On regularity properties of solutions to hysteresis-type problems *

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1 Introduction.

We consider the bounded solutions of the following parabolic free boundary problem:

$$Lu = \mathcal{H}u \quad \text{in} \quad Q_1 = B_1\times]-1,1]. \tag{1}$$

Eq. (1) is understood in the weak (distributional) sence. Here $L = \partial_{xx} - \partial_t$ is the heat operator, $B_1 = \{x \in \mathbb{R} : |x| < 1\}$, and $\mathcal{H}$ is a hysteresis-type operator which is defined as follows.

We fix two numbers α and β ($\alpha < \beta$) and consider a **multivalued** function

$$f(s) = \begin{cases} -1, & \text{for} \quad s \in]-\infty, \alpha], \\ 1, & \text{for} \quad s \in [\beta, +\infty[, \\ -1 \text{ or } 1, & \text{for} \quad s \in]\alpha, \beta[. \end{cases}$$

For $u \in C(\overline{Q}_1)$ we suppose that on the bottom of the cylinder Q_1 the initial values $\mathcal{H}u(x,-1) := f(u(x,-1))$ are prescribed.

After that for every point $z = (x,t) \in \overline{Q}_1$ the corresponding value of $\mathcal{H}u(z)$ is uniquely defined in the following manner. Let us denote by E a set of points

$$E := \{z \in Q_1 : u(z) \leqslant \alpha\} \cup \{z \in Q_1 : u(z) \geqslant \beta\} \cup \{B_1 \times \{-1\}\}.$$

In other words, E is a set where $f(u(z))$ is well-defined.

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If $z \in E$ then $\mathcal{H}u(z) = f(u(z))$. Otherwise, for $z = (x,t) \in Q_1$ such that $\alpha < u(z) < \beta$ we set

$$\mathcal{H}u(x,t) = \mathcal{H}u(x, \tau(x)). \tag{2}$$

Here

$$\tau(x) = \max_{[0,t]} \{\tau : (x,\tau) \in E\}$$

Roughly speaking, condition (2) means that the hysteresis function $\mathcal{H}u(x,t)$ takes for $u(x,t) \in (\alpha, \beta)$ the same value as at the previous moment (see Figure 1).

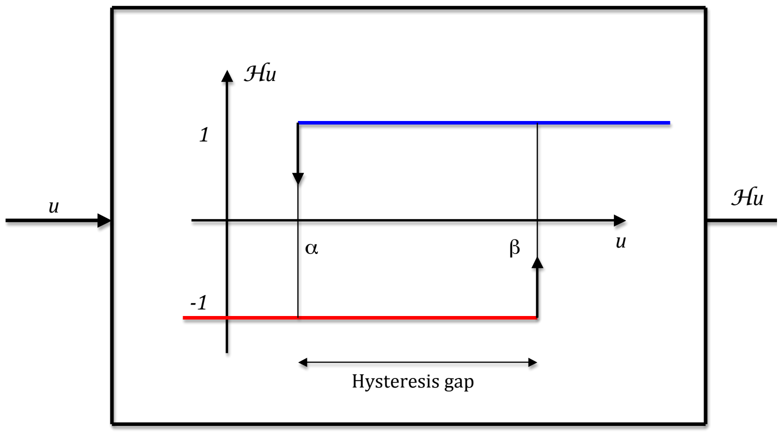

Figure 1: The hysteresis operator $\mathcal{H}$

We suppose also that

$$\sup_{Q_1} |u| \leqslant M \quad \text{with} \quad M \geqslant 1.$$

Since the right-hand side of (1) is bounded, the general parabolic theory (see, e.g. [LSU67]) implies

$$\|u_t\|_{q,Q_{3/4}} + \|u_{xx}\|_{q,Q_{3/4}} \leqslant N_1(q,M) \quad \forall q < \infty.$$

In particular, u satisfies (1) a.e. in $Q_{3/4}$.

Moreover, functions u and u_x are continuous in $Q_{3/4}$ and $u_x \in C_{x,t}^{\delta,\delta/2}\left(Q_{3/4}\right)$ for arbitrary $\delta \in (0;1)$. It is evident that the $(n+1)$-dimensional Lebesgue measure of the sets $\{u = \alpha\}$ and $\{u = \beta\}$ equal zero.

In this paper we are interested in local L^∞-estimates for the derivatives u_{xx} and u_t of the function u satisfying (1).

2 Notation and Preliminaries.

Throughout this article we use the following notation:
$z = (x, t)$ are points in $\mathbb{R}^2_{x,t}$;
$B_r(x^0)$ denotes the open ball in $\mathbb{R}^1$ with center x^0 and radius r;
$Q_r(z^0) = Q_r(x^0, t^0) = B_r(x^0) \times]t^0 - r^2, t^0 + r^2]$;
$Q_r^-(z^0) = Q_r(z^0) \cap \{t < t^0\}$;
When omitted, x^0 (or $z^0 = (x^0, t^0)$, respectively) is assumed to be the origin. We emphasize that in this paper the top of the cylinder $Q_r(z^0)$ (as well as the top of $Q_r^-(z^0)$ is included in the set $Q_r(z^0)$ (in the set $Q_r^-(z^0)$).
$u_t = \dfrac{\partial u}{\partial t}, \quad u_x = \dfrac{\partial u}{\partial x}, \quad u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$;

We define the parabolic distance dist_p between a point $z = (x, t)$ and a set $\mathcal{D} \subset \mathbb{R}^2$ by

$$\mathrm{dist}_p\,(z, \mathcal{D}) := \sup \left\{ r > 0 : Q_r^-(z) \cap \mathcal{D} = \emptyset \right\}.$$

We use letters M, N, and C (with or without indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in the parentheses: $C(\dots)$.

