Some more remarks on Grothendieck-Lidskiĭ trace formulas

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ABSTRACT. **Theorem:** *Let $r \in (0,1], 1 \leq p \leq 2,$ $u \in X^*\widehat{\otimes} X$ and u admits a representation*

$$u = \sum_i \lambda_i x'_i \otimes x_i,$$

with $(\lambda_i) \in l_r,$ (x'_i) bounded and $(x_i) \in l^w_{p'}(X)$. If $1/r + 1/2 - 1/p = 1,$ then the system (μ_k) of all eigenvalues of the corresponding operator $\widetilde{u}$ (written according to their algebraic multiplicities) is absolutely summable and

$$\text{trace } u = \sum_k \mu_k.$$

One of the main aim of these notes is not only to give a proof of the theorem but also to show that it could be obtained by A. Grothendieck in 1955.

In 1955, A. Grothendieck [4] has shown that if the linear operator T in a Banach space is 2/3-nuclear then the trace of T is well defined and is equal to the sum of all eigenvalues $\{\mu_k(T)\}$ of T. V.B. Lidskiĭ [8], in 1959, proved his famous theorem on the coincidence of the trace of the S_1-operator in an (infinite dimensional) Hilbert space with its spectral trace $\sum_{k=1}^{\infty} \mu_k(T)$.

In 1970's and in early 1980's, the interest to the trace formulas (and, generally, to the distribution of eigenvalues of some classes of operators) has been increased (A. Pietsch, H. König and others). The trace formula was established for such ideals of operators as $\mathfrak{L}_1^{app}$, $\mathfrak{P}_2 \circ \mathfrak{P}_2$, $(\mathfrak{P}_2)^{app}_{2,1}$ $\mathfrak{L}_1^{gel}$, $\mathfrak{L}_1^{kol}$, $\mathfrak{L}_1^{weil}$, $\mathfrak{L}_1^{ent}$ (see [12], p. 404). In the book [10] by A. Pietsch, one can find a generalization of Grothendieck-Lidskiĭ theorem to the case of the quasinormed operator ideal $N_{1,1,2}$ of the so called (1,1,2)-nuclear operators (see [10], Th. 27.4.11). In 1996, M. White [15] has obtained a very general theorem on the spectral trace for a wide classes of quasi-normed operator ideals. What about concrete Banach spaces, it was shown recently by Oleg Reinov and Qaiser Latif [14] that the Grothendieck-Lidskiĭ formula can be "interpolated" between L_∞–L_2 (or, between L_1–L_2) cases. More precisely, they have shown that for $p \in [1,\infty]$ and $s \in (0,1]$ with $1/s = 1 + |1/2 - 1/p|$, and for every s-nuclear operator

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T in every subspace of any $L_p(\nu)$-space the trace of T is well defined and equals the sum of all eigenvalues of T. The same is true for quotients of $L_p(\nu)$-spaces. Note that for $p = 2$ one has $s = 1$, and for $p = \infty$ one has $s = 2/3$.

In this note, we are going to give some more examples of such a kind (see Theorem below). Let us mention that the proof of the theorem consists of a "reconsideration" of some Grothendieck's arguments from [4], Chap. II, of proving by him his famous trace formula in the case of 2/3-nuclear operators. In the case where $X = H$ is a Hilbert space and $T \in S_1(H)$ (nuclear case; so, $p = 2$ in the theorem below), our theorem gives the Lidskiǐ formula; in the case where X is any Banach space and T is 2/3-nuclear (so, $p = +\infty$ in the theorem below), we obtain the Grothendieck 2/3-theorem (with an analogues proof!). If X is any and $p = 2$ in our theorem, we obtain the above mentioned $N_{1.1.2}$-result. We give also (after the proof of the theorem) some new consequences and make some remarks on Grothendieck's considerations in Chapter II of his famous work [4]. Let us note now only that, in particular, A. Pietsch writes (concerning Lidskiǐ's 1959 formula) in the book [12], p. 404: "a remark in [GRO1, Chap. II, p. 13] indicates that by 1955, Grothendieck was aware of this fact". We will give a citation from [4], which shows that A. Grothendieck (in 1955) was aware of a more stronger result than the Lidskiǐ theorem (but the result was given there without any proof). Surely, that work by A. Grothendieck was unknown to V. Lidskiǐ, so, the famous Lidskiǐ's formula is Lidskiǐ's formula forever.

