On linear operators with s-nuclear adjoints, $0 < s \leq 1$

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Abstract

We prove that if $s \in (0,1]$ and T is a linear operator with s-nuclear adjoint from a Banach space X to a Banach space Y and if one of the spaces X^* or Y^{***} has the approximation property of order s, then the operator T is nuclear. The result is in a sense exact. For example, it is shown that for each $r \in (2/3, 1]$ there exist a Banach space Z_0 and a non-nuclear operator $T : Z_0^{**} \to Z_0$ so that Z_0^{**} has a Schauder basis, Z_0^{***} has the AP_s for every $s \in (0, r)$ and T^* is r-nuclear.

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2010 MSC: 47B10. Operators belonging to operator ideals (nuclear, p-summing, in the Schatten-von Neumann classes, etc.)

1. Introduction

We will be interested in the following question from [6, Problem 10.1]: Suppose T is a (bounded linear) operator acting between Banach spaces X and Y, and let $s \in (0,1)$. Is it true that if T^* is s-nuclear then T is s-nuclear too?

As is well known, for $s = 1$, a negative answer was obtained already by T. Figiel and W.B. Johnson in [4]. For $s \in (2/3, 1]$ the negative answer can be found, e.g., in [14]. Here we are going to give some (partially) positive results in this direction as well as to show the sharpness of them.

It is not difficult to see that if T^* is s-nuclear, then T is p-nuclear with $1/s = 1/p + 1/2$ (see e.g., [13], [14]). This is the best possible general result one can obtain without imposing any conditions on the Banach spaces involved. The sharpness of the assertion $1/s = 1/p + 1/2$, for $s \in (2/3, 1]$, can be seen, for instance, in [14].

Below we consider a slightly different question: Under which conditions on the Banach spaces involved is it valid that

$(*)$ an operator $T \in L(X,Y)$ is nuclear if its adjoint T^* is s-nuclear?

One of the possibilities for getting some positive answers to $(*)$ is to apply the so-called approximation properties of order s, $s \in (2/3, 1]$ (we assume that $s > 2/3$ since for $s \leq 2/3$ the answer is positive for any Banach spaces; see e.g., [5] or [14]).

We will prove that $(*)$ is true if either X^* or Y^{***} has the AP_s (Theorem 1). Some examples, given after Theorem 1, will show that these assumptions are, in a sense, necessary (for example, it is not enough to assume that X or Y^{**} or even both enjoy the AP_s, not even for $s = 1$). Let us note that the case where $s = 1$ was first investigated in the paper [8] by Eve Oja and the author.

2. Notations and preliminaries

Our main reference is [9]. All the spaces under considerations, $(X, Y, \ldots)$ are real or complex Banach spaces, all linear mappings (operators) are continuous; as usual, $X^*, X^{**}, \ldots$ are Banach duals (of X), and $x', x'', \ldots$ (or $y', \ldots$) are the functionals acting on $X, X^*, \ldots$ (or on $Y, \ldots$). By π_Y we denote the natural isometric injection of Y into its second dual. If $x \in X, x' \in X^*$ then $\langle x, x' \rangle = \langle x', x \rangle = x'(x)$. $L(X,Y)$ as usual stands for the Banach space of all linear bounded operators from X to Y.

An operator $T : X \to Y$ is s-nuclear ($0 < s \leq 1$) if it is of the form

$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle y_k$$

for all $x \in X$, where $(x'_k) \subset X^*, (y_k) \subset Y$, $\sum_k ||x'_k||^s \, ||y_k||^s < \infty$. We use the notation $N_s(X,Y)$ for the quasi-Banach space of all such operators, equipped with the quasi-norm

$$||T||_{N_s} := \inf(\sum_k ||x'_k||^s \, ||y_k||^s)^{1/s},$$

where the infimum is taken over all representations of T in the above form (see [9, 6.1 and 18.1]). If $s = 1$, then $N_1(X,Y)$ is a Banach space, which is usually denoted by $N(X,Y)$. The operators from $N(X,Y)$ are called also "nuclear operators". It is clear that for $0 < s_1 < s_2 \le 1$ one has the natural continuous injection $N_{s_1}(X,Y) \subset N_{s_2}(X,Y)$.

