Approximation of p-summing operators by adjoints

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Abstract. We consider the following question for the ideals Π_p of absolutely p-summing operators: Is it true that, for given Banach spaces X and Y, the unit ball of the space $\Pi_p(X,Y)$ is dense, for some natural topology, in the unit ball of the space $\Pi_p(X,Y^{**})$ or in the unit ball of the corresponding space $\Pi_p^{dual}(Y^*,X^*) := \{U : Y^* \to X^* \mid U^*|_X \in \Pi_p(X,Y^{**})\}$? As "natural topologies", we consider strong and weak operator topologies, compact–open topology, topology of $X \times Y^*$-convergence etc.

We discuss some questions of the following type. Let J be a normed operator ideal. Is it true that, for given Banach spaces X and Y, the unit ball of the space $J(X,Y)$ is dense, for some natural topology, in the unit ball of the space $J(X,Y^{**})$ or in the unit ball of the corresponding space $J^t(Y^*,X^*) := \{U : Y^* \to X^* \mid U^*|_X \in J(X,Y^{**})\}$? As "natural topologies", we consider strong and weak operator topologies, compact–open topology, topology of $X \times Y^*$-convergence etc.

In this paper, we consider the case where the ideals under investigations are the injective ideals Π_p of absolutely p-summing operators (here $1 \le p \le \infty$).

§1. Preliminaries

All the spaces $X, Y, Z, W, \ldots$ are Banach. For a bounded subset B of X, we denote by X_B the Banach space generated by B, with the unit ball $\overline{\Gamma}(B)$ (= the closed absolutely convex hull of B); $\Phi_B : X_B \to X$ is the natural embedding (see, e.g., [1]). $L(X,Y)$ is the space of all (bounded) linear operators from X to Y with its natural operator norm. For $Y = \mathbb{K}$ (scalar field), we write X^* instead of $L(X,\mathbb{K})$; we always consider the space X as the subspace $\pi_X(X)$ of its second dual X^{**} (denoting, if needed, by π_X the canonical injection). Some other notations (below $p, q \in [1,\infty]$).

If X is a Banach space and μ is a measure then by $L_p(\mu;X)$ we understand the L_p-space of all (equivalent classes of) strongly μ-measurable p-summable functions. In the case where the measures are discrete, we use also the notations of type $l_p(X)$, $l_p(\Gamma;X)$, $c_0(X)$, etc. For the quasinorm of a sequence $(x_k)_k$ from the space $l_p(X)$, we use the notation $\alpha_p(x_k) := \left(\sum_k \|x_k\|^p\right)^{1/p}$ (if $p = \infty$, the corresponding

The research was supported by the Higher Education Commission of Pakistan.

AMS Subject Classification 2010: 47B10, 47A58.

Key words: absolutely p-summing operators, Banach tensor product, pointwise convergence.

changes are needed). Recall that $l_p(\Gamma;X)^* = l_{p'}(\Gamma;X^*)$ and $c_0(\Gamma;X)^* = l_1(\Gamma;X)$ for all finite $p \geqslant 1$. A family $(x_k)_{k=1}^{\infty} \subset X$, for which the value $\alpha_p(x_k)$ is finite, is said to be *absolutely p-summable.* A family $(x_k)_{k=1}^{\infty}$ is called *weakly p-summable,* if $(\langle x_k, x'\rangle) \in l_p$ for all $x' \in X^*$. We set

$$\varepsilon_p(x_j) := \sup_{\|x'\|\leqslant 1} \left(\sum_j |\langle x_j, x'\rangle|^p \right)^{1/p}.$$

This is a norm in the space $l_p\{X\}$ of all weakly p-summable sequences in X.

We use the following notations for the classical operator ideals (all the information on the theory of operator ideals can be found in [3]; in our work, we follow, however, the other notation and terminology; see [4, 5]):

$[\Pi_p, \pi_p]$ — the ideal of the absolutely p-summing operators;

$[\mathrm{QN}_p, \pi_p]$ — the ideal of quasi-p-nuclear operators;

$[\mathrm{N}_p, \nu_p]$ — the ideal of p-nuclear operators;

$[\mathrm{I}_p, i_p]$ — the ideal of (strictly) p-integral operators;

Recall some important definitions (to be selfcontained; see [3, 4, 7]).

