# Grothendieck-Lidskiĭ theorem for subspaces of $L_p$-spaces

Oleg Reinov and Qaisar Latif

Abstract. In 1955, A. Grothendieck [1] has shown that if the linear operator $T$ in a Banach subspace of an $L_\infty$-space is 2/3-nuclear then the trace of $T$ is well defined and is equal to the sum of all eigenvalues $\{\mu_k(T)\}$ of $T$. V.B. Lidskiĭ [2], in 1959, proved his famous theorem on the coincidence of the trace of the $S_1$-operator in $L_2(\nu)$ with its spectral trace $\sum_{k=1}^\infty \mu_k(T)$. We show that *for* $p \in [1,\infty]$ *and* $s \in (0,1]$ *with* $1/s = 1 + |1/2 - 1/p|$, *and for every* $s$*-nuclear operator* $T$ *in every subspace of any* $L_p(\nu)$*-space the trace of* $T$ *is well defined and equals the sum of all eigenvalues of* $T$. Note that for $p = 2$ one has $s = 1$, and for $p = \infty$ one has $s = 2/3$.

## §1. Definitions and a theorem

All the terminology and facts (now classical), given here without any explanations, can be found in [7–10].

Let $X, Y$ be Banach spaces. For $s \in (0,1]$, denote by $X^*\widehat{\otimes}_s Y$ the completion of the tensor product $X^* \otimes Y$ (considered as a linear space of all finite rank operators) with respect to the quasi-norm

$$||z||_s := \inf\{\left(\sum_{k=1}^N ||x'_k||^s\, ||y_k||^s\right)^{1/s} :\ z = \sum_{k=1}^N x'_k \otimes y_k\}.$$

Let $\Phi_p$, for $p \in [1,\infty]$, be the ideal of all operators which can be factored through a subspace af an $L_p$-space. Put $N_s(X,Y) :=$ image of $X^*\widehat{\otimes} Y_s$ in the space $L(X,Y)$ of all bounded linear transformations under the canonical factor map $X^*\widehat{\otimes}_s Y \to N_s(X,Y) \subset L(X,Y)$. We consider the (Grothendieck) space $N_s(X,Y)$ of all $s$-nuclear operators from $X$ to $Y$ with the natural quasi-norm, induced from $X^*\widehat{\otimes}_s Y$.

Finally, let $\Phi_{p,s}$ (respectively, $\Phi_{s,p}$) be the quasi-normed product $N_s \circ \Phi_p$ (respectively, $\Phi_p \circ N_s$) of the corresponding ideals equipped with the natural quasi-norm $\nu_{p,s}$ (respectively, $\nu_{s,p}$): if $A \in N_s \circ \Phi_p(X,Y)$ then $A = \varphi \circ T$ with $T = \beta\alpha \in \Phi_p$, $\varphi = \delta\Delta\gamma \in N_s$ and

$$A : X \overset{\alpha}{\to} X_p \overset{\beta}{\to} Z \overset{\gamma}{\to} c_0 \overset{\Delta}{\to} l_1 \overset{\delta}{\to} Y,$$

The research was supported by the Higher Education Commission of Pakistan.
AMS Subject Classification 2010: 47B06.
Key words: $s$-nuclear operators, eigenvalue distributions.

where all maps are continuous and linear, $X_p$ is a subspace of an $L_p$-space, constructed on a measure space, and $\Delta$ is a diagonal operator with the diagonal from $l_s$. Thus, $A = \delta\Delta\gamma\beta\alpha$ and $A \in N_s$. Therefore, if $X = Y$, the spectrum of $A$, $sp\,(A)$, is at most countable with only possible limit point zero. Moreover, $A$ is a Riesz operator with eigenvalues of finite algebraic multiplicities and $sp\,(A) \equiv sp\,(B)$, where $B := \alpha\delta\Delta\gamma\beta : X_p \to X_p$ is an $s$-nuclear operator, acting in a subspace of an $L_p$-space.

Let $T$ be an operator between Banach spaces $Y$ and $W$. The operator $\mathbf{1} \otimes T : X^*\widehat{\otimes}_s Y \to X^*\widehat{\otimes}_s W$ is well defined and can be considered also as an operator from $X^*\widehat{\otimes}_s Y$ into $X^*\widehat{\otimes} W$ (the Grothendieck projective tensor product), the last space having the space $L(W, X^{**})$ as dual.

**Definition.** We say that $T$ *possesses the property* $AP_s$ (written down as "$T \in AP_s$") if for every $X$ and any tensor element $z \in X^*\widehat{\otimes}_s Y$ the operator $T \circ z : X \to W$ is zero iff the corresponding tensor $(\mathbf{1} \otimes T)(z)$ is zero as an element of the space $X^*\widehat{\otimes} W$. If $Y = W$ and $T$ is the identity map, we write just $Y \in AP_s$ (the approximation property of order $s$).