Every s-nuclear operator is a canonical image of an element of a projective tensor product. Namely, denote by $X^*\widehat{\otimes}_s Y$ the s-projective tensor product of X^* and Y consisting of all tensor elements z which admit a representation of the kind

$$z = \sum_{k=1}^{\infty} x'_k \otimes y_k \text{ with } \sum_{k=1}^{\infty} ||x'_k||^s \, ||y_k||^s < \infty.$$

Then every s-nuclear operator from X to Y is an image of an element of $X^*\widehat{\otimes}_s Y$ via the canonical mappings

$$X^*\widehat{\otimes}_s Y \xrightarrow{j_s} X^*\widehat{\otimes}_1 Y \xrightarrow{j} L(X,Y).$$

Note that $X^*\widehat{\otimes}_1 Y$ is exactly the projective tensor product $X^*\widehat{\otimes} Y$ of A. Grothendieck [5]. If $z \in X^*\widehat{\otimes} Y$ then we denote the corresponding operator (from X to Y) by $\tilde{z}$.

We say, following, e.g., [13] or [14], that a Banach space Y has the AP_s (the approximation property of order s), if for every Banach space X the natural map jj_s (from above) is one-to-one (note that $AP_1 = AP$ of A. Grothendieck). It can be seen that, just like for the classical AP, Y has the AP_s if the natural map $Y^*\widehat{\otimes}_s Y \to L(Y,Y)$ is one-to-one. Therefore, if Y has the AP_s, we have $X^*\widehat{\otimes}_s Y = N_s(X,Y)$ whatever Banach space X we consider. Clearly, $AP_s \Rightarrow AP_t$ for $0 < t < s \le 1$. Every Banach space has the $AP_{2/3}$ (Grothendieck's Theorem [5], see also [11], [13] or [14]).

Remark 1: Let $s \in (0,1]$. If the space Y is a dual space, say $Y = Z^*$, then the tensor product $X^*\widehat{\otimes}_s Y = X^*\widehat{\otimes}_s Z^*$ can be naturally (isometrically) identified with the tensor product $Z^*\widehat{\otimes}_s X^*$ by definition (e.g., we can consider an element $x' \otimes z'$ also as the element $z' \otimes x'$). It follows that the dual space Z^* has the AP_s iff for every Banach space X the natural map $Z^*\widehat{\otimes}_s X^* \to L(Z,X^*)$ is one-to-one. In this case if Z^* has the AP_s, then for any X we have $N_s(Z,X^*) = Z^*\widehat{\otimes}_s X^*$ and $N_s(X,Z^*) = X^*\widehat{\otimes}_s Z^*$. We will use this remark in the proof of Theorem 1 (Section 3).

If $w \in Y^*\widehat{\otimes} X$ and $U \in L(X,Y)$, then $w \circ U$ denotes the image of w in $X^*\widehat{\otimes} X$ under the map $U^* \otimes \mathrm{id}_X$ (see e.g., [5]). So if $w = \sum \varphi'_m \otimes \psi_m$ is a representation of w in $Y^*\widehat{\otimes} X$, then $w \circ U = \sum U^*\varphi'_m \otimes \psi_m$ is a representation of $w \circ U$ in $X^*\widehat{\otimes} X$. Then $\operatorname{trace} w \circ U = \sum \langle U^*\varphi'_m, \psi_m\rangle = \sum \langle \varphi'_m, U\psi_m\rangle = \operatorname{trace} U \circ w$, where $U \circ w$ is the image of w in $Y^*\widehat{\otimes} Y$ under the map $\mathrm{id}_{Y^*} \otimes U$.

If $U \in L(X, Y^{**})$, then the tensor element $U \circ w \in Y^* \widehat{\otimes} Y^{**}$ is defined by the same way.

Recall that the dual space of $X^* \widehat{\otimes} Y$ is $L(X, Y^{**})$ and the duality is defined by the "trace": If $z \in X^* \widehat{\otimes} Y$ and $U \in L(Y, X^{**})$, then

$$\langle U, z \rangle := \operatorname{trace} U \circ z$$

($= \sum_k \langle x'_k, U y_k \rangle$ for a projective representation $z = \sum_k x'_k \otimes y_k$). So, the element $z \in X^* \widehat{\otimes}_s Y$ is zero iff it it zero in the projective tensor product $X^* \widehat{\otimes} Y$ iff for every $U \in L(Y, X^{**})$ $\operatorname{trace} U \circ z = 0$. If $z \in X^* \widehat{\otimes} Y$, then the corresponding operator $\tilde{z}$ is zero iff for every $R \in Y^* \otimes X$ we have $\operatorname{trace} R \circ z = 0$ (evidently).

