{T}$ then the set $\{t : t \geq s\}$ is called a subtree growing from s. The branch growing from a vertex s (of the n-th level) is any totally ordered set in which s is the smallest element and that contains a vertex of the m-th level for every $m, m \geq n$. By subtrees (branches) of the n-th level we understand any subtrees (branches), growing from the vertexes of a n-th level.

Branches of the zero level are in natural bijective correspondence with the sequences of zeros and ones, that is, with the points of the dyadic Cantor set $\mathcal{C}$. In this correspondence (it is allowed some freedom of speech here), subtrees of the n-th level correspond to 2^n dyadic intervals of the n-th rank (which form a partition of $\mathcal{C}$), which we denote by $I_1^n, \dots, I_{2^n}^n$. In what follows, if I is any dyadic interval then the corresponding subtree is denoted by $\mathcal{T}_I$; F_s^n is the branch of n-th level, corresponding to $s, s \in \mathcal{C}$. Every branch F can be considered as a sequence (if numbering its elements in ascending order) and, therefore, the expression of the form $||g|_F||$ has a sense, where g is a (finite) function on $\mathcal{T}$ and $||\cdot||$ is a norm in some sequence space.

The definition of the classical James's space J can be found in [**11**]. Recall it. The space J is the completion of the set of all finite sequences with respect to the norm $||\cdot||_J$:

$$||x||_J := \sup \left\{ \left(\sum_{j=1}^{m} | \sum_{k=n_j}^{n_{j+1}-1} x_k|^2 \right)^{1/2} : \ 1 \leq n_1 < n_2 < \dots < n_m, \ m = 1, 2, \dots \right\}.$$

Finally, we define the space $JT_r, \ \ 1 \leq r < \infty$. JT_r is the space of functions on $\mathcal{T}$, obtained by completion of the set of finite functions with respect to the norm $|||\cdot|||_r$,

$$|||x|||_r := \sup_n \sup \left\{ \left(\sum_{j=1}^{2^n} ||x|_{F_{s_j}^n}||_J^r \right)^{1/r} : \ s_j \in I_j^n, \ 1 \leq j \leq 2^n \right\}.$$

We consider only the proof of Part B of Theorem II.2, — moreover, only the case where $r \geq 2$ and $2 > p \geq 1$. It is *this case where the Grothendieck's inequality is used.*

So, let us consider the part B of Theorem II.2 for $r \geq 2$ and $2 > p \geq 1$. Define an operator S from JT_r into $C(\mathcal{C})$ by $(Sx)(s) := \lim_{a \in F_s^0} \sum_{b \leq a} x(b)$. Let μ be the Lebesgue measure on $\mathcal{C}$. We shall show that $I_{\mu,p}S \notin \mathrm{N}_p^Q$.

Assume that the latter is not true. Let $\varepsilon > 0$, and find such a finite dimensional (say, m-dimensional) operator U that $\nu_p^Q(I_{\mu,p}S - U) < \varepsilon$. Put

$$X_N := \mathrm{span}_{JT_r}\{e_1^{(N)}, \dots, e_{2^N}^{(N)}\};$$

X_N is isometric to $l_r^{(2^N)}$ (note that the vectors $e_j^{(N)}$ correspond under this isometry to the standard basis of $l_r^{(2^N)}$ and $I_{\mu,p}S(e_j^{(N)})$ are the characteristic functions of dyadic intervals of N-th rank). Let P_N be the natural projection from $L_p(\mu)$ onto span$\{I_{\mu,p}S(e_j^{(N)})\}_{1\le j\le 2^N}$, $||P_N|| = 1$. Operators $P_N I_{\mu,p}S|_{X_N}$ and $P_N U|_{X_N} =: u_N$ can be considered as the operators acting from $l_r^{(2^N)}$ into $l_p^{(2^N)}$, the first one being "identical" with $2^{-N/p}\, h_{r,p}^{(2^N)}$, where $h_{r,p}^{(2^N)}$ is the identity embedding of $l_r^{(2^N)}$ into $l_p^{(2^N)}$, and $\operatorname{rank} u_N \le m$. Since $\nu_p^Q(I_{\mu,p}S - U) < \varepsilon$, we have $2^{-N/p}\,\pi_p(h_{r,p}^{(2^N)} - 2^{N/p}\,u_N) \le \varepsilon$.

