Sharp constants in Poincaré, Steklov and related inequalities (a survey)

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On the occasion of the 150th anniversary of V. A. Steklov's birth

The 150th anniversary of the birth of the outstanding Russian mathematician Vladimir Andreevich Steklov falls on 9 January 2014. All over the world, researchers in all areas of mathematics know this name. Indeed, widely known mathematical institutes in Moscow and St. Petersburg are named after Steklov (before the unfortunate recent reform of the Russian Academy of Sciences, they were among its leading institutions). This commemorates the fact that he was the founding father of their predecessor — the Physical-Mathematical Institute established in 1921 in Petrograd (now St. Petersburg). Steklov was the first director of the institute until his unexpected and untimely death on 30 May 1926. Meanwhile, Steklov's scientific contributions (in particular, to analysis, mathematical physics and mechanics) are less known even in present-day Russia. (The reason might be that his papers were published mainly in French.) In this paper, we describe the work of Steklov and his contemporaries on inequalities of mathematical physics and some further advances concerning sharp constants in these inequalities. Steklov's results in other areas and their development are presented in the recent papers [52] and [53].

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The work of Poincaré, Steklov and his disciples

In this section, we outline the early work on inequalities with sharp constants.

One-dimensional inequalities of V. A. Steklov, J. D. Tamarkin and N. M. Krylov

In 1896, Steklov [94] proved that the following inequality

$$\int_0^\ell u^2(x)\,\mathrm{d}x \le \left(\frac{\ell}{\pi}\right)^2 \int_0^\ell [u'(x)]^2\,\mathrm{d}x \tag{1}$$

holds for all functions which are continuously differentiable on $[0,\ell]$ and have zero mean there. For this purpose he used the closedness equation for the Fourier coefficients of u; the corresponding system is $\{\cos(k\pi x/\ell)\}_{k=0}^{\infty}$ normalised on $[0,\ell]$. (One finds similar considerations in [10, Ch. 5, Sect. 11], where the so-called Wirtinger inequality is proved.) Steklov's extensive work on the closedness equation lasted for 30 years until his death. For this reason A. Kneser [48] referred to this equation as "Steklov's favorite formula". It should be mentioned that Steklov introduced the term *closedness equation* for general orthonormal systems much later (see the brief announcement [98] and the full-length paper [99] published in 1910 and 1911, respectively).

Inequality (1) was among the earliest inequalities with sharp constant that appeared in mathematical physics. Steklov applied it to justifying the Fourier method for initial-boundary value problems for the heat equation in two dimensions with variable coefficients independent of time. Later, he also justified the Fourier method for the wave equation under similar assumptions. The fact that the constant in (1) is sharp was emphasized by Steklov in [96, pp. 294–296], where he gave an alternative proof of this inequality. Another result proved in [96, pp. 292–294] says that (1) is true for continuously differentiable functions vanishing at the interval's end-points, and again the constant is sharp. (It is worth mentioning that the latter result appeared in the widely known book [42, Sect. 7.7] without any reference concerning its authorship.) A further generalization of inequality (1) was given by Steklov in [97]. In the monograph [100], the generalized form of (1) is given along with proofs for both types of assumptions about u.

Mitrinović *et al.* [65, Ch. II] investigated the history of (1) and related inequalities. This 48 pages long chapter entitled "An Inequality Ascribed to Wirtinger and Related Results" includes more than 200 references. In particular, the authors cite [96] along with Steklov's note published under the same title in *Comptes Rendus* in 1898. Moreover, it is said that Steklov proved (1) under both conditions guaranteeing its validity; the generalization obtained in [97] is also mentioned. However, the first proof of (1) for functions vanishing at the interval's end-points is ascribed to L. Scheeffer (see [65, p. 67]). Indeed, his paper [88] was published posthumously as early as 1885 (the author died that year aged 26), but it is inaccurate to think that (1) is a result of this note concerned with the simplest problem of variational calculus. Applying the so-called Jacobi transformation to the second variation, Scheeffer obtained as an intermediate formula an identity from which inequality (1) immediately follows. Unfortunately, he, unlike Steklov, did not notice the importance of this inequality and it is not even written explicitly in [88].

Let us turn to results obtained by Steklov's disciples. In his article [104] published in 1910, J. D. Tamarkin (at that time he was a student whom Steklov made interested in boundary value problems of mathematical physics; see [43] and [52]) generalized (1) in the following way. Multiplying two inequalities of this form and combining both conditions imposed on u, he proved that for every function $u \in \mathcal{C}^2([0,\ell])$, satisfying the conditions

$$u(0) = u(\ell) \quad \text{and} \quad \int_0^\ell u(x)\,\mathrm{d}x = 0, \tag{2}$$

the following inequality holds:

$$\int_0^\ell u^2(x)\,\mathrm{d}x \leq \left(\frac{\ell}{\pi}\right)^4 \int_0^\ell [u''(x)]^2\,\mathrm{d}x. \tag{3}$$

It allowed Tamarkin to apply Steklov's method for studying the transversal vibrations of a homogeneous elastic rod.