Let us mention some examples of Banach spaces with AP_s : For $s \in [2/3, 1]$ and $1/p + 1/2 = 1/s$, every quotient of any subspace of any L_p-space (and every subspace of any quotient of any $L_{p'}$-space) has the AP_s (as well as all their duals; see, e.g., [13] or [14]; for a more general fact, see Lemma 3 below). Here $1/p + 1/p' = 1$.

All Banach spaces have AP_s for $s \in (0, 2/3]$, however if $2/3 \leq s_1 < s_2 \leq 1$ then $AP_{s_2} \Rightarrow AP_{s_1}$ but AP_{s_1} does not imply AP_{s_2}. It is known that for every $p \neq 2$, $p \in [1, \infty]$, there exists a subspace (a lot of them) of l_p without the Grothendieck approximation property; thus, for example, $AP_1 \neq AP_s$ if $s \in (0, 1)$. Indeed, firstly, for $s \in (0, 2/3]$ any Banach space has the AP_s, but not every Banach space has the AP_1. Secondly, for $s \in (2/3, 1)$ and p with $1/p + 1/2 = 1/s$ every subspace of l_p (as was said) has the AP_s, but there is a subspace of l_p without the $AP_1 (= AP)$. Some more information about the statement $AP_{s_2} \neq AP_{s_1}$ can be deduced from our Examples 1 and 2, as well as Theorems 2 and 3.

We will use later the following facts (surely well known, but maybe not mentioned in the literature):

Lemma 1. If $T \in L(X, Y^{**})$ then $||T^*|_{\pi_{Y^*}(Y^*)}|| = ||T||$ and $(T^*|_{\pi_{Y^*}(Y^*)})^*|_X = T$. So, we may write $L(X, Y^{**}) = L(Y^*, X^*)$, where the equality sign refers to the above identification.

Lemma 2. Let $0 < s \leq 1$. If $T \in L(X, Y)$ then

1) $\pi_Y T \in N_s(X, Y^{**}) \iff T^* \in N_s(Y^*, X^*)$;

2) $T \in N_s(X, Y) \Rightarrow T^* \in N_s(Y^*, X^*)$.

Lemma 3. If E is a Banach space of type 2 (respectively, of cotype 2) and of cotype q_0 (respectively, of type q'_0) then E has the AP_s, where $1/s = 3/2 - 1/q_0$.

The proof of Lemma 3 can be found in [13] or [14].

Lemma 4. Let $s \in (0, 1]$. If Y^* has the AP_s, then Y has the AP_s too.

Proof. We use the fact (mentioned above) that Y has the AP_s iff the natural map $Y^* \widehat{\otimes}_s Y \to L(Y, Y)$ is one-to-one. As is known [5], the projective tensor product $Y^* \widehat{\otimes} Y$ is a Banach subspace of the tensor product $Y^* \widehat{\otimes} Y^{**}$. The tensor product $Y^* \widehat{\otimes}_s Y$ is a linear subspace of $Y^* \widehat{\otimes} Y$, as well as $Y^* \widehat{\otimes}_s Y^{**}$ is a linear subspace of $Y^* \widehat{\otimes} Y^{**}$. Therefore, the natural map $Y^* \widehat{\otimes}_s Y \to Y^* \widehat{\otimes}_s Y^{**}$ is one-to-one. Now if Y^* has the AP_s, then the canonical map $Y^{**} \widehat{\otimes}_s Y^* \to L(Y^*, Y^*)$ is one-to-one. Since we can identify the tensor product $Y^{**} \widehat{\otimes}_s Y^*$ with the tensor product $Y^* \widehat{\otimes}_s Y^{**}$ (see Remark 1), it follows that the natural map $Y^* \widehat{\otimes}_s Y \to L(Y, Y)$ is one-to-one. Thus, if Y^* has the AP_s, then Y has the AP_s too.

We will use Lemma 4 in the proof of Theorem 1 in the next section.

Remark 2: For any $s \in (2/3, 1]$ there exists a Banach space, possessing the Grothendieck approximation property, whose dual does not have the AP_s (it is well known for the case where $s = 1$). Moreover, if $s \in (2/3, 1]$, then we can find a Banach space W such that W has a Schauder basis and W^* does not have the AP_s. Indeed, let E be a separable reflexive Banach space without the AP_s (see [13] or [14]). Let Z be a separable space such that Z^{**} has a basis and there exists a linear homomorphism φ from Z^{**} onto E^* so that the subspace $\varphi^*(E)$ is complemented in Z^{***} and, moreover, $Z^{***} \cong \varphi^*(E) \oplus Z^*$ (see [7, Proof of Corollary 1]). Put $W := Z^{**}$. This (second dual) space W has a Schauder basis and its dual W^* does not have the AP_s.