On the other hand, if $r \ge 2$ and $2 > p \ge 1$ then $\underline{\lim}_N\, 2^{-N/p}\,\pi_p(h_{r,p}^{(2^N)} - 2^{N/p}\,u_N) \ge C > 0$, where C is an absolute constant. Indeed, to prove this we may and do assume that $r = 2$. Denote the operator $h_{2,p}^{(2^N)} - 2^{N/p}\,u_N$ by A, and let $M := 2^N$ and $b_1, \dots, b_M$ be the rows of the matrix A. Since $p < 2$,

$$\pi_p(A) \ge \pi_2(A) \ge \gamma_\infty(A) = \gamma_1(A^*) \ge K_G^{-1}\,\pi_1(A^*) \ge K_G^{-1}\,\pi_p(A^*)$$

(the second and third norms are the norms in the ideals of operators which can be factored through $C(K)$ and L_1, respectively). Since $\operatorname{rank} 2^{N/p}\,u_N \le m$ for every N, we can find (for all N large enough) at least $M/2$ vectors b_j with l_2-norms greater then, say, $1/4$. Now, it is enough to recall that $M = 2^N$ and to apply the definition of π_p-operators to get $\pi_p(A^*) \ge \left(\sum_{1\le j\le M} ||b_j||_2^p\right)^{1/p} \ge C_0\, 2^{N/p}$. Done.

III. AN APPLICATION TO S_p-ALGEBRAS.

Let us recall that a uniform algebra is a closed subalgebra of $C(K)$ for some compact space K. One says that a Banach algebra is a Q-algebra [**12**] if there exists a uniform algebra A and a closed ideal I of A so that B is isomorphic as a Banach algebra to the quotient algebra A/I. An operator algebra is a Banach algebra which can be identify up to norm equivalence with a closed subalgebra of L(H) for some Hilbert space H. It is known that a quotient algebra of an operator algebra is also an operator algebra [**12**], [**13**], [**14**]. This is a result of B. Cole who proved it for the quotient algebras of the uniform algebras; but the proof goes essentially even for the general case (see, e.g., [**14**]). Thus, every Q-algebra is an operator algebra. It is also clear that a closed subalgebra of an operator algebra is an operator algebra. What is nice and what has been proved by N. Varopoulos [**14**] is the fact that if a Banach $\mathcal{L}_\infty$ space admits a Banach algebra structure then it is necessarily the structure of an operator algebra. The theorem was stated for Banach $C(K)$-spaces B, but as was mentioned in [**14**], the proof has a local Banach spaces technique character, so it suffice to suppose that B is a $\mathcal{L}_\infty$-space in the sense of Lindenstrauss and Pelzcinski [**18**]

It is interesting that the first application of Grothendieck's theorem in the theory of operator algebras seems to be applied in [**14**] by N. Varopoulos. More precisely, Varopoulos has proved a "multidimensional" analogue of the following theorem due to A.Grothendieck [**1**] (see also [**18**]): *there is a $C > 0$, for which every complex bilinear form on $C(X)$ of norm 1, where X is a compact, can be extended to a bilinear form on $L_2(X,\mu) \times L_2(X,\nu)$ of norm $\le C$, for some probability measures μ, ν on X.*

Unfortunately, the formulation of this generalization of the Grothendieck theorem is a little bit complicated and there is no place to bring this nice result of Varopoulos to here (see Lemma 3.1 in [**14**].) One can say that the heart of the proof of Varopoulos Theorem [**14**] is his criterium for a Banach algebra to be an operator algebra and a very clever application of the Grothendieck theorem.

In [**15**] N. Varopoulos has proved a criterion for a Banach algebra to be a Q-algebra (the criterion was very close to the one of A.M. Davie [**19**], as Varopoulos noted, and the proof was also analogous). Varopoulos has introduced the new notion of so called *injective algebras* and used his criterion for proving ([**15**], Theorem 1) that any injective algebra is a Q-algebra. Recall the definition. A Banach algebra R is said to be an injective algebra if the linear mapping induced by the algebra multiplication

$$m: \ R \otimes R \to R \quad (m(x \otimes y) = xy; \, x, y \in R)$$

is continuous for the injective norm of the tensor product $R \hat{\hat{\otimes}} R$ of A. Grothendieck.

The next nice theorem in [**15**] is an interpolation theorem, which was used not only by Varopoulos, but also by many other authors in considering of S_p-algebras. This result (Theorem 2 [**15**]) asserts that for two Q-algebras that form an interpolation pair, the intermediate algebra is also a Q-algebra.

As examples of applications, following Varopoulos, let us consider the algebras $l_p, \ 1 \leq p \leq \infty$ (with pointwise multiplications). A.M. Davie [**19**] proved that the spaces l_p are Q-algebras for $1 \leq p \leq 2$. It was communicated to Varopoulos by Sten Kaijser (see [**15**], p. 4) that l_1 is an injective algebra and thus, by Varopoulos, a Q-algebra . So, interpolating between l_1 and l_∞, Varopoulos got the fact that l_p is Q-algebra for every $p \in [1, \infty]$.