Let $T \in \mathrm{L}(X,Y)$. The operator $T: X \to Y$ is called *absolutely p-summing*, if there is a constant $C > 0$ such that, for any finite family $\{x_n\}_{n=1}^M \subset X$, the following inequality holds $\alpha_p(Tx_n) \leqslant C\varepsilon_p(x_n)$; corresponding norm $(\inf C)$ is denoted by $\pi_p(T)$. Note that $[\Pi_\infty, \pi_\infty]$ is exactly the operator ideal $[\mathrm{L}, \|\cdot\|]$. The operator $T: X \to Y$ is called *quasi-p-nuclear*, $T \in \mathrm{QN}_p(X,Y)$, if for some isometric embedding $i: Y \to L_\infty(\mu)$ the composition iT is in the closure of the space of all finite rank operators in $\mathrm{N}_p(X, L_\infty(\nu))$. The norm in $\mathrm{QN}_p(X,Y)$ is induced from the space $\Pi_p(X, L_\infty(\nu))$.

We say that an operator $T \in \mathrm{L}(X,Y)$ *belongs to the ideal* N_p (p-nuclear), if it can be represented in the form

$$T := \sum_{n=1}^{\infty} x'_n \otimes y_n,$$

where the sequences (x'_n) and (y_n) are such that $\varepsilon_{p'}(y_n) < \infty$, $\alpha_p(x'_n) < \infty$. With the norm, given by $\nu^p(T) := \inf \varepsilon_{p'}(y_n)\alpha_p(x'_n)$, the class N_p is a normed operator ideal. Let us give other characteristics of the operators T from $\mathrm{N}_p(X,Y)$. An operator T is an N_p-operator iff it factors in the following way:

$$X \xrightarrow{A} l_\infty \xrightarrow{\Delta} l_p \xrightarrow{B} Y,$$

where Δ is a diagonal operator with a diagonal from l_p, A and B are the operators of norm 1; moreover, $\nu_p(T)$ is just the infimum (over all possible factorizations) of the norms of the diagonals in l_p.

Let us say that an operator $T \in \mathrm{L}(X,Y)$ *belongs to the ideal* I_p (strictly p-integral), if it admits a factorization of the kind

$$X \xrightarrow{A} L_\infty(\mu) \xhookrightarrow{j} L_p(\mu) \xrightarrow{B} Y,$$

where μ is a probability measure, j is the identity injection, A and B are continuous operators. We put $i_p(T) = \inf \|A\|\|B\|$, where the inf is taken over all the factorizations of T of the mentioned kind.

Now, finally, some important words on the notions of the Banach tensor products. We consider, mainly, the tensor norms on the tensor products of the kind $X^* \otimes Y$. In this case, the tensor product $X^* \otimes Y$ can be identified naturally with the linear space of all finite dimensional operators from X to Y. In the general case, the tensor product $X \otimes Y$ can be considered as the linear space of all *weak*-to-weak* continuous finite rank linear mappings from X^* to Y (or from Y^* to X).

On the class $\mathfrak{T}$ of all such tensor products there is a maximal (the strongest) and a minimal (the weakest) tensor norms [3]. The strongest tensor norm ν_1^0 on $X^* \otimes Y$ generates (after completion with respect to this norm) the *projective tensor product of Grothendieck* $X^*\widehat{\otimes}Y$ [3], and the weakest one — $\|\cdot\|$ — *injective tensor product* $X^*\widehat{\widehat{\otimes}}Y$, which can be considered as the completion of the linear space of all finite dimensional operators from X to Y, equipped with the usual operator norm $\|\cdot\|$ (for the case of $X \otimes Y$, we can say the analogous words). Thus, $X^*\widehat{\widehat{\otimes}}Y$ can be identified with the closed linear subspace of $\mathrm{L}(X,Y)$. Therefore, for any tensor norm α between ν_1^0 and $\|\cdot\|$, the natural mapping $X^* \otimes_\alpha Y \to \mathrm{L}(X,Y)$ can be extended to the *canonical map* from $X^*\widehat{\otimes}_\alpha Y$ to $\mathrm{L}(X,Y)$. Analogously, in the general case of products of the kind $X \otimes Y$, the natural mapping $X \otimes_\alpha Y \to \mathrm{L}(X^*,Y)$ can be extended to the *canonical map* from $X\widehat{\otimes}_\alpha Y$ to $\mathrm{L}(X^*,Y)$, and the image of this map belongs to the subspace (of $\mathrm{L}(X^*,Y) = \mathrm{L}(Y^*,X)$) of all *weak*-to-weak* continuous operators from X^* to Y (or from Y^* to X).