Note that the constant in (3) is not sharp and this drawback was exterminated by N. M. Krylov — another disciple of Steklov. (He graduated from the St. Petersburg Institute of Mines in 1902 and after studies in Paris and Pisa in 1908–1910 completed his mathematics education through personal contacts with Steklov and by reading his articles.) In his paper [51] published in 1915, Krylov proved that the inequality

$$\int_0^\ell u^2(x)\,\mathrm{d}x \le \left(\frac{\ell}{2\pi}\right)^4 \int_0^\ell [u''(x)]^2\,\mathrm{d}x \tag{4}$$

holds for any function with the following properties:

• conditions (2) are fulfilled;

• u' is absolutely continuous and its Fourier expansion converges uniformly on $[0,\ell]$;

• u'' existing almost everywhere is square integrable.

It is clear that the constant in (4) is sharp which fact was emphasized by Krylov. His proof of this inequality is based on the completeness equation for trigonometric functions, but it is applied in more sophisticated way than in the cases considered by Steklov. Inequality (4) also holds for functions vanishing at the interval's end-points (see [54]), but this result was proved only in 1955.

An inequality ascribed to Wirtinger

What is presented here confirms the Arnold Principle [5]: "If a notion bears a personal name, then this name is not the name of the discoverer".

In the same paper [51], Krylov notes that if u satisfies only the first two of the above listed conditions, then his method gives the following inequality:

$$\int_0^\ell u^2(x)\,\mathrm{d}x \le \left(\frac{\ell}{2\pi}\right)^2 \int_0^\ell [u'(x)]^2\,\mathrm{d}x. \tag{5}$$

Again, the constant is sharp and less than that in (1) which is a consequence of the first condition (2) added to the second one. Krylov also mentions that for $u \in \mathcal{C}^2([0,\ell])$ (this is more restrictive than the second condition imposed by Krylov) satisfying conditions (2) inequality (5) was obtained by E. Almansi [4] in 1905 in connection with his investigation of stability of the equilibrium of the Plateau figures in capillary theory.

However, in accordance with the Arnold Principle, inequality (5) for functions satisfying (2) is usually referred to as *Wirtinger's inequality.* No wonder that in Blaschke's book [13, p. 105] (5) is ascribed to Wirtinger because the latter was Blaschke's teacher. One finds the same attribution and reference to Blaschke's book in [42, Sect. 7.7] and in [10], where Sections 10–13 of Chapter 5 are devoted to this and related inequalities. The question of priority of Wirtinger was discussed by Mitrinović and Vasić in 1969 in their interesting article [66] and again in [65, Ch. II]. One finds only the reference [4] in [66], but in [65] the results obtained by Steklov, Tamarkin and Krylov are also presented.

The Poincaré and Steklov inequalities

In the same volume of the *Communications of the Kharkov Mathematical Society* in which inequality (1) was published, the Steklov's paper [93] had appeared even a little bit earlier. (One has to keep in mind that another article having the same title as [93], namely [95], was published in 1897.) In [93], he considered the following analogue of (1):

$$\int_\Omega u^2 \, \mathrm{d}x \leq C \int_\Omega |\nabla u|^2 \, \mathrm{d}x. \tag{6}$$

Here ∇ stands for the gradient operator and the integral on the right-hand side is called the Dirichlet integral. Assuming that Ω is a bounded three-dimensional domain whose boundary is piecewise smooth and u is a real $\mathcal{C}^1$-function on $\bar{\Omega}$ with zero mean, Steklov found that the sharp constant in (6) is λ_1^{-1}, where λ_1 is the smallest positive eigenvalue of the Neumann Laplacian in Ω:

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega; \qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega.$$

Here $\partial / \partial \mathbf{n}$ stands for differentiation with respect to the exterior unit normal.

Under the same assumptions about u, inequality (6) was first proved by H. Poincaré [82] in 1890 provided Ω is a smooth, convex domain. He also estimated C from above for this class of domains. Moreover, he demonstrated that if a homogeneous, isotropic body occupies a domain for which (6) is valid, then solutions of the heat equation in this domain tend to the equilibrium temperature distribution at the exponential rate. In the second article [83] published by Poincaré on this topic in 1894, he obtained that (6) is true provided Ω is the union of a finite number of smooth, convex domains.