Returning to the paper [**14**] by Varopoulos, let us mention, among the other results, the following one which gave a rise to a 35 years standing open question in the theory of operator algebras. To formulate this result we need some notations from Proposition 4.2 of [**14**]. Let H be a separable Hilbert space and fix $E := \{e_n\}_{n=1}^\infty$, an orthonormal basis in H. Let

$$M := \{m = (m_{ij}; \ i, j = 1, 2, \dots)\}$$

be the space of matrix representations of bounded operators on H with respect to the basis E (that is, $m_{ij} = \langle Te_i, e_j \rangle$ for $T \in L(H)$). *Then one can give on M commutative Banach algebra structure by defining*

$$m \cdot n = (m_{ij} \cdot n_{ij} : \ i, j = 1, 2, \dots)$$

for $m, n \in M$, and that algebra is then an operator algebra.

For a proof that M under the pointwise multiplication is *a normed algebra* see [**16**]. Combined with a criterion of Varopoulos [**14**] for a Banach algebra to be an operator algebra, the proof in [**16**] gives more, namely, that M is operator algebra. As mentioned in [**14**], M appeared in the literature for the first time in [**17**] in 1911.

Thus, taking in account that the space $S_2(H)$ of all Hilbert–Schmidt operators with Schur ("pointwise") multiplication is evidently an operator algebra, Varopoulos in [**14**] proved essentially that $S_p(H)$ (under the Schur multiplication) is an operator algebra for every $p \in [2, \infty]$. The fact that S_p is an operator algebra for all $p \in [1, \infty]$ was proved later by D.P. Blecher and C. Le Merdy [**20**].

The main question, leaving open in [**14**] was:

Is the above algebra M a Q-algebra or is it not?

The mathematical community was solving, step by step, the problem of Varopoulos, or more generally, the problem of whether the commutative Schur S_p-algebras were Q-algebras for $1 \le p \le \infty$. The crucial thing was, surely, to solve the problem in the main cases where $p = 1$ and $p = \infty$ (having in minds the beautiful interpolation result of Varopoulos).

I would like to emphasize 3 main steps.

1) The case where $p = 4$ was settled in 1998 by C. Le Merdy [**21**].

2) The case where $p = 1$ was settled in 2006 by David Pérez-García [**22**].

3) The case where $p = \infty$ was settled in 2009 by J. Briët, H. Buhrman, T. Lee and T. Vidick [**23**].

Thus, by Varopoulos, there were solved, step by step, the cases 1) $2 \le p \le 4$, 2) $1 \le p \le 2$ and 3) $1 \le p \le \infty$.

In any case, one of the main tool in proving the corresponding result was the Grothendieck's inequality in one or another formulation, or some of its generalizations. For instance, in [**21**], among the other different and difficult facts, the author has used the little Grothendieck theorem (see the paper for details). We do not touch the technique from the last nice paper [**23**] (see the short note by Jop Briët, "A problem of Varopoulos - Short survey on Schatten-Schur algebras"). Let us mention only that Grothendieck theorem was also one of the crucial point in the proof.

When I was giving this small lecture, I was unaware of the result on the case where $p = \infty$. So, a question, I recalled during the lecture, was "Is S_p a Q-algebra for $4 \le p \le \infty$?". Now, as we said, the problem is closed.

For me (and, hope, not only for me), it is interesting to consider a main part of David Pérez-García's proof for the trace class. In fact, it is fairly to compare a multidimensional inequality proved in [**22**] and a A. M. Davie's criterion which was used by David Pérez-García. Here they are.

A multilinear Grothendieck's inequality (with a simple proof in that paper):

Theorem III.1 [**22**](Theorem 2.2.) *For every $m \in \mathbb{N}$, $n \ge 2$, $(a_{i_1 \dots i_n}) \subset \mathbb{C}$ and $x^1_{i_1}, \dots, x^n_{i_n} \in B_{l_2^m}$ we have*

$$\left| \sum_{i_j=1}^{m} a_{i_1 \dots i_n} \sum_{k=1}^{m} x^1_{i_1}(k) \dots x^n_{i_n}(k) \right| \le K_G^{n-1} \sup_{|t_{i_j}| \le 1} \left| \sum_{i_j=1}^{m} a_{i_1 \dots i_n} t_{i_1} \dots t_{i_n} \right|$$

(here $B_{l_2^m}$ is the closed unit ball of the m-dimensional Euclidean space l_2^m).

Remark: Compare with Corollary II.1.

A. M. Davie's criterion [**19**]:

Theorem III.2. *A commutative Banach algebra A is a Q-algebra if and only if there exists a positive constant K such that*

$$\left\| \sum_{i_j=1}^{m} a_{i_1 \dots i_n} x_{i_1} \dots x_{i_n} \right\| \le K^n \sup_{|t_{i_j}| \le 1} \left| \sum_{i_j=1}^{m} a_{i_1 \dots i_n} t_{i_1} \dots t_{i_n} \right|,$$

for every sequence $x_1, \dots, x_m \in A$ with $||x_i|| \le 1$ and for every choice of $a_{i_1 \dots i_n} \in \mathbb{C}$.

I can guess that David Pérez-García was successfully looking at both of the theorems and then it was nothing for him to solve the problem for the case where $p = 1$ (for the details, see the paper **[22]** itself).

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