Let us give the main examples of the tensor products we will working with.

The finite p-nuclear tensor norm $\|\cdot\|_p$ for $p \in [1,+\infty]$ is defined on the product $X \otimes Y$ by the following way: if $z \in X \otimes Y$, then

$$\|z\|_p := \inf \left(\sum_{k=1}^N \|x_k\|^p \right)^{1/p} \sup_{\|y'\|\leqslant 1} \left\{ \left(\sum_{k=1}^N |\langle y_k, y'\rangle|^{p'} \right)^{1/p'} \right\},$$

where $1/p+1/p' = 1$ and the infimum is taken over all representations of the tensor element z in the space $X \otimes Y$ in the form $z = \sum_{k=1}^N x_k \otimes y_k$ (formally, (**) has sense only for finite exponents $p > 1$, and for the case $p = 1$ and $p = +\infty$, the definition have to be modified). The completion of the tensor product $X \otimes Y$ with respect to the norm $\|\cdot\|_p$, $1 \le p \leqslant \infty$, is denoted by $X\widehat{\otimes}_p Y$.

If $p \in [1,\infty]$, then the conjugate space to the tensor product $X\widehat{\otimes}_p Y$ is equal to $\Pi_{p'}(Y, X^{**})$ (with the natural duality defined by trace).

For a tensor $z \in X^*\widehat{\otimes}X$, the trace of z is well defined: if $z = \sum_{k=1}^\infty x'_k \otimes x_k$ is a representation of z in the space $X^*\widehat{\otimes}X$ (see [1]) then

$$\operatorname{trace} z = \sum_{k=1}^\infty \langle x'_k, x_k\rangle.$$

For $T \in \Pi_p(X, Y^{**})$ and $z \in Y^*\widehat{\otimes}_{p'}X$, the trace of $T \circ z$ is well defined (since $T \circ z$ belongs to the projective tensor product $Y^*\widehat{\otimes}Y^{**}$). This trace gives us a possibility to consider the space $\Pi_p(X, Y^{**})$ as the Banach dual to $Y^*\widehat{\otimes}_{p'}X$. Moreover, the dual to any p-projective tensor product $Z\widehat{\otimes}_{p'}W$ is $\Pi_p(W, Z^*)$ (again, with duality defined by trace). Analogously, let $Y^*\widehat{\widehat{\otimes}}_pX$ be the closure of finite rank operators in $\Pi_p(Y, X)$; then the dual to the space $Y^*\widehat{\widehat{\otimes}}_pX$ is $\mathrm{I}_{p'}(X, Y^{**})$ (and the Banach dual to any product $Z\widehat{\widehat{\otimes}}_pW$ of such a kind is $\mathrm{I}_{p'}(W, Z^*)$). Note that in the case $p = 1$ we can write $X\widehat{\otimes}_1Y = Y\widehat{\otimes}_1X$ (and only in this case, generally). For $p = \infty$, $(\Pi_\infty, \pi_\infty) = (L, || \cdot ||)$.

§2. Results

We need the following notation. For an operator $T \in \mathrm{L}(X, Y)$, we write $T \in \Pi_p^d(X, Y)$ iff $T^* \in \Pi_p(Y^*, X^*)$, and we define a norm on the linear space $\Pi_p^d(X, Y)$ by setting $\pi_p^d(T) := \pi_p(T)$ (so, Π_p^d is the ideal which is dual to the ideal Π_p in the sense of [3]).