This is equivalent to the fact that if $z \in X^*\widehat{\otimes}_s Y$ then it follows from

$$\operatorname{trace}\,(\mathbf{1} \otimes T)(z) \circ R = 0, \quad \forall\, R \in W^* \otimes X$$

that trace $U \circ (\mathbf{1} \otimes T)(z) = 0$ for every $U \in L(W, X^{**})$. There is a simple characterization of the condition $T \in AP_s$ in terms of the approximation of $T$ on some sequences of the space $Y$, but we omit it now, till the next time. We need here only one example which is crucial for our note (other examples, as well as more general applications will appear elsewhere).

**Example.** Let $s \in (0, 1]$, $p \in [1, \infty]$ and $1/s = 1 + |1/p - 1/2|$. Any subspace as well as any factor space of any $L_p$-space have the property $AP_s$ (this means that, for that space $Y$, $\mathrm{id}_Y \in AP_s$). Thus, in the case of such a space $Y$, we have the quasi-Banach equality $X^*\widehat{\otimes}_s Y = N_s(X, Y)$, whichever the space $X$ was.

**Lemma.** Let $s \in (0, 1]$, $p \in [1, \infty]$ and $1/s = 1 + |1/2 - 1/p|$. Then the system of all eigenvalues (with their algebraic multiplicities) of any operator $T \in N_s(Y, Y)$, acting in any subspace $Y$ of any $L_p$-space, belongs to the space $l_1$. The same is true for the factor spaces of $L_p$-spaces.

**Corrolary.** If $s \in (0, 1]$, $p \in [1, \infty]$ with $1/s = 1 + |1/2 - 1/p|$ then the quasi-normed ideals $\Phi_{p,s}$ and $\Phi_{s,p}$ are of (spectral) type $l_1$.

**Theorem.** Let $Y$ be a subspace of an $L_p$-space, $1 \le p \le \infty$. If $T \in N_s(Y, Y)$, $1/s = 1 + |1/2 - 1/p|$, then

1. the (nuclear) trace of $T$ is well defined,
2. $\sum_{n=1}^{\infty} |\lambda_n(T)| < \infty$, where $\{\lambda_n(T)\}$ is the system of all eigenvalues of the operator $T$ (written in according to their algebraic multiplicities)

and

$$\operatorname{trace} T = \sum_{n=1}^{\infty} \lambda_n(T).$$

## §2. Proofs

*Proof* of Lemma. Let $Y$ be a subspace or a factor space of an $L_p$-space and $T \in N_s(Y, Y)$ with an s-nuclear representation

$$T = \sum_{k=1}^{\infty} \mu_k y_k' \otimes y_k,$$

where $||y_k'||, ||y_k|| = 1$ and $\mu_k \geq 0$, $\sum_{k=1}^{\infty} \mu_k^s < \infty$. The operator $T$ can be factored in the following way:

$$T : Y \xrightarrow{A} c_0 \xrightarrow{\Delta_{1-s}} l_r \overset{j}{\hookrightarrow} c_0 \xrightarrow{\Delta_s} l_1 \xrightarrow{B} Y,$$

where $A$ and $B$ are linear bounded, $j$ is the natural injection, $\Delta_s \sim (\mu_k^s)_k$ and $\Delta_{1-s} \sim (\mu_k^{1-s})$ are the natural diagonal operators from $c_0$ into $l_1$ and from $c_0$ into $l_r$, respectively. Here, $r$ is defined via the conditions $1/s = 1 + |1/p - 1/2|$ and $\sum_k \mu_k^s < \infty$ : we have to have $\sum_k \mu_k^{(1-s)r} < \infty$, for which $(1-s)r = s$ is good. Therefore, put $1/r = 1/s - 1$, or $1/r = |1/p - 1/2|$.

From now and on in the proof we assume (surely, without loss of generality) that $p \geq 2$. Then $1/r = 1/2 - 1/p$ and $r(1-s) = s$. Note that if $s = 1$ then $r = \infty, p = 2$ and $j\Delta_{1-s} \equiv j$; and if $s = 2/3$ then $r = 2, p = \infty$ and $\Delta_{1-s} \sim (\mu_k^{1/3})_k \in l_2$.

Now, let us factorize the diagonal $\Delta_s$ as $\Delta_s = \Delta_1 \Delta_2 : c_0 \xrightarrow{\Delta_2} l_2 \xrightarrow{\Delta_1} l_1$ in such a (clear) way that diagonals $\Delta_2$ is in $\Pi_2$ and $\Delta_1^*$ is in $\Pi_2$ too, respectively.