§1. Preliminaries and a theorem

All the terminology and facts (now classical), given here without any explanations, can be found in [1, 2, 4, 7, 10, 11].

Let X, Y be Banach spaces. For the Banach dual of X, we use the notation X^*. If $x \in X$ and $x' \in X^*$, then we use the notation $\langle x', x\rangle$ for $x'(x)$.

Denote by $X^*\widehat{\otimes}Y$ the completion of the tensor product $X^* \otimes Y$ (considered as a linear space of all finite rank operators from X to Y) with respect to the projective norm

$$||w|| := \inf\{\left(\sum_{k=1}^N ||x'_k||\,||y_k||\right):\ w = \sum_{k=1}^N x'_k \otimes y_k\}$$

(see, e.g., [4], [1]). For $X = Y$, the natural linear continuous functional "trace" on $X^* \otimes X$ has a unique continuous extension to the space $X^*\widehat{\otimes}X$, which we still will denote by "trace".

Put $N(X, Y) :=$ image of $X^*\widehat{\otimes}Y$ in the space $L(X, Y)$ of all bounded linear transformations under the canonical factor map $X^*\widehat{\otimes}Y \to N(X, Y) \subset L(X, Y)$. We consider the (Grothendieck) space $N(X, Y)$ of all nuclear operators from X to Y with the natural norm, induced from $X^*\widehat{\otimes}Y$. For a tensor element $u \in X^*\widehat{\otimes}Y$, we denote by $\widetilde{u}$ the corresponding nuclear operator from X to Y.

For $q \in (0, +\infty]$, we denote by $l_q^w(X)$ the space of all weakly q-summable sequences $(x_i) \subset X$ (see, e.g., [9], [10]) with a quasi-norm

$$\varepsilon_q((x_i)) := \sup\{\left(\sum_i |\langle x', x_i\rangle|^q\right)^{1/q} : \ x' \in X^*, \ ||x'|| \leq 1\}$$

(in the case where $q = \infty$, we suppose (x_i) to be just bounded and tending to zero, i.e., $\varepsilon_\infty((x_i)) = \sup_i ||x_i||$).

We are going to prove

Theorem. *Let $r \in (0, 1], 1 \leq p \leq 2$, $u \in X^*\widehat{\otimes}X$ and u admits a representation*

$$u = \sum_i \lambda_i x_i' \otimes x_i,$$

with $(\lambda_i) \in l_r$, (x_i') bounded and $(x_i) \in l_{p'}^w(X)$. If $1/r + 1/2 - 1/p = 1$, then the system (μ_k) of all eigenvalues of the operator $\widetilde{u}$ (written according to their algebraic multiplicities) is absolutely summable and

$$\text{trace}\ u = \sum_k \mu_k.$$

We obtained this result rather casually, just analyzing the arguments, given by A. Grothendieck [4, Ch. II] for getting his trace formula for 2/3-nuclear operators, and noting that Hadamard's inequality for determinants may be improved in some L_p situations (this idea appeared after considerations again of arguments from [14] and the facts that the Hilbert spaces are the best Banach spaces, but the Banach spaces of type L_p for $p \in (1, \infty)$ are, maybe, worse than an H but better that any X (or, the same, in a sence, than L_∞)).

In the proof of Theorem, we shall use, in particular, the "related operators theorem" [10, p. 375], namely, in the following situation. If u is as in Theorem then it is easy to see that it admits a factorization

$$u = AB : \ X \to l_p \to X,$$

where B is s-nuclear (is generated by "un noyau de Fredholm de puissance s.ème sommable dans $X^*\widehat{\otimes}l_p$" in terms of [4]), A maps the unit vector basis of l_p to the sequence (x_i) (which is weakly p'-summable). Therefore, the set of all eigenvalues of u is the same as the set of all (with their algebraic multiplicities) eigenvalues of the operator BA, which maps l_p into l_p (so, results of [6] and [14] may be applied).