3. A positive result

Theorem 1. Let $s \in (0, 1]$, $T \in L(X, Y)$ and assume that either $X^* \in AP_s$ or $Y^{***} \in AP_s$. If $T \in N_s(X, Y^{**})$, then $T \in N_1(X, Y)$. In other words, under these conditions, from the s-nuclearity of the conjugate operator T^*, it follows that the operator T is nuclear.

Proof. Suppose there exists an operator $T \in L(X, Y)$ such that $T \notin N_1(X, Y)$, but $\pi_Y T \in N_s(X, Y^{**})$. Since either X^* or Y^{**} has the AP_s, $N_s(X, Y^{**}) = X^* \widehat{\otimes}_s Y^{**}$ (see Remark 1 in Section 2). Therefore the operator $\pi_Y T$ can be identified with the tensor element $t \in X^* \widehat{\otimes}_s Y^{**} \subset X^* \widehat{\otimes}_1 Y^{**}$. In addition, by the choice of T, $t \notin X^* \widehat{\otimes}_1 Y$ (the space $X^* \widehat{\otimes}_1 Y$ is considered as a closed subspace of the space $X^* \widehat{\otimes}_1 Y^{**}$). Hence there is an operator $U \in L(Y^{**}, X^{**}) = \left(X^* \widehat{\otimes}_1 Y^{**}\right)^*$ with the properties that $\operatorname{trace} U \circ t =$

trace $(t^* \circ (U^*|_{X^*})) = 1$ and trace $U \circ \pi_Y \circ z = 0$ for each $z \in X^* \widehat{\otimes}_1 Y$. From the last observation it follows that, in particular, $U\pi_Y = 0$ and $\pi_Y^* U^*|_{X^*} = 0$. In fact, if $x' \in X^*$ and $y \in Y$, then

$$\langle U\pi_Y\, y, x'\rangle = \langle y, \pi_Y^* U^*|_{X^*} x'\rangle = \operatorname{trace} U \circ (x' \otimes \pi_Y(y)) = 0.$$

Evidently, the tensor element $U \circ t \in X^* \widehat{\otimes}_s X^{**}$ induces the operator $U\pi_Y T$, which is equal to the 0-operator.

If $X^* \in AP_s$ then $X^* \widehat{\otimes}_s X^{**} = N_s(X, X^{**})$ and, therefore, this tensor element is zero, which contradicts the equality trace $U \circ t = 1$.

Now let $Y^{***} \in AP_s$. In this case

$$V := (U^*|_{X^*}) \circ T^* \circ \pi_Y^* :\ Y^{***} \to Y^* \to X^* \to Y^{***}$$

uniquely determines a tensor element t_0 from the s-projective tensor product $Y^{****} \widehat{\otimes}_s Y^{***}$. Let us take any representation $t = \sum x'_n \otimes y''_n$ for t as an element of the space $X^* \widehat{\otimes}_s Y^{**}$. Denoting the operator $U^*|_{X^*}$ by U_*, we obtain

$$\begin{aligned} V y''' &= U_*\ (T^* \pi_Y^*\, y''') = U_*\ \left(\left(\sum y''_n \otimes x'_n\right) \pi_Y^*\, y''' \right) \\ &= U_* \left(\sum \langle y''_n, \pi_Y^*\, y''' \rangle\, x'_n \right) = \sum \langle \pi_Y^{**} y''_n, y''' \rangle\, U_* x'_n. \end{aligned}$$

So, the operator V (or the element t_0) has in the space $Y^{****} \widehat{\otimes}_s Y^{***}$ the representation

$$V = \sum \pi_Y^{**}(y''_n) \otimes U_*(x'_n).$$

Therefore,

$$\operatorname{trace} t_0 = \operatorname{trace} V = \sum \langle \pi_Y^{**}(y''_n), U_*(x'_n) \rangle = \sum \langle y''_n, \pi_Y^*\, U_* x'_n \rangle = \sum 0 = 0$$

(since $\pi_Y^* U_* = 0$, see above).