Lemma 1. Let $T \in \mathrm{L}(X, Y)$. We have: $T \in \Pi_p(X, Y)$ iff $T^* \in \Pi_p^d(Y^*, X^*)$.

Proof. If $T^* \in \Pi_p^d(Y^*, X^*)$ then $T^{**} \in \Pi_p(X^{**}, Y^{**})$; so, by injectivity of Π_p, one has $T \in \Pi_p(X, Y)$. If $T \in \Pi_p(X, Y)$ then $T^{**} \in \Pi_p(X^{**}, Y^{**})$. It is clear, but let us explain this: it is enough, e.g., to consider an isometric imbedding of Y into an L_∞-space and to use the fact that the second adjoint to a p-integral (with values in the space with the metric approximation property) is p-integral itself.

Lemma 2. Let $C > 0$, $\{A_\beta\}_{\beta\in\mathcal{B}}$ be a net in $\Pi_p(Z, W)$, $A \in \Pi_p(Z, W)$. The following are equivalent:

1) for each β $\pi_p(A_\beta) \le C$ and for every $x \in Z$ we have $A_\beta x \underset{\beta}{\to} Ax$ in W;

2) for each β $\pi_p(A_\beta) \le C$ and for every $\varepsilon > 0$ and any compact subset $K \subset Z$ there is a β_ε so that for each $\beta > \beta_\varepsilon$ and for every $k \in K$ $||A_\beta k - Ak|| \le \varepsilon$.

Proof. We have to prove only that the second part of the assertion 1) is equivalent to the second part of the assertion 2), if we suppose only that the usual norms of all operators from the net $\{A_\beta\}_{\beta\in\mathcal{B}}$ are bounded by C. But in this case, the lemma is proved, e.g., in [6].

Let us say that a net $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ of operators from X to $Y\,(\subset Y^{**})$ is $X \times Y^*$-pointwise convergent to an operator $B \in \mathrm{L}(X, Y^{**})$ (or, to an operator $D \in \mathrm{L}(Y^*, X^*)$), if for all $x \in X$ and $y' \in Y^*$ one has $\langle B_\alpha x, y'\rangle \underset{\alpha}{\to} \langle Bx, y'\rangle$ (or, $\langle B_\alpha x, y'\rangle \underset{\alpha}{\to} \langle x, Dy'\rangle$).

Recall also the definition of the topology τ_p of π_p-compact convergence. For Banach spaces X, Y, the *topology* τ_p *of* π_p*-compact convergence* in the space $\Pi_p(Y, X)$ is the topology, a local base (in zero) of which is defined by sets of type

$$\omega_{K,\varepsilon} = \{U \in \Pi_p(Y, X) : \ \pi_p(U\Phi_K) < \varepsilon\}\,,$$

where $\varepsilon > 0$, $K = \overline{\Gamma}(K)$ is a compact subset of Y.

Proposition 3. Let $C > 0$, $B \in \Pi_p(X, Y^{**})$ and $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ be a net in $\Pi_p(X, Y^{**})$. The following are equivalent:

(i) for every α $\pi_p(B_\alpha) \le C$ and the net $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ is $X \times Y^*$-pointwise convergent to the operator B;

(ii) for every α $\pi_p(B_\alpha) \le C$ and the net $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ is $\sigma(\Pi_p(X, Y^{**}), Y^*\widehat{\otimes}_{p'}X)$ convergent to the operator B.

In both cases, we have $\pi_p(B) \le C$.

If, in addition, $\{B_\alpha\}_{\alpha\in\mathcal{A}} \subset \Pi_p(X, Y)$ and $B \in \Pi_p(X, Y)$, then every of these two assertions implies the following ones (and (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v)):

(iii) the operator B is in the closure of the ball of radius C of the space $\Pi_p(X, Y)$ in the strong operator topology, i.e. there exists a net $\{A_\beta\}_{\beta\in\mathcal{B}}$ such that for every $x \in X$ we have $A_\beta x \underset{\beta}{\to} Bx$ in Y;

(iv) the operator B is in the closure of the ball of radius C of the space $\Pi_p(X, Y)$ in the topology of compact convergence;

(v) the operator B is in the closure of the ball of radius C of the space $\Pi_p(X, Y)$ in the topology of τ_p-convergent to the operator B.