Moreover, he improved and extended his estimate of 1890 for smooth, convex domains; namely, he obtained that

$$C \le \frac{9(\operatorname{diam}\Omega)^2}{16} \quad \text{in three and} \quad C \le \frac{7(\operatorname{diam}\Omega)^2}{24} \quad \text{in two dimensions.}$$

Here $\operatorname{diam}\Omega$ is the diameter of Ω, that is, its maximal chord.

In his article [95] published in 1897, Steklov proved the following new results about (6). First, this inequality is valid provided u is a real $\mathcal{C}^1$-function on $\bar{\Omega}$ vanishing on $\partial\Omega$; again Ω is supposed to be a bounded three-dimensional domain whose boundary is piecewise smooth. Second, under these assumptions the sharp constant in (6) is equal to λ_1^{-1}, but in this case λ_1 is the smallest eigenvalue of

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega; \tag{7}$$

that is, of the Dirichlet Laplacian in Ω.

Further development

The problem of finding and estimating sharp constants in functional inequalities attracted much attention from those who work in theory of functions and mathematical physics (see, for example, the classical monographs [42] and [84]). More than thirty years ago, the role of sharp constants was emphasized in the book [64] by S. G. Mikhlin. Let us quote the review [80]:

> [This book] is devoted to appraising the (best) constants — exact results or explicit (numerical) estimates — in various inequalities arising in "analysis" (=PDE). [...] This is a most original work, a bold attack in a direction where still very little is known.

Our aim is to outline main achievements in this area. Since integral inequalities (as well as integration by parts) are at the heart of theory of differential equations arising in mathematical physics, one might expect that the interest to sharp constants in these inequalities will only intensify in the future.

Scope of this section and preliminary material

We restrict ourselves to the direct generalizations of (1) and (6), that is, to inequalities of the following form:

$$\|u\|_{L^q(\Omega)} \le C\, \|\nabla u\|_{L^p(\Omega)}. \tag{8}$$

Here Ω is a domain in $\mathbb{R}^n$, $n \geq 1$, whereas $p, q \geq 1$ satisfy the following restrictions:

$$\begin{aligned} &q \leq p^* = \frac{np}{n-p}, && \text{if} \quad 1 \leq p < n; \\ &q < \infty, && \text{if} \quad p = n > 1; \\ &q \leq \infty, && \text{if} \quad p > n \quad \text{or} \quad n = 1. \end{aligned}$$

It is assumed that u belongs to $L^{1,p}(\Omega)$, that is, $u \in L^p_{loc}(\Omega)$, its Sobolev derivatives of the first order belong to $L^p(\Omega)$, and $\|\nabla u\|_{L^p(\Omega)}$ is the norm of $|\nabla u|$ in $L^p(\Omega)$.

Weighted inequalities — the Hardy inequality and its generalizations such as the Hardy–Sobolev inequality, the Maz'ya inequality, the Caffarelly–Kohn–Nirenberg inequality — are beyond our scope. We also do not consider inequalities involving derivatives of higher order which received much attention during the past few years.

If u vanishes on $\partial\Omega$ (this is understood as follows: u can be approximated in the norm $\|\nabla u\|_{L^p(\Omega)}$ by smooth functions having compact support in Ω), then (8) is true with some positive constant C for any domain of finite volume[1] and for an arbitrary domain in the critical case $p < n$, $q = p^*$. For these functions, inequality (8) often appears under various names for different values of p and q. In particular, it is referred to as:

- the *Steklov inequality* when $p = q = 2$;
- the *Friedrichs inequality* when $p = q$;
- the *Sobolev inequality* when $p < n$, $q = p^*$.

Note that a slightly different inequality was obtained by K.-O. Friedrichs [39] under the assumption that $\Omega \subset \mathbb{R}^2$. Namely, his inequality is as follows:

$$\int\limits_{\Omega} u^2 \, \mathrm{d}x \leq C \left[\int\limits_{\Omega} |\nabla u|^2 \, \mathrm{d}x + \int\limits_{\partial\Omega} u^2 \, \mathrm{d}S \right], \tag{9}$$

where $\mathrm{d}S$ denotes the element of length of $\partial\Omega$. Generally speaking, (9) holds for all bounded domains in $\mathbb{R}^n$ ($\mathrm{d}S$ denotes the element of area when $n > 2$), for which the divergence theorem is true (see [63, p. 24]). Furthermore, the

[1] This condition is not sharp; in the recent papers [44] and [45], the necessary and sufficient condition is given for the validity of (8) with $p = q$.

Sobolev inequality was proved by S. L. Sobolev himself only for $p > 1$ and E. Gagliardo proved it for $p = 1$ (see [90] and [40], respectively).