On the other hand,

$$V y''' = U_*\, (\pi_Y T)^*\, y''' = U_* \circ t^*(y''') = U_* \left(\sum \langle y''_n, y''' \rangle\, x'_n \right) = \sum \langle y''_n, y''' \rangle\, U_* x'_n,$$

whence $V = \sum y''_n \otimes U_*(x'_n)$. Therefore

$$\operatorname{trace} t_0 = \operatorname{trace} V = \sum \langle y''_n, U_* x'_n \rangle = \sum \langle U y''_n, x'_n \rangle = \operatorname{trace} U \circ t = 1.$$

The obtained contradiction completes the proof of the theorem.

4. Examples

We need two examples to show that all conditions, imposed on X and Y in Theorem 1, are essential. Ideas of such examples are taken from the author's work [13] (not translated, as far as we know, from Russian).

Example 1. Let $r \in (2/3, 1], q \in [2, \infty), 1/r = 3/2 - 1/q$. There exist a separable reflexive Banach space Y_0 and a tensor element $w \in Y_0^* \widehat{\otimes}_r Y_0$ so that $w \neq 0, \tilde{w} = 0$, the space Y_0 (as well as Y_0^*) has the AP_s for every $s < r$ (but, evidently, does not have the AP_r). Moreover, Y_0 is of type 2 and of cotype q_0 for any $q_0 > q$.

Proof. We will use a variant of Per Enflo's example [3] of a Banach space without the approximation property given in the book [9, 10.4.5]. Namely, it follows from the constructions in [9] that there exist a Banach space X and a tensor element $z \in X^* \widehat{\otimes} X$ so that $\operatorname{trace} z \neq 0$, the operator $\tilde{z}$, generated by z, is identically zero and z can be represented in the following form:

$$(1) \quad z = \sum_{N=1}^{\infty} \sum_{n=1}^{3 \cdot 2^N} N^{1/2} 2^{-3N/2} x'_{nN} \otimes x_{nN},$$

where the sequences (x'_{nN}) and (x_{nN}) are norm bounded by 1 (see [2] or [9, 10.4.5]).

Fix $r \in (2/3, 1]$ and put $1/q = 3/2 - 1/r$ (thus, $q \in [2, \infty)$). Let $\{\varepsilon_N\}_1^\infty$ be a sequence of numbers such that $\sum N^{-1-\varepsilon_N} < +\infty$, put $\gamma_N = 2 + 3\varepsilon_N/2$ and let q_N be a number such that $1/q - 1/q_N = N^{-1} \log_2 N^{\gamma_N}$ (therefore, $q_N > q$). Set $Y = \left(\sum_N l_{q_N}^{3 \cdot 2^N} \right)_{l_q}$.

Denote by e_{nN} (and e'_{nN}) the unit vectors in Y and in Y^* respectively ($N = 1, 2, \ldots; n = 1, 2, \ldots, 3 \cdot 2^N$), and put

$$z_1 = \sum_N \sum_n 2^{-N/r} \, N^{-(1+\varepsilon_N)/r} \, e'_{nN} \otimes x_{nN};$$

$$T = \sum_N \sum_n 2^{-(3/2-1/r)N} \, N^{(1/2+1/r+\varepsilon_N/r)} \, x'_{nN} \otimes e_{nN}.$$

Let us show that $z_1 \in Y^* \widehat{\otimes}_r X$ and $T \in Ł(X, Y)$. Then $z = z_1 \circ T$. The first inclusion is evident (because of the choice of ε_N).

If $||x|| \leqslant 1$ then

$$||Tx|| \leqslant 3 \left(\sum_N \left(N^{1/2+1/r+\varepsilon_N/r} / 2^{(3/2-1/r-1/q_N)N} \right)^q \right)^{1/q}.$$

Since $3/2 - 1/r - 1/q_N = 1/q - 1/q_N = N^{-1} \log_2 N^{\gamma_N}$, we get from the last inequality:

$$||T|| \leqslant 3 \left(\sum N^{(1/2+1/r+\varepsilon_N/r-\gamma_N)q} \right)^{1/q} = 3 \left(\sum N^{-1-\varepsilon_N} \right)^{1/q} < \infty.$$

Hence,

$$z = z_1 \circ T, \quad \tilde{z} : \ X \overset{T}{\to} Y \overset{\tilde{z}_1}{\to} X.$$

Now, let $Y_0 := \overline{T(X)} \subset Y$, $T_0 : X \to Y_0$ be induced by T and $z_0 := z_1 \circ j$ where $j : Y_0 \hookrightarrow Y$ is the natural embedding. Then $T_0 \in L(X, Y_0), z_0 \in Y_0^* \widehat{\otimes}_r X$, $z = z_1 \circ T = z_0 \circ T_0$, $\operatorname{trace} z_0 \circ T_0 \neq 0$ (so, $z_0 \neq 0$) and $\tilde{z}_0 = 0$.