*Case* (i). $Y$ is a subspace of an $L_p$-space. Denoting by $l : Y \hookrightarrow L_p$ an isomorphic embedding of $Y$ into a corresponding $L_p = L_p(\nu)$, we obtain that the map $\Delta_2^* B^* l^*$ : $L_{p'} \xrightarrow{l^*} Y^* \xrightarrow{B^*} l_\infty \xrightarrow{\Delta_2^*} l_2$ is of type $\Pi_2$, so is in $\Pi_p$. Thus its preadjoint $lB\Delta_2$ : $l_2 \xrightarrow{\Delta_2} l_1 \xrightarrow{B} Y \xrightarrow{l} L_p$ is order bounded and, therefore, $p$-absolutely summing.

*Case* (ii). $Y$ is a factor space of an $L_p$-space. Denoting by $q : L_p \to Y$ a factor map from a corresponding $L_p = L_p(\nu)$ onto $Y$ and taking a lifting $Q : l_1 \to L_p$ for $B$ with $B = qQ$, we obtain that the map $\Delta_2^* Q^* : L_{p'} \xrightarrow{Q*} l_\infty \xrightarrow{\Delta_2^*} l_2$ is of type $\Pi_2$, so is in $\Pi_p$. Thus its pre-adjoint $Q\Delta_2 : l_2 \xrightarrow{\Delta_2} l_1 \xrightarrow{Q} L_p$ is order bounded and, therefore, $p$-absolutely summing. Hence, $B\Delta_2 : l_2 \xrightarrow{\Delta_2} l_1 \xrightarrow{Q} L_p \xrightarrow{q} Y$ is also $p$-absolutely summing.

It folows from all that's said that in all the cases our operator $T : Y \to Y$ can be written as a composition:

$$T = U_1 U_2 U_3 \;\; \text{with} \;\; U_3 \in \Pi_r, U_2 \in \Pi_2, U_1 \in \Pi_p,$$

all the exponents being not less than 2. Now, $1/r + 1/2 + 1/p = (1/2 - 1/p) + 1/2 + 1/p = 1$. □

*Proof* of the statement of Example. It follows from:

($\alpha$) every finite dimensional subspace $E$ of any factor space of any $L_p$-space is $c_p\,(\dim E)^{|1/2-1/p|}$-complemented.

For more general statements on $AP_s$ and their proofs, we refer to [4] and [5]; see also an old paper of O.I. Reinov [3] for the idea to apply the projections in the questions which are under consideration in this note. □

*Proof* of Corrolary. Apply Lemma. □

*Proof* of Theorem. Apply Lemma, Example, Corrolary and the main result of M.C. White [6].

*Remark*: Since finite rank operators are dense in $N_s$, Theorem can be proved without referring to the paper of M.C. White; but this would take a little bit longer explanations.

## References


[1] A. Grothendieck: *Produits tensoriels topologiques et éspaces nucléaires*, Mem. Amer. Math. Soc., **16**(1955).

[2] V. B. Lidskiĭ: *Nonselfadjoint operators having a trace*, Dokl. Akad. Nauk SSSR, **125**(1959), 485–487.

[3] O.I. Reinov: *A simple proof of two theorems of A. Grothendieck*, Vestn. Leningr. Univ. 7 (1983), 115-116.

[4] O.I. Reinov: *Disappearance of tensor elements in the scale of p-nuclear operators*, Theory of operators and theory of functions (LGU) 1(1983), 145-165.

[5] O.I. Reinov: *Approximation properties $AP_s$ and p-nuclear operators (the case* $0 < s \leq 1$), Journal of Mathematical Sciences **115**, No. 3 (2003), 2243-2250.

[6] M.C. White: *Analytic multivalued functions and spectral trace*, Math. Ann. 304 (1996), 665-683.

[7] A. Pietsch: *Operator Ideals*, North Holland (1980).

[8] P. Wojtaszczyk: *Banach Spaces for Analysts*, Cambridge Univ. Press (1991).

[9] Hermann König: *Eigenvalues of Operators and Applications*, Handbook of the geometry of Banach spaces, vol. 1, Chapter 22 (2001), 941-974.

[10] A. Pietsch: *History of Banach Spaces and Linear Operators*, Birkhäuser (2007).



Department of Mathematics and Mechanics, St. Petersburg State University, Saint Petersburg, RUSSIA.

Abdus Salam School of Mathematical Sciences, 68-B, New Muslim Town, Lahore 54600, PAKISTAN.

*E-mail address*: `orein51@mail.ru`

Abdus Salam School of Mathematical Sciences, 68-B, New Muslim Town, Lahore 54600, PAKISTAN.