Inequality (8) for u with zero mean value over Ω is equivalent to the following

$$\|u - \langle u \rangle\|_{L^q(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \langle u \rangle = \frac{\int_\Omega u(x)\,\mathrm{d}x}{\mathrm{meas}_n\,\Omega} \quad \text{for all } u \in L^{1,p}(\Omega). \tag{10}$$

Here the n-dimensional measure of Ω stands in the denominator. Moreover, some requirements must be imposed on Ω for the validity of (10). Indeed, as early as 1933 O. Nikodým [77] (see also [63, p. 7]) constructed a bounded two-dimensional domain Ω and a function with the finite Dirichlet integral over Ω such that inequality (10) is not true for $p = q = 2$. Another example of a domain with this property is given in [25, Ch. 7, Sect. 8.2] (see also [63, Sect. 6.10.3]). On the other hand, if $p = q$, then (10) (it is called the *Poincaré inequality* in this case) is valid for all domains such that their boundaries are locally graphs of continuous functions in Cartesian coordinates (see, for example, the classical book [25] by R. Courant and D. Hilbert for the proof which can be easily extended from $p = 2$ to any p).

Furthermore, if $p < n$ and $q = p^*$, then (10) (it is called the *Poincaré–Sobolev inequality* in this case) holds for any bounded n-dimensional Lipschitz domain. Moreover, the inequality is true provided Ω belongs to the class of so-called *John's domains* as was proved by B. Bojarski [15]. We recall that this class was introduced by F. John [46] and domains belonging to it are more general than the Lipschitz ones. Finally, if $q \neq p^*$, then (10) holds if and only if $L^{1,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$. This was established by J. Deny and J.-L. Lions [29] for $p = q$, whereas the general case was considered in [73].

Thus, the major point to be clarified about inequality (10) is smoothness of $\partial\Omega$. To a great extent, this was made by V. G. Maz'ya in his comprehensive monograph *Sobolev Spaces* in which he presented his own results and surveyed those of other authors. (Originally this book was published in Russian in 1985 by the Leningrad State University. Recently, the 2nd revised and augmented English edition [63] appeared; its bibliography exceeds 800 entries. Moreover, several sections deal with the question of exact constants in some inequalities.) Proofs of basic facts concerning inequality (10) can be also found in the recent textbook [72]; its English translation is currently in preparation.

Almost everything known about sharp constants in various versions of inequality (8) mainly comes under one of the following four conditions:

- $p = q = 2$ (the quadratic case);
- $\Omega = (0, \ell)$ (the one-dimensional case);
- $p < n$, $q = p^*$ (the critical case);
- $p = q = 1$ (the "geometric" case).

The quadratic case

It was mentioned above that the sharp constant in (8) is $\lambda_1^{-1/2}$ in the quadratic case. Here $\lambda_1 = \lambda_1^D (\lambda_1^N)$ is the smallest positive eigenvalue of the Dirichlet (Neumann, respectively) Laplacian for the Steklov (Poincaré, respectively) inequality. Explicit values of these eigenvalues are known only for several particular domains. Among them, one finds the following (see [84]):

- Rectangle $a \times b$: $\lambda_1^D = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$, $\lambda_1^N = \left[\frac{\pi}{\max\{a,b\}}\right]^2$;
- 45° right triangle: $\lambda_1^D = 5\left(\frac{\pi}{a}\right)^2$, $\lambda_1^N = \left(\frac{\pi}{a}\right)^2$, where a is the leg length;
- 30° right triangle: $\lambda_1^D = \frac{112}{9}\left(\frac{\pi}{a}\right)^2$, $\lambda_1^N = \frac{16}{3}\left(\frac{\pi}{a}\right)^2$, where a is the hypotenuse length;
- Equilateral triangle: $\lambda_1^D = \frac{16}{3}\left(\frac{\pi}{a}\right)^2$, $\lambda_1^N = \frac{16}{9}\left(\frac{\pi}{a}\right)^2$, where a is the side length;
- Disk of the radius a: $\lambda_1^D = \left(\frac{j_{0,1}}{a}\right)^2$, $\lambda_1^N = \left(\frac{j_{1,1}}{a}\right)^2$.

Here $j_{0,1}$ ($j_{1,1}$) is the first positive zero of the Bessel function J_0 (J_1, respectively). The Dirichlet and Neumann eigenvalues for sectors and annuli can also be expressed in terms of Bessel functions.

Furthermore, there are simple formulae for the fundamental eigenvalues in domains that are Cartesian products of two domains of different dimensions. Let $\Omega_1 \subset \mathbb{R}^m$ and $\Omega_2 \subset \mathbb{R}^n$ be bounded domains. If the fundamental Dirichlet and Neumann eigenvalues in Ω_j ($j = 1, 2$) are $\lambda_1^{(j),D}$ and $\lambda_1^{(j),N}$, respectively, then

$$\lambda_1^D = \lambda_1^{(1),D} + \lambda_1^{(2),D} \quad \text{and} \quad \lambda_1^N = \min\{\lambda_1^{(1),N}, \lambda_1^{(2),N}\}$$

are the corresponding eigenvalues in $\Omega = \Omega_1 \times \Omega_2$.