In all the cases, we have $\pi_p(B) \le C$.

Proof. The natural mapping $j_p : Y^*\widehat{\otimes}X \to Y^*\widehat{\otimes}_{p'}X$ has a dense image, so its adjoint gives us a homeomorphism from the unit ball of $\Pi_p(X, Y^{**})$ with its weak*-topology onto a weak*-compact subset of $\mathrm{L}(X, Y^{**})$. To prove $(i) \iff (ii)$, it remains to note that $X \times Y^*$-pointwise convergence of our net to B is just its convergence to the operator B in the topology $\sigma(\mathrm{L}(X, Y^{**}), Y^*\widehat{\otimes}X)$.

In the cases (iii)–(v), the topology $\sigma(\mathrm{L}(X, Y^{**}), Y^*\widehat{\otimes}X)$, considered on the linear subspace $\Pi_p(X, Y)$, gives the same closure of the ball of radius C of the space $\Pi_p(X, Y)$ as the topologies of compact convergence (follows from [1]) and τ_p [5]. Therefore, (i)–(ii) imply (iv) and (v), and (iv) $\Longleftrightarrow$ (v). The fact, that these last two assertions are equivalent to the assertion (iii), follows from Lemma 2.

By Lemma 1, we can identify the Banach space $\Pi_p(X, Y^{**})$ with the Banach space $\Pi_p^d(Y^*, X^*)$. Also, these two Banach spaces give us realizations of the dual space to the tensor product $Y^*\widehat{\otimes}_{p'}X$. Considering in the first part of the proof of Proposition 3 the space $\Pi_p^d(Y^*, X^*)$ instead of $\Pi_p(X, Y^{**})$, we get

Proposition 4. Let $C > 0$, $B \in \Pi_p^d(Y^*, X^*)$ and $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ be a net in $\Pi_p^d(Y^*, X^*)$. The following are equivalent:

(i) for every α $\pi_p^d(B_\alpha) \le C$ and the net $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ is $Y^* \times X$-pointwise convergent to the operator B;

(ii) for every α $\pi_p^d(B_\alpha) \le C$ and the net $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ is $\sigma(\Pi_p^d(Y^*, X^*), Y^*\widehat{\otimes}_{p'}X)$ convergent to the operator B.

In both cases, we have $\pi_p^d(B) \le C$.

Corollary 5. 1) With notations of Proposition 3, if a π_p-bounded net $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ converges in the space $\Pi_p(X, Y^{**})$ to B in the topology of compact convergence (or strongly), then this net is $\sigma(\Pi_p(X, Y^{**}), Y^*\widehat{\otimes}_{p'}X)$ convergent to the operator B.

2) With notations of Proposition 4, if a π_p^d-bounded net $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ converges in the space $\Pi_p^d(Y^*,X^*)$ to B in the topology of compact convergence (or strongly), then this net is $\sigma(\Pi_p^d(Y^*,X^*), Y^*\widehat{\otimes}_{p'}X)$ convergent to the operator B.

Corollary 6. With notations of Proposition 3, if a π_p-bounded net $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ converges $X\times Y^*$-pointwise to B, then $B_\alpha^*|_{Y^*} \underset{\alpha}{\to} B^*|_{Y^*}$ $Y^*\times X$-pointwise and vice versa.

We are going now to reformulate the definitions of the approximation properties of order q, $q\in[1,\infty]$, which were considered in [7] and, e.g., in [4], in terms of some convergences (by analogue with the approximation properties of Grothendieck; see [1]). Recall the "tensor language" definitions. We say that the Banach space X has the AP_q, if for every Banach space Y the natural mapping $Y^*\widehat{\otimes}_qX\to N(Y,X)$ is one-to-one (and thus, an isometric isomorphism between these spaces). In [4], there were introduced the notions of bounded approximation properties of order q. To recall them, we need some new notations.