§2. Proofs

Let u be an element of the projective tensor product $X^*\widehat{\otimes}X$. It can be represented in the form

$$u = \sum_i \lambda_i x_i' \otimes x_i,$$

where $(\lambda_i) \in l_1$ and $||x_i'|| \leq 1$, $||x_i|| \leq 1$ (see [4], [1]). Recall that the Fredholm determinant $\det(1 - zu)$ of u (see [4], [5], [10], [11]) is an entire function

$$\det(1 - zu) = 1 - z\ \text{trace}\ u + \dots + (-1)^n z^n \alpha_n(u) + \dots,$$

all zeros of which are exactly (according to their multiplicities) the inverses of nonzero eigenvalues (μ_k) of the operator $\widetilde{u}$, associated with the tensor element u. If u has a form $u = \sum_i \lambda_i x'_i \otimes x_i$ as above, the coefficients $\alpha_n(u)$ in the previous formula are defined explicitly by

$$\alpha_n = \sum_{i_1<\dots<i_n} \lambda_{i_1} \dots \lambda_{i_n} \det(\langle x'_{i_\alpha}, x_{i_\beta}\rangle)_{1\le\alpha,\beta\le n}$$

(see [4, Chap. II, p.13, (5bis)], [5]).

Suppose now, that u has a representation

$$u = \sum_i \lambda_i x'_i \otimes x_i,$$

with $(\lambda_i) \in l_r$, $\lambda_i \ge 0$, $r \in (0,1]$, $||x'_i|| \le 1$, $(x_i) \in l^w_{p'}(X)$, $\varepsilon_{p'}((x_i)) \le 1$ (here $1 \le p \le 2$). We have:

$$f(z) := \det(1 + zu) = \sum_{n=0}^{\infty} \alpha_n(u)\, z^n,$$

where $\alpha_n(u)$ are as above; therefore, taking in account that for every $\alpha = 1, \dots, n$

$$\left(\sum_{\beta=1}^{n} |\langle x'_{i_\alpha}, x_{i_\beta}\rangle|^{p'}\right)^{1/p'} \le 1$$

and thus

$$\left(\sum_{\beta=1}^{n} |\langle x'_{i_\alpha}, x_{i_\beta}\rangle|^{2}\right)^{1/2} \le n^{1/p-1/2},$$

by Hadamard's inequality for determinants (see, e.g., [16], 8.7.4 Problems and Exercises, Ex. 9c), or [2, p. 1018]), we get

$$|\alpha_n(u)| \le n^{n(1/p-1/2)}\, \alpha_n(\lambda),$$

where

$$\alpha_n(\lambda) = \sum_{i_1<\dots<i_n} \lambda_{i_1} \dots \lambda_{i_n}.$$

Since the function $g(z) = \prod_i (1 + \lambda_i z)$ is of order $\le r$ (see, e.g., [7], p. 30, Th. 3 (Borel)) and since its coefficients are exactly α_n, we obtain for these coefficients the estimates, for each $t > r$,

$$\alpha_n(\lambda) \le M_t n^{-n/t}$$

(see the same book [7], p. 6). Hence,

$$|\alpha_n(u)| \le M_t n^{-n(1/t-1/p+1/2)} = M_t n^{-n/\omega},$$

where $1/\omega = 1/t + 1/2 - 1/p$. By a classical result of Hadamard (see, e.g., [7], pp. 5-6), the function $f(z)$ is of order $\le \omega$ and, therefore, of order $\le \nu$, where $1/\nu = 1/r + 1/2 - 1/p$ (since $t > r$ was arbitrary).