It is worth mentioning that the Dirichlet eigenvalues have the following integral representation (see [85]):

$$\lambda_k^D = \frac{1}{4} \int\limits_{\partial\Omega} \left(\frac{\partial u_k}{\partial \mathbf{n}} \right)^2 \frac{\partial |x|^2}{\partial \mathbf{n}} \, \mathrm{d}S, \quad k = 1, 2, \dots , \tag{11}$$

where u_k is the kth eigenfunction normalized in $L^2(\Omega)$. Formula (11) with $k = 1$ allows us to express the sharp constant in the Steklov inequality in terms of the normalized fundamental eigenfunction of the Dirichlet Laplacian.

In order to estimate λ_1^D one can use its monotonicity with respect to domain variation and the Steiner symmetrization (see [84]). In particular, among all quadrilaterals of the same area the least value of λ_1^D is delivered by the square, whereas the equilateral triangle has the least value of λ_1^D among all triangles of the same area (see [37]). Finally, a ball in $\mathbb{R}^n$ has the least value of λ_1^D among all figures of the same area/volume. In 1877, the two-dimensional version of the last assertion was conjectured by Lord Rayleigh (see [101, pp. 339–340]). It was proved independently by G. Faber [33] and E. Krahn [49], [50]. It must be emphasized that all estimates involving symmetrization for their derivation are true for arbitrary p and q. Thus, under the condition that u vanishes on $\partial\Omega$ the sharp constant in (8) has the largest value for a ball in $\mathbb{R}^n$ (comparing other domains of the same area/volume). Unfortunately, bounds for sharp constants are implicit unless $p = q = 2$.

Less is known about estimates of the first positive Neumann eigenvalue. The classical result of G. Szegő [102] ($n = 2$) and H. F. Weinberger [107] (higher dimensions) says that a ball in $\mathbb{R}^n$ has the largest value of λ_1^N among all domains of the same area/volume (see also [6]). Analogous result for triangles was obtained recently in [56].

A global lower bound for λ_1^N was obtained for *convex* domains by L. E. Payne and H. F. Weinberger [79] ($n = 2$) and by M. Bebendorf [9] ($n \geq 3$); namely, $\lambda_1^N > \big(\frac{\pi}{\operatorname{diam}\Omega}\big)^2$ unless $n = 1$ when Ω is an interval. A generalization of this result for arbitrary $p = q > 1$ was established recently in [35] (see also [32] and [105]).

There are also inequalities between the Dirichlet and Neumann eigenvalues (see, for example, the recent paper [36], where background is also briefly described). Furthermore, it is shown in [91] that if (10) holds in $\Omega_1 \subset \mathbb{R}^m$ and $\Omega_2 \subset \mathbb{R}^n$ with an arbitrary $p = q$ and the sharp constants C_1 and C_2, respectively, then the sharp constant in the same inequality in $\Omega_1 \times \Omega_2$ is less than or equal to $\sqrt{2}\,(C_1 + C_2)$. We also mention the recent survey [81] where, in particular, the results on the upper estimates for sharp constants in the Poincaré inequality in quadratic case on surfaces without boundary are collected.

The one-dimensional case

Without loss of generality we assume that $\Omega = (0,1)$ and begin with the case when u vanishes at the end-points for which the sharp constant in (8) is as follows:

$$C = C_1(p,q) = \frac{\mathfrak{F}\,(q^{-1} + p'^{-1})}{2\,\mathfrak{F}\,(q^{-1})\,\mathfrak{F}\,(p'^{-1})}\,, \tag{12}$$

where $\mathfrak{F}(s) = \frac{\Gamma(s+1)}{s^s}$ and $p' = \frac{p}{p-1}$ is the Hölder conjugate exponent to p. This constant was obtained by E. Schmidt [89] as early as 1940 (the case $p = q$ was considered even earlier by V. I. Levin [58]; see also [42, Sect. 7.6]). This classical result still remains unnoticed by some researchers. It was rediscovered in 2002 (see [11]), whereas its particular case considered in [16] was recently referred to as "the best one in the literature" (see [3]).

The function U delivering the extremal value (12) is symmetric with respect to $x - \frac{1}{2}$, can be expressed in quadratures and is usually referred to as the Lindqvist $\cos_{p,q}$ function (see [59]). Besides, it is well known in the stability theory as the Lyapunov cosine being introduced (for $p = 2$, $q = 2m$, $m \in \mathbb{N}$) by A. M. Lyapunov in 1893 (see [60]).

The one-dimensional Poincaré-type inequality has even more complicated story. It took several years after the pioneering paper [27][2] to establish the following result (see [19], [68] and also the recent paper [41] for a more general problem and a historical survey).