Consider, in the space $I_q(Y,X^{**})$, the subspace $Y^*\widetilde{\otimes}_qX$, the closure of the space of all finite rank operators from Y to X in that space $I_q(Y,X^{**})$. Let $C\in[1,\infty)$. We say that the space X has the property C-MAP_q, if for each Banach space Y the natural map $i_q: Y^*\widehat{\otimes}_qX\to Y^*\widetilde{\otimes}_qX$ is C-isometric, i.e., i_q is one-to-one and $||i_q^{-1}||\le C$. This is the same as to tell that for each Banach Y the adjoint map i_q^* takes the unit ball of the space $G_{q'}(X,Y^{**})$, dual to $Y^*\widetilde{\otimes}_qX$ (it is, evidently, some Banach space of operators), into the weak*-compact set, containing the ball of radius $1/C$ of the space $\Pi_{q'}(X,Y^{**})$ (which is a representation of the dual space to the space $Y^*\widehat{\otimes}_qX$). If the space X has C-MAP_q for some constant C, we say that it has the property BAP_q. Clearly, BAP_q implies AP_q (the inverse is not true [4]).

Now, consider the subspace $X^*\otimes_{q'}Y$ of the space $X^*\widehat{\widehat{\otimes}}_{q'}Y$, which is, in turn, the closure of the space of all finite rank operators $X^*\otimes Y$ in $\Pi_{q'}(X,Y^{**})$. The dual to the space $X^*\otimes_{q'}Y$ is just $I_q(Y,X^{**})$, so, by the bipolar theorem, the unit ball of the space $X^*\otimes_{q'}Y$ is weak*-dense in the unit ball of the space $G_{q'}(X,Y^{**})$. Thus, the space X has the C-metric approximation property of order q, C-MAP_q, if and only if *for any Banach space* Y the ball of radius C of the space $X^*\otimes_{q'}Y$ is weak*-dense in the unit ball of the space $\Pi_{q'}(X,Y^{**})$ (or, if one wishes, of the space $\Pi_{q'}^d(Y^*,X^*)$). Also, we can see that the assertion *"for each reflexive Banach space Y, the natural map $i_q: Y^*\widehat{\otimes}_qX\to Y^*\widetilde{\otimes}_qX$ is C-isometric"* is equivalent to the assertion *"for any reflexive Banach space Y, the ball of radius $\le C$ of the space $X^*\otimes_{q'}Y$ is weak*-dense in the unit ball of the space $\Pi_{q'}(X,Y)$"*. By Proposition 3, the last is equivalent to the assertion *"for any reflexive Banach space Y, the ball of radius C of the space $X^*\otimes_{q'}Y$ is dense in the unit ball of the space $\Pi_{q'}(X,Y)$ in the topology of compact convergence (or, in $\tau_{q'}$)"*. Consider the case where $q\in(1,\infty)$. Since the subspaces of $L_{q'}$ in Grothendieck-Pietsch factorizations for absolutely q'-summing operators are reflexive (see also [3], 17.3.11), we get easily (taking in account the second part of Proposition 3):

Proposition 7. Let $q\in(1,\infty)$. A Banach space X has the C-metric approximation property of order q, C-MAP_q, if and only if, for any Banach space Y, the ball of radius $\le C$ of the space $X^* \otimes_{q'} Y$ is dense in the unit ball of the space $\Pi_{q'}(X,Y)$" in the topology of compact convergence (or, in the topology $\tau_{q'}$). Also, this is equivalent to the fact that for each reflexive Banach space Y, the natural map $i_q : Y^*\widehat{\otimes}_q X \to Y^*\widetilde{\otimes}_q X$ is C-isometric.

Of course, the proposition is valid in the case $q = 1$ (A. Grothendieck [1]). What about the case $q = \infty$, we have no place to discuss this at the moment.

We are able to get now, as a consequence of our considerations, the first result, concerning the question in the very beginning of our paper.