Write z_0 as $z_0 = \sum_{m=1}^{\infty} f'_m \otimes f_m$, where $(f'_m) \subset Y_0^*, (f_m) \subset X$ and $\sum_m ||f'_m||^r ||f_m||^r < \infty$. We get

$$\operatorname{trace} z_0 \circ T_0 = \sum \langle T_0^* f'_m, f_m \rangle = \sum \langle f'_m, T_0 f_m \rangle = \operatorname{trace} T_0 \circ z_0.$$

Therefore, $w := T_0 \circ z_0 \in Y_0^* \widehat{\otimes}_r Y_0$, $\operatorname{trace} w \neq 0$ and $\tilde{w} = 0$.

Since the space Y is of type 2 and of cotype q_0 for every $q_0 > q$ (and Y^* is of cotype 2 and of type q'_0), the space Y_0 (respectively, Y_0^*) has the AP_s, where $1/s = 3/2 - 1/q_0$, for every $s < r$ (Lemma 3).

Remark 3: We have a nice "by-product consequence" of Example 1. For $q = 2$ (that is, $r = 1$), the space Y_0 is a subspace of a space of the type $\left(\sum_j l_{p_j}^{k_j} \right)_{l_2}$ with $p_j \searrow 2$ and $k_j \nearrow \infty$. Every such space is an asymptotically Hilbertian space (for definitions and some discussion, see [1]). So we have obtained

Corollary. There exists an asymptotically Hilbertian space without the Grothendieck approximation property.

The first example of such a space was constructed by the author in 1982 [10], where A. Szankowski's results were used (let us note that in that time there was not yet such notion as "asymptotically Hilbertian space"). Later, in 2000, by applying Per Enflo's example in a version due to A.M. Davie [2], P. G. Casazza, C. L. García and W. B. Johnson [1] gave another example of an asymptotically Hilbertian space which fails the approximation property. Here we have got it (accidentally) by using the construction from [9].

Example 2. Let $r \in [2/3, 1), q \in (2, \infty], 1/r = 3/2 - 1/q$. There exist a subspace Y_q of the space l_q and a tensor element $w_q \in Y_q^* \widehat{\otimes}_1 Y_q$ so that $w_q \in Y_q^* \widehat{\otimes}_s Y_q$ for each $s > r$, $w_q \neq 0, \tilde{w}_q = 0$ and the space Y_q (as well as Y_q^*) has the AP_r (but, evidently, does not have the AP_s if $1 \geq s > r$). Clearly, Y_q is of type 2 and of cotype q for $q < \infty$.

Proof. We are going to follow the way indicated in the proof of Example 1. Let X and z be as in that proof, so that z has the form (1). Fix $r \in [2/3, 1)$. Now $1/q = 3/2 - 1/r$, and we put $\varepsilon_N = 0$ and $\gamma_N = 2$ for all N; all q_n's are equal to q. Let us fix also an $\alpha = \alpha(q) > 0$ (to be specified later). Consider the space $Y := \left(\sum_N l_q^{3\cdot 2^N}\right)_{l_q}$ (in the case $q = \infty$ "l_q" means "c_0").

Denote by e_{nN} (and e'_{nN}) the unit vectors in Y and in Y^* respectively ($N = 1, 2, \ldots; n = 1, 2, \ldots, 3 \cdot 2^N$), and set this time

$$z_1 = \sum_N \sum_n 2^{-N/r} N^\alpha e'_{nN} \otimes x_{nN};$$

$$T = \sum_N \sum_n 2^{-(3/2-1/r)N} N^{1/2-\alpha} x'_{nN} \otimes e_{nN}.$$

Then $z_1 \in Y^* \widehat{\otimes}_s X$ for every $s > r$. Indeed, if $s \in (r, 1]$ then

$$\sum_N \sum_n [2^{-N/r} N^\alpha]^s = \sum_N 3 \cdot 2^N \cdot 2^{-Ns/r} N^{\alpha s} = \sum_N 3 \cdot 2^{-\varepsilon_0 N} N^{\alpha s} < \infty,$$

where $\varepsilon_0 = s/r - 1 > 0$.