Let $n = 1$ and $\Omega = (0,1)$. If $q \leq 3p$, then the sharp constant in (10) *is equal to $C_1(p,q)$ defined by* (12)*, whereas the corresponding extremal function V is as follows:*

$$V(x) = \begin{cases} U\Big(x + \frac{1}{2}\Big) & \text{when } x \leq \frac{1}{2}, \\ -U\Big(x - \frac{1}{2}\Big) & \text{when } x \geq \frac{1}{2}, \end{cases}$$

where U is Schmidt's function. In particular, V is antisymmetric with respect to $x - \frac{1}{2}$. The constant in (10) *is greater than $C_1(p,q)$ and V has no symmetry provided $q > 3p$.*

[2] In [63, Sect. 1.1.19], the first result for $p = q$ is attributed to A. Stanoyevitch. However, the proof in his PhD thesis (1990) turned out to be incorrect.

Note that a particular case $q \le 2p$ considered in [27] was also rediscovered in 2004 (see [12]).

The critical case

First we note that the sharp constant in the Sobolev inequality is invariant with respect to dilations of the domain Ω. Since it is obviously monotone with respect to inclusion of domains, in fact, it is independent of Ω.

In 1960, V. G. Maz'ya [61] and H. Federer and W. H. Fleming [34] found the sharp constant in the Sobolev inequality with $p = 1$. Its value is as follows:

$$\omega_{n-1}^{-\frac{1}{n}} \cdot n^{\frac{1-n}{n}}, \quad \text{where} \ \ \omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)},$$

the latter is equal to the $(n-1)$-dimensional measure of the unit sphere in $\mathbb{R}^n$.

It was G. Rosen [87], who made the next step in 1971. Namely, he proved that the exact constant in the Sobolev inequality for $n = 3$, $p = 2$ (and $q = 6$) is $2^{\frac{2}{3}}3^{-\frac{1}{2}}\pi^{\frac{2}{3}} \approx 0.4273$.

Four years later, T. Aubin [7] and G. Talenti [103] independently considered the case of arbitrary $n \ge 2$ and $1 < p < n$. It is worth emphasizing that the Bliss inequality [14] and symmetrization — the key ingredients of the proof — were known for a long time before that. The corresponding sharp constant is equal to

$$C_2(n,p) = \omega_{n-1}^{-\frac{1}{n}}\, n^{-\frac{1}{p}} \left(\frac{p-1}{n-p}\right)^{\frac{1}{p'}} \left[\mathfrak{B}\left(\frac{n}{p}, \frac{n}{p'} + 1\right)\right]^{-\frac{1}{n}}, \tag{13}$$

where $\mathfrak{B}$ stands for the Euler beta function. This constant is *not attained* unless $\Omega = \mathbb{R}^n$. In the paper [24] published ten years ago, the constant $C_2(n,p)$ was obtained by virtue of the mass transportation approach (the generalized Monge–Kantorovich problem).

The situation is again more complicated for the Sobolev–Poincaré inequality. It is known that for any John domain the sharp constant is greater than or equal to $2^{\frac{1}{n}} \cdot C_2(n,p)$, where $C_2(n,p)$ is defined by (13). Moreover, if Ω is a $\mathcal{C}^2$-domain and C in (10) is strictly greater than $2^{\frac{1}{n}} \cdot C_2(n,p)$, then the sharp constant *is attained* for this Ω. In particular, for any bounded $\mathcal{C}^2$-domain there exists $\beta > 0$ such that the sharp constant in the Sobolev–Poincaré inequality is attained when $1 < p < \frac{n+1}{2} + \beta$ (see [28] for the proof;

the case $p = 2$ was considered earlier in [2] and [106]). In the survey article [69], the question when the sharp constant is attainable is discussed for various critical inequalities.

The "geometric" case

In some sense, the case $p = 1$ is simpler than those considered above. Indeed, by linearity one rearranges the gradient along level lines and then uses the coarea formula. On the other hand, since $L^1(\Omega)$ is not reflexive, the sharp constant usually is not attained in $L^{1,1}(\Omega)$, and one has to solve the corresponding extremal problem in the space $BV(\Omega)$ of functions with bounded variation.

Results outlined in this section originate from J. Cheeger's insight dating back to his pioneering paper [20]. Twenty years later, A. Cianchi [21] obtained the following expression for the sharp constant in the Poincaré inequality for $p = q = 1$:

$$C = \sup_{F \subset \Omega} \frac{2}{\operatorname{meas}_{n-1} F} \frac{\operatorname{meas}_n E \cdot \operatorname{meas}_n(\Omega \setminus \bar{E})}{\operatorname{meas}_n \Omega}.$$

Here F ranges over all surfaces dividing Ω into two connected subsets E and $\Omega \setminus \bar{E}$; $\operatorname{meas}_{n-1} F$ is the $(n-1)$-dimensional measure of F (generally speaking, its Hausdorff measure).