Now, suppose that $\nu = 1$ (that is, $1/r + 1/2 - 1/p = 1$). By Hadamard (see [7], p. 26, Th. 1),

$$\det(1 - zu) = e^{-az} \prod_i (1 - z\mu_i) e^{z\mu_i}$$

(recall that (μ_k) is a sequence of all eigenvalues of $\widetilde{u}$, counted according to their algebraic multiplicities). On the other hand, as was said above,

$$\det(1 - zu) = 1 - z \text{ trace } u + \dots + (-1)^n z^n \alpha_n(u) + \dots,$$

and we get (considering the expansion of the entire function $e^{-az} \prod_i (1 - z\mu_i) e^{z\mu_i}$) that $a =$ trace u. Therefore,

$$\det(1 - zu) = e^{-z \text{ trace } u} \prod_i (1 - z\mu_i) e^{z\mu_i}.$$

Now we apply Theorem 2.6 of [6] or results from [14] to get that $(\mu_k) \in l_1$, from which it follows (see, e.g., [7], p. 25-26) that

$$\det(1 - zu) = e^{-\alpha z} \prod_i (1 - z\mu_i), \quad \text{where} \quad \alpha = \text{trace } u - \sum_k \mu_k$$

and

$$\text{the function} \quad \det(1 - zu) \quad \text{is of minimal type}$$

(by the same Hadamard's theorem; see also [7], pp. 25-26 or the second part of the proof of Borel theorem in [7], p. 30). Whence, $\alpha = 0$, i.e. trace $u = \sum_k \mu_k$.

§3. Corollaries and remarks.

Corollary 1. *Let r, p, u be as in Theorem. The operator $\widetilde{u} : X \to X$ is equal to zero iff the tensor element u is zero.*

The same proof as the one of Theorem, with evident changes, gives us

Corollary 2. *Let $r \in (0, 1], 1 \le p \le 2$, $u \in X^* \widehat{\otimes} X$ and u admits a representation*

$$u = \sum_i \lambda_i x_i' \otimes x_i,$$

with $(\lambda_i) \in l_r$, (x_i) bounded and $(x_i') \in l^w_{p'}(X^*)$. If $1/r + 1/2 - 1/p = 1$, then the system (μ_k) of all eigenvalues of the operator $\widetilde{u}$ (written according to their algebraic multiplicities) is absolutely summable and

$$\text{trace } u = \sum_k \mu_k.$$

Corollary 3. *Let r, p, u be as in the previous corollary. The operator $\widetilde{u} : X \to X$ is equal to zero iff the tensor element u is zero.*

Remark. For the case where $r = 2/3$ and $p = \infty$, we get 2/3-theorems of A. Grothendieck ([4]; for a simple proof of the 2/3-theorems, see [13]). For the case $r = 1$ and $p = 2$, we get the $N_{1,1,2}$-results of [10, p. 381]. Also, Corollaries 1 and 3 are valid if we consider the operators $\widetilde{u}$ from X to Y, for any Banach X, Y.

As was said above, in our proof we just used the ideas of A. Grothendieck from [4]. Let us mention that our Theorem could be proved by A. Grothendieck in 1955, as well as the Lidskiĭ's result. Namely, in [4, Ch. II, Remark 4, p. 21], A. Grothendieck writes:

"*Soit* $0 < p \leq 1$. *Pour tout* u*..., soit* $\widehat{u}$ *la suite non ordonnee des valeurs propres de* u *... ... ce qui permet facilement, ..., de se ramener à un résultat plus fin sur les espaces de Hilbert: Si* H *est un espace de Hilbert, l'application* $u \to \widehat{u}$ *de* $H' \overset{(p)}{\otimes} H$ *dans* (l'espace) $\Sigma_{(p)}$ (des suites non ordonnées d'ordre $\leq p$) *est continue.*"

Here $\overset{(p)}{\otimes}$ denotes the tensor product which corresponds to the space of the p-nuclear operators. In the case where $p = 1$, we have the class S_1 of Schatten and von Neumann. Thus, it seems that the S_1-trace-formula indeed was known to A. Grothendieck in 1955, but he (we can only guess, why) did not pay any more attention to the Hilbert case.

Concluding the paper (on March 28, 2012), I would like to bring my deep acknowledgments to Alexander Grothendieck for his ideas from [4] (which are all in these notes) on the day of his Birth.

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