We show that $T \in L(X, Y)$ (clearly, then $z = z_1 \circ T$). Indeed, if $||x|| \leqslant 1$ then, for $q < \infty$,

$$||Tx||_{l_q}^q \leq \sum_N \sum_n \left(2^{-N/q} N^{1/2-\alpha} |\langle x'_{nN}, x\rangle|\right)^q \leq \sum_N 3{\cdot}2^{N-(N/q)q} N^{(1/2-\alpha)q} = \sum_N 3{\cdot}1{\cdot}N^{\alpha_0},$$

where $\alpha_0 = (1/2 - \alpha)q$. Now, take $\alpha > 0$ such that $\alpha_0 = -2$. For $q = \infty$ take $\alpha = \alpha(\infty) = 1$.

Therefore,

$$z = z_1 \circ T, \quad \tilde{z} : \; X \xrightarrow{T} Y \xrightarrow{\tilde{z}_1} X;$$

As in the case of the previous proof (in Example 1), let $Y_q := \overline{T(X)} \subset Y, T_q : X \to Y_q$ be induced by T and $z_q := z_1 \circ j$ where $j : Y_q \hookrightarrow Y$ is the natural embedding. Then $T_q \in L(X, Y_q), z_q \in Y_q^* \widehat{\otimes}_s X$ for all $s > r$, $z = z_1 \circ T = z_q \circ T_q$, trace $z_q \circ T_q \neq 0$ (so, $z_q \neq 0$) and $\tilde{z}_q = 0$.

Write z_q as $z_q = \sum_{m=1}^{\infty} f'_m \otimes f_m$, where $(f'_m) \subset Y_q^*, (f_m) \subset X$ and $\sum_m ||f'_m||\, ||f_m|| < \infty$. We get:

$$\operatorname{trace} z_q \circ T_q = \sum \langle T_q^* f'_m, f_m \rangle = \sum \langle f'_m, T_q f_m \rangle = \operatorname{trace} T_q \circ z_q.$$

Therefore, $w_q := T_q \circ z_q \in Y_q^* \widehat{\otimes}_s Y_q$ for every $s > r$, trace $w_q \neq 0$ and $\tilde{w}_q = 0$. Finally, Lemma 3 says that the space Y_q has the AP_r if $q < \infty$. If $q = \infty$ then, as we know, any Banach space has the $AP_{2/3}$.

Remark: The space Y_∞ from Example 2 not only does not have the AP_s for any $s \in (2/3, 1]$, but also does not have the AP_p (in the sense of the paper [12]) for any $p \in [1, 2)$ (this follows from some facts proved in [13]).

5. Applications of Examples

The next two theorems show that the conditions "X^* has the AP_s" and "Y^{***} has the AP_s" are essential in Theorem 1 and can not be replaces by the weaker conditions (see Remark 2) "X has the AP_s" (even by "X has the AP_1") or "Y^{**} has the AP_s"; moreover, even "both X and Y^{**} have the AP_1" is not enough for the conclusion of Theorem 1 to hold.

Theorem 2. Let $r \in (2/3, 1]$. There exist a Banach space Z_0 and an operator $T \in L(Z_0^{**}, Z_0)$ so that

(1) Z_0^{**} has a Schauder basis;
(2) all the duals of Z_0 are separable;
(3) Z_0^{***} has the AP_s for every $s \in (0, r)$;
(4) $\pi_{Z_0} T \in N_r(Z_0^{**}, Z_0^{**})$;
(5) $T \notin N_1(Z_0^{**}, Z_0)$;
(6) Z_0^{***} does not have the AP_r.

Proof. Let us fix $r \in (2/3, 1], q \in [2, \infty), 1/r = 3/2 - 1/q$ and take the pair (Y_0, w) from Example 1. Let Z_0 be a separable space such that Z_0^{**} has a basis and there exists a linear homomorphism φ from Z_0^{**} onto Y_0 with the kernel $Z_0 \subset Z_0^{**}$ so that the subspace $\varphi^*(Y_0^*)$ is complemented in Z_0^{***} and, moreover, $Z_0^{***} \cong \varphi^*(Y_0^*) \oplus Z_0^*$ (see [7, Proof of Corollary 1]). Lift the tensor element w, lying in $Y_0^*\widehat{\otimes}_r Y_0$, to an element[1] $w_0 \in Y_0^*\widehat{\otimes}_r Z_0^{**}$, so that $\varphi\circ w_0 = w$, and set $T := w_0\circ\varphi$. Since $\operatorname{trace} w_0\circ\varphi = \operatorname{trace}\varphi\circ w_0 = \operatorname{trace} w = 1$ and Z_0^{**} has the AP, then $\widetilde{w_0} = w_0 \neq 0$. Besides, the operator $\widetilde{\varphi\circ w_0} : Y_0 \to Z_0^{**} \to Y_0$, associated with the tensor $\varphi\circ w_0$, is equal to zero. Therefore $w_0(Y_0) \subset \operatorname{Ker}\varphi = Z_0 \subset Z_0^{**}$, that is, the operator w_0 acts from Y_0 into Z_0.