For functions vanishing on $\partial\Omega$ the sharp constant in (8) with $p = q = 1$ was found by L. Lefton and D. Wei [57] (see also [47]):

$$C = \sup_{E \subset \Omega} \frac{\operatorname{meas}_n E}{\operatorname{meas}_{n-1} \partial E}.$$

Here E ranges over all subsets of Ω such that $\partial E \cap \partial\Omega = \emptyset$.

Let $C(p)$ denote the sharp constant in the Friedrichs inequality, that is, in inequality (8) with $p = q$ valid for u vanishing on $\partial\Omega$. The following estimate $C(p) \le p \cdot C(1)$ was also proved in [57]; the case $p = 2$ was considered earlier by J. Cheeger [20].

Concerning estimates of the constant in the Poincaré inequality, we mention the sharp inequality $C(1) \le \frac{\operatorname{diam}\Omega}{2}$ obtained by G. Acosta and R. Durán [1] for *convex domains*. This L^1-analogue of the Payne–Weinberger–Bebendorf estimate is widely used in studies of finite element approximations. A survey of related results for the "geometric" case can be found, for example, in [78] and in [92].

The "twisted" Steklov–Poincaré inequality

Let us consider inequality (8) for functions u satisfying *both* Steklov and Poincaré conditions, that is, vanishing on $\partial\Omega$ and having zero mean value, respectively. We will refer to the corresponding inequality as the "twisted" Steklov–Poincaré.

In the one-dimensional case, this estimate arises (mainly for $p = 2$) in various applications. We mention just two of them: estimating the fundamental eigenvalue in the Lagrange problem about the shape of strongest column (see, for example, [30]); the characterization problem in nonparametric statistics (see, for example, [76, Sect. 6.2]). It is noted in [68] that the sharp constant in the twisted inequality is the one-half of the sharp constant in the usual Poincaré inequality (10) (cf. (1) and (5) as well).

In the quadratic case considered by L. Barbosa and P. Bérard ([8]), it was shown that the sharp constant is equal to $\lambda_1^{-1/2}$, where $\lambda_1 = \lambda_1^T$ is the smallest positive eigenvalue of the following "twisted" Dirichlet problem:

$$-\Delta u = \lambda u - \langle \Delta u \rangle \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega.$$

In [8], it was proved that this problem has the following property along with some others. Its spectrum interlaces with that of problem (7). In particular, this implies that

$$\lambda_1^D \le \lambda_1^T \le \lambda_2^D. \tag{14}$$

In their paper [38], P. Freitas and A. Henrot showed that if Ω is a pair of equal disjoint balls, then λ_1^T has the least value comparing with those for open sets of the same area/volume. Note that both inequalities (14) are equalities for this Ω. In the recent paper [26], Henrot and his coauthors tried to obtain a similar result for a more general range of values of p and q, but, unfortunately, there is a gap in their proof as is shown in [71].

The case $p = 1$ was considered recently in [17]. As in the quadratic case, a pair of disjoint balls yields the largest sharp constant among all open sets of given area/volume. However, their radii depend on q; namely, if q is close to 1, then the optimal set consists of two equal balls, whereas two different balls give the optimal set for q close to $1^* = \frac{n}{n-1}$.

The “boundary” Poincaré inequality

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and let G be an open part of $\partial\Omega$ possibly coinciding with $\partial\Omega$. Then the “boundary analogue” of inequality (10) is as follows:

$$\|u - \langle u\rangle_G\|_{L^q(G)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad \langle u\rangle_G = \frac{\int_G u(x)\,\mathrm{d}S}{\mathrm{meas}_{n-1}\, G}. \tag{15}$$

This inequality holds for $u \in L^{1,p}(\Omega)$ provided

$$\begin{aligned} &q \leq p^{**} = \frac{(n-1)p}{n-p}, && \text{if} \quad 1 \leq p < n; \\ &q < \infty, && \text{if} \quad p = n; \\ &q \leq \infty, && \text{if} \quad p > n. \end{aligned}$$

In the *quadratic* case (that is, $p = q = 2$), the sharp constant in (15) is again equal to $\lambda_1^{-1/2}$, but now $\lambda_1 = \lambda_1^S$ is the smallest positive eigenvalue of the following mixed (unless $G = \partial\Omega$) Steklov problem:

$$\Delta u = 0 \ \text{ in } \Omega, \quad \frac{\partial u}{\partial \mathbf{n}} = \lambda u \ \text{ on } G, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega \setminus G.$$

We recall that $\mathbf{n}$ is the exterior unit normal existing almost everywhere on $\partial\Omega$. For $n = 2$ ($n = 3$) and particular choices of Ω and G the eigenvalues of the above problem give *sloshing frequencies* of the free oscillations of a liquid in a channel (container, respectively); see, for example, [55, Ch. IX].