Theorem 8. Let $p \in [1,\infty]$. If the Banach space X has the metric approximation property of order p (i.e., the 1-MAP_p), then, for every Banach space Y, the unit ball of $\Pi_p(X,Y)$ is dense in the unit ball of the space $\Pi_p(X,Y^{**})$ in the topology of compact convergence.

By Corollaries 5 and 6, we get also

Theorem 9. Let $p \in [1,\infty]$. If the Banach space X has the metric approximation property of order p then, for every Banach space Y, the unit ball of $\Pi_p(X,Y)$ is dense in the unit ball of the space $\Pi_p(Y^*,X^*)$ in the topology of $Y^* \times X$-pointwise convergence.

The proof of the following result use the main theorem from J. Lindenstrauss' paper [2].

Theorem 10. Let $p \in [1,\infty]$. Let Z be such a space that for every Banach space W, each operator $U \in \Pi_p^d(W^*,Z^*)$ with $\pi_p^d(U) = 1$ belongs to the closure in the topology of $W^* \times Z$-convergence of the unit ball of the space $\Pi_p(Z,W)$. Then, for every Banach space E, each operator $S \in \Pi_p(Z,E)$ with $\pi_p(S) = 1$ belongs to the closure in the topology of compact convergence (or, in the τ_p-topology) of finite rank operators with π_p-norms ≤ 1.

Proof. Fix E and S. We can assume that the space E is separable. By [2], there is a separable Banach space Y such that Y^* has the metric approximation property of Grothendieck and $Y^{**} = E^* \oplus Y$, with the natural projector P from Y^{**} onto E having the norm one, and with the natural inclusion $j : E^* \hookrightarrow Y^{**}$ being weak*-to-weak* continuous. Consider the operator $S^*P : Y^{**} \to E^* \to Z^*$. It can be approximated, $Y^{**} \times Z$-pointwisely, by T's: $Z \to Y^*$ with $\pi_p(T) \le 1$. Apply Proposition 4 with $C = 1$ to the operator $S^*P \in \Pi_p^d(Y^{**},Z^*)$ to get that there is a net $\{T_\alpha\} \subset \Pi_p(Z,Y^*)$ such that $\pi_p(T_\alpha) \le 1$ for all α and $T_\alpha \underset{\alpha}{\to} S^*P$ is in the weak*-topology $\sigma(\Pi_p^d(Y^{**},Z^*), Y^{**}\widehat{\otimes}_{p'}Z)$. The family $\{T_\alpha\}$ belongs to the closure in $\sigma(\Pi_p^d(Y^{**},Z^*), Y^{**}\widehat{\otimes}_{p'}Z)$ of finite rank operators from the unit ball of the space $\Pi_p(Z,Y^*)$ (since the space Y^* has the MAP). The operator S^*P belongs to the closure in the same topology of the set $\{T_\alpha\}$. Thus, S^*P belongs to the closure in $\sigma(\Pi_p^d(Y^{**},Z^*), Y^{**}\widehat{\otimes}_{p'}Z)$ of finite rank operators from the unit ball of the space $\Pi_p(Z,Y^*)$. Since $S^* = S^*Pj : E^* \to Y^{**} \to E^* \to Z^*$, we have that S lies in the closure in the weak*-topology $\sigma(\Pi_p(Z,E^{**}), E^*\widehat{\otimes}_{p'}Z)$ of finite rank operators from the unit ball of the space $\Pi_p(Z,E)$, hence in the closure of finite rank operators from the unit ball of the space $\Pi_p(Z, E)$ in $\tau_p$topology (or, if one wish, in the topology of compact convergence).

Taking in account the last three theorems and Proposition 7 before them, we obtain, as by-product (at least, for $C = 1$), that the proposition 7 is true also in the cases where $q = 1$ or ∞. It is not hard to see that the theorems can be formulated and proved for general cases "for any constant $C \geq 1$", — e.g., for C-MAP_p, for "closures of C-balls" etc. But this is the theme of the next papers (as well as the considerations of some "(counter)examples").

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