Since the subspace $\varphi^*(Y_0^*)$ is complemented in Z_0^{***}, we have that $w_0\circ\varphi \in Z_0^{***}\widehat{\otimes}_r Z_0 = N_r(Z_0^{**}, Z_0)$ iff $w_0 \in Y_0^*\widehat{\otimes}_r Z_0 = N_r(Y_0, Z_0)$.

If $w_0 \in N_r(Y_0, Z_0)$, then, for its arbitrary (nonzero!) N_r-representation of the form $w_0 = \sum y_n' \otimes z_n$, the composition $\varphi\circ w_0$ is a zero tensor element in $Y_0^*\widehat{\otimes}_r Y_0$; but this composition represents the element w, which, by its choice, can not be zero. Thus, $w_0 \notin N_r(Y_0, Z_0)$ and, thereby, $w_0\circ\varphi \notin Z_0^{***}\widehat{\otimes}_r Z_0 = N_r(Z_0^{**}, Z_0)$. On the other hand, certainly, $w_0\circ\varphi \in Z_0^{***}\widehat{\otimes}_r Z_0^{**} = N_r(Z_0^{**}, Z_0^{**})$.

Finally, since $Z_0^{***} \cong \varphi^*(Y_0^*) \oplus Z_0^*$, one has that the space Z_0^{***} has the AP_s for every $s \in (0, r)$.

Theorem 3. Let $r \in [2/3, 1), q \in (2, \infty], 1/r = 3/2 - 1/q$. There exist a Banach space Z_q and an operator $T \in L(Z_q^{**}, Z_q)$ so that

(1) Z_q^{**} has a Schauder basis;
(2) if $q < \infty$, then all the duals of Z_q are separable;
(3) Z_q^{***} has the AP_r;
(4) $\pi_{Z_q} T \in N_s(Z_q^{**}, Z_q^{**})$ for every $s \in (r, 1]$;
(5) $T \notin N_1(Z_q^{**}, Z_q)$;
(6) Z_q^{***} does not have the AP_s for any $s \in (r, 1]$;

Proof. Let us fix $r \in [2/3, 1), q \in (2, \infty], 1/r = 3/2 - 1/q$ and take the pair (Y_q, w_q) from Example 2. Let Z_q be a separable space such that Z_q^{**} has a basis and there exists a linear homomorphism φ from Z_q^{**} onto Y_q with the kernel $Z_q \subset Z_q^{**}$ so that the subspace $\varphi^*(Y_q^*)$ is complemented in Z_q^{***} and, moreover, $Z_q^{***} \cong \varphi^*(Y_q^*) \oplus Z_q^*$ (as in the proof of Theorem 2, see [7, Proof of Corollary 1]).

Construct $w_0 \in Y_q^*\widehat{\otimes}_1 Z_q^{**}$ (following the way of the proof of Theorem 2) so that $w_0 \in Y_q^*\widehat{\otimes}_s Z_q^{**}$ for every $s \in (r, 1]$ and $\varphi\circ w_0 = w_q$ (it is possible to apply a "simultaneous lifting" procedure — see Footnote 1 — since w has a form from the proof of the assertion of Example 2). Set $T := w_0\circ\varphi$. From

[1]If $w = \sum_{k=1}^\infty y_k' \otimes y_k$ is any representation of w in $Y_0^*\widehat{\otimes}_r Y_0$ with $(y_k) \in l_r(Y_0)$, then we take $\{z_n''\} \subset Z_0^{**}$ in such a way that the last sequence is absolutely r-summing and $\varphi(z_n'') = y_n$ for every n.

this point, the proof repeats the arguments of the proof of Theorem 2, and we have to mention only: since $Z_q^{***} \cong \varphi^*(Y_q^*) \oplus Z_q^*$, one has that the space Z_q^{***} has the AP_r.

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