In [74], λ_1^S is found for several simple domains with different sets chosen as G. For example, let Ω be a 45° right triangle with leg equal to a, then:

- if G is the hypotenuse, then $\lambda_1^S = \frac{\sqrt{2}}{a}$;
- if G is a leg, then $\lambda_1^S = \frac{z_1^{(1)} \tanh z_1^{(1)}}{a} \approx \frac{2.3236}{a}$, where $z_1^{(1)}$ is the smallest positive zero of $\tan z + \tanh z = 0$;
- if two legs form G, then $\lambda_1^S = \frac{2 z_1^{(2)} \tanh z_1^{(2)}}{a} \approx \frac{1.3765}{a}$, where $z_1^{(2)}$ is the smallest positive zero of $\tan z \cdot \tanh z = 1$.

In [74] (see also [86]), some applications of sharp constants in (10) and (15) are considered. These applications concern quantitative analysis of solutions and a posteriori error estimation for partial differential equations.

When $G = \partial\Omega$, the estimate analogous to that obtained by Szegő–Weinberger was found by R. Weinstock [108] ($n = 2$) and by F. Brock [18] (higher dimensions). Namely, a ball in $\mathbb{R}^n$ has the largest value of λ_1^S among all domains of the same area/volume.

It should be emphasized that in the *critical* case (that is, $p < n$, $q = p^{**}$) the sharp constant in (15) is related to that in the following *trace Sobolev inequality* for the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,:\, x_n > 0\}$:

$$\|u(\cdot, 0)\|_{L^{p^{**}}(\mathbb{R}^{n-1})} \le C_3(n,p)\, \|\nabla u\|_{L^p(\mathbb{R}^n_+)}, \quad u \in L^{1,p}(\mathbb{R}^n_+). \tag{16}$$

In particular, $C_3(n,p) = 1$ for $p = 1$ which follows from [62, Sect. 1.3].

J. F. Escobar [31] conjectured that if $p > 1$ in (16), then the extremal function is equal to $|x - x^*|^{-(n-p)/(p-1)}$, where $x^* \notin \mathbb{R}^n_+$ is arbitrary, but he proved this assertion only for $p = 2$. Afterwards, the general case was established in the remarkable paper [67] based on the mass transportation approach (see also [70]). This result implies that

$$C_3(n,p) = \left(\frac{p-1}{n-p}\right)^{\frac{1}{p'}} \left[\frac{\omega_{n-2}}{2}\, \mathfrak{B}\Big(\frac{n-1}{2}, \frac{n-1}{2(p-1)}\Big)\right]^{-\frac{1}{(n-1)p'}}.$$

As for the Sobolev–Poincaré inequality, the following is true in the critical case. The sharp constant in (15) is greater than or equal to $C_3(n,p)$ for Lipschitz domains. Moreover, if Ω is a $\mathcal{C}^2$-domain and $C > C_3(n,p)$, then the sharp constant *is attained* for this Ω. In particular, for any bounded $\mathcal{C}^2$-domain in $\mathbb{R}^n$, $n \ge 3$, there exists $\delta > 0$ such that the sharp constant is attained for $1 < p < \frac{n+1}{2} + \delta$ (see [75] for the proof).

Finally, we mention two recent papers dealing with the *"geometric"* case $p = q = 1$. In the first of them [22], A. Cianchi obtained the following formula for the sharp constant in (15):

$$C = \frac{2}{\operatorname{meas}_{n-1} \partial\Omega} \sup_{E \subset \Omega} \frac{\operatorname{meas}_{n-1}(\partial E \cap \partial\Omega) \cdot \operatorname{meas}_{n-1}(\partial\Omega \setminus \partial E)}{\operatorname{meas}_{n-1} \partial E \cap \Omega},$$

where E ranges over all subdomains of Ω with Lipschitz boundary. He also found the sharp constant for balls. It turns out that for $n \ge 3$ the optimal choice of E is a half-ball and $C = \frac{(n-1)\omega_{n-1}}{2\omega_{n-2}}$, whereas $C = 2$ and the supremum is *not attained* for $n = 2$.

In the second paper [23], it is proved that the least sharp constant for Lipschitz domains is attained for balls. (Note that the "geometric" case here is, at the same time, the critical one, and so the sharp constant depends on the shape of Ω, but not on its size.) Moreover, for $n \geq 3$ balls are the only optimal domains, whereas if $n = 2$, then some nearly circular stadium-shaped domains yield the same value of the sharp constant.

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