We denote by

$$\begin{aligned} \Omega_\pm(u) &:= \{z \in Q_1, \text{ where } \mathcal{H}u(z) = \pm 1\}, \\ \Gamma(u) &:= \partial\Omega_+ \cap \partial\Omega_- \text{ is the free boundary.} \end{aligned}$$

The latter means that $\Gamma(u)$ is the set where the function $\mathcal{H}u(z)$ has a jump.

We also introduce special notation for the different parts of $\Gamma(u)$

$$\begin{aligned} \Gamma_\alpha(u) &:= \Gamma(u) \cap \{u = \alpha\}, \\ \Gamma_\beta(u) &:= \Gamma(u) \cap \{u = \beta\}. \end{aligned}$$

Observe that the sets $\{u = \alpha\}$ and $\{u = \beta\}$ not alsways are the parts of the free boundary $\Gamma(u)$ (see Figure 2). Moreover, by definition,

$$\{u \leqslant \alpha\} \subset \Omega_- \quad \text{and} \quad \{u \geqslant \beta\} \subset \Omega_+.$$

It is also easy to see that the sets $\{u = \alpha\}$ and $\{u = \beta\}$ are separated from each other. In other words, there exists a positive constant d_0 completely determined by M such that

$$\mathrm{dist}\, \{\{u = \alpha\}\, ; \{u = \beta\}\} \geqslant d_0 > 0.$$

The latter guarantees that Γ_α and Γ_β are the isolated components of $\Gamma(u)$.

Consider a part of $\partial\Omega_-$ satisfying $\partial\Omega_- \cap \Gamma_\beta = \emptyset$. We see that this part of $\partial\Omega_-$ may contain several components of Γ_α connected by open segments parallel to t-axis. Similar statement is true for a part of $\partial\Omega_+$ satisfying $\partial\Omega_+ \cap \Gamma_\alpha = \emptyset$. We will denote by Γ_v the set of all points z lying in such open vertical segments (see Figure 2).

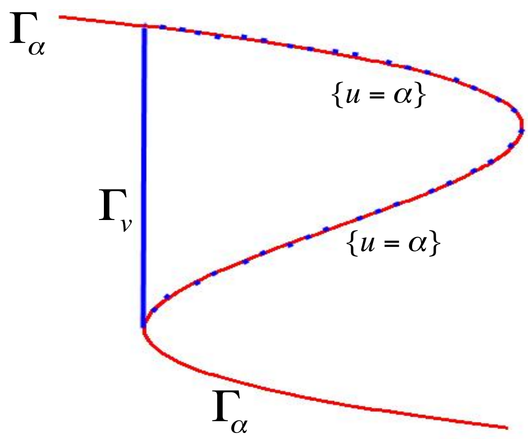

Figure 2: Structure of the free boundary

We will also distinguish the following parts of Γ:

$$\Gamma^0_\alpha(u) = \Gamma_\alpha(u) \cap \{u_x = 0\}, \qquad \Gamma^*_\alpha(u) = \Gamma(u) \setminus \Gamma^0_\alpha(u).$$

The sets Γ^0_β and Γ^*_β are defined analogously.

Remark 2.1. *It is obvious that $u \in C^\infty$ in the interior of the sets $\Omega_\pm$.*

3 Main Result

Lemma 3.1. *Let u be a bounded solution of Eq. (1) and let $z^* \in \left(\Gamma^*_\alpha \cup \Gamma^*_\beta\right) \cap Q_{1/2}$. Then u_t is a continuous function in a some neigborhood of z^* and u_{xt} is locally a L^2-function.*

Lemma 3.2. *Let u be a bounded solution of Eq. (1). Then there exists a positive constant $N = N(M)$ such that*

$$|u_t(z)| \leqslant N \qquad \textit{for all} \quad z \in \left(\Gamma_\alpha^* \cup \Gamma_\beta^*\right) \cap Q_{1/2}.$$

Lemma 3.3. *Let u be a bounded solution of Eq. (1). The following statements hold true.*

(i) Let $z^0 \in \Gamma_\alpha^0 \cap Q_{1/2}$ and $z^0 \notin \Gamma_v$. Then there exists a positive constant N_α completely defined by $\rho_0 := dist_p\{z^0, \Gamma_v\}$ and M such that

$$\sup_{Q_r^-(z^0)} |u - \alpha| \leqslant N_\alpha r^2 \qquad \forall r \leqslant \rho_0.$$

(ii) Let $z^0 \in \Gamma_\beta^0 \cap Q_{1/2}$ and $z^0 \notin \Gamma_v$. Then there exists a positive constant N_β completely defined by $\rho_0 := dist_p\{z^0, \Gamma_v\}$ and M such that

$$\sup_{Q_r^-(z^0)} |u - \beta| \leqslant N_\beta r^2 \qquad \forall r \leqslant \rho_0.$$

Theorem 1. *Let u be a bounded solution of Eq. (1), let z be a point in $Q_{1/2} \setminus \Gamma(u)$, and let $\rho_0 := dist_p\{z, \Gamma_v\}$. Then there exists a positive constant C completely defined by the values of ρ_0 and M such that*

$$|u_{xx}(z)| + |u_t(z)| \leqslant C.$$

References

[LSU67] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.