Operator frames in Banach spaces

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Definition

A system $\mathcal{F} := ((x'_k)_{k=1}^{\infty}, (y_k)_{k=1}^{\infty})$ is an O-frame (*operator frame*) for $T \in L(X,Y)$, if for every $x \in X$ the series $\sum_{k=1}^{\infty} \langle x'_k, x\rangle y_k$ converges in Y and

$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x\rangle y_k, \quad x \in X.$$

Examples. 1. $\Delta : l_\infty \to l_1$, a diagonal, $(\delta_k) \in l_1$. Then $\Delta x = \sum \delta_k \langle e_k, x\rangle e_k$.
2. X has a basis $(f_k)_{k=1}^{\infty}$ If $T : X \to W$, then

$$Tx = \sum_{k=1}^{\infty} \langle f'_k, x\rangle Tf_k, \quad x \in X.$$

3. W has a basis (w_k). If $T : X \to W$, then $\langle Tx, w'_k\rangle = \langle x, T^* w'_k\rangle$, hence $Tx = \sum_{k=1}^{\infty} \langle T^* w'_k, x\rangle w_k$.
4. If X (or Y) is separable and has BAP. Every $T \in L(X,W)$ has O-frame.

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4. If X (or Y) is separable and has BAP. Every $T \in L(X, W)$ has O-frame.

Ideal property

Proposition

$T \in L(X, W)$, $A \in L(W, V)$, $B \in L(Z, X)$. If T has an O-frame, then $ATB : Z \to V$ has an O-frame.

Corollary

If $T \in L(X, W)$ factors through a Banach space with a basis, then T has an O-frame.

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If $T \in L(X, W)$ factors through a Banach space with the BAP, then T has an O-frame.

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One more property:

Proposition

Let $\mathcal{F} := ((x'_k), (w_k))$ be an O-frame for $T \in L(X.W)$. Then the dual system $\mathcal{F}^d := ((w_k), (x'_k))$ is a weak* O-frame for T^*, i.e.

$$T^* w' = w^*\text{-}\lim_N \sum_{k=1}^{N} \langle w', w_k \rangle x'_k, \quad w' \in W^*.$$

Proof. For $w' \in W^*$ and $x \in X$ we have:

$$\langle Tx, w' \rangle = \langle \sum_{k=1}^{\infty} \langle x'_k, x \rangle w_k, w' \rangle = \langle \sum_{k=1}^{\infty} \langle w', w_k \rangle x'_k, x \rangle,$$

hence $T^* w' = w^*\text{-}\lim_N \sum_{k=1}^{N} x'_k \langle w', w_k \rangle$ (the limit is in the topology $\sigma(X^*, X)$).

Dual situation

One more property:

Proposition

Let $\mathcal{F} := ((x_k'), (w_k))$ be an O-frame for $T \in L(X.W)$. Then the dual system $\mathcal{F}^d := ((w_k), (x_k'))$ is a weak O-frame for T^*, i.e.*

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Dual O-frame

Definition

O-frame $((x'_k),(w_k))$ *for* T *is shrinking if for every* $w' \in W^*$ *the norm* $||\sum_{k=n+1}^{\infty} x'_k \langle w', w_k \rangle|| \to 0$ *as* $n \to \infty$.

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Boundedly complete O-frame

Definition

O-frame $((x'_k), (w_k))$ *for* T *is boundedly complete if for every* $x'' \in X^{**}$ *the series* $\sum_{k=1}^{\infty} \langle x'', x'_k \rangle w_k$ *converges in the space* W.

Proposition

Let $\mathcal{F} := ((x'_k), (w_k))$ *be an O-frame for* $T \in L(X.W)$. *TFAE:*
1) *O-frame* $\mathcal{F}$ *is boundedly complete;*
2) *for every* $x'' \in X^{**}$, *it follows from the boudedness of the partial sums* $(\sum_{k=1}^{N} \langle x'', x'_k \rangle w_k)_{N=1}^{\infty}$ *the convergence of the series* $\sum_{k=1}^{\infty} \langle x'', x'_k \rangle w_k$ *in the space* W.

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O-frames and weak compactness

Theorem

Let $\mathcal{F} := ((x'_k), (w_k))$ be an O-frame for $T \in L(X.W)$. If this O-frame $\mathcal{F}$ is boundedly complete and shrinking, then the operator T is weakly compact.

O-frames and basis-factorization

Theorem

Let $T \in L(X, W)$. TFAE:

1) *T has an O-frame;*

2) *the operator T factors through a Banach space with a basis;*

3) *T factors through a Banach sequence space with a basis.*

Proof.

T, O-frame $\mathcal{F} := ((x'_k), (w_k))$, $w_k \neq 0$.
$\exists\, K > 0 : \ \forall N \,||\sum_{k=1}^N x'_k \otimes w_k|| \leq K$.
$t := \{a = (a_k)_{k=1}^\infty : \text{ series } \ \sum_{k=1}^\infty a_k w_k \text{ converges in } W\}$,
$|||a|||_t := \sup_N ||\sum_{k=1}^N a_k w_k|| \ \ (\geq \lim_N ||\sum_{k=1}^N a_k w_k||)$. For
$a = (a_1, a_2, \dots, a_{N+s}, 0, 0, \dots)$, $|||\sum_{k=1}^N a_k e_k||| \leq |||\sum_{k=1}^{N+s} a_k e_k|||$
and the linear span of $(e_k)_{k=1}^\infty$ is dense in t. Thus, (e_k) is a monotonr basis in the Banach space t. If $j : t \to W$ is a natural map $a \mapsto \sum_{k=1}^\infty a_k w_k$, then $||j|| \leq 1$. Set $Ax := (\langle x'_k, x\rangle)_{k=1}^\infty$; then $Ax \in t$. Furthermore,

$$|||Ax|||_t = \sup_N ||\sum_{k=1}^N \langle x'_k, x\rangle w_k|| \leq K\,||x||, \quad \forall\, x \in X.$$

Thus, $A \in L(X, t)$ and $T = jA : X \to t \to W$. □

Unconditional O-frames

Definition

Let $T \in L(X, W)$, $(x'_k)_{k=1}^{\infty} \subset X^*, (w_k)_{k=1}^{\infty} \subset W$. We say that $\mathcal{F} := ((x'_k)_{k=1}^{\infty}, (w_k)_{k=1}^{\infty})$ is an UO-frame (unconditional operator frame) for T, if for every $x \in X$ the series $\sum_{k=1}^{\infty} \langle x'_k, x\rangle w_k$ converges unconditionally in W and

$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x\rangle w_k, \quad x \in X.$$

Theorem

Let $T \in L(X, W)$. TFAE:
1) T has a UO-frame;
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O-frames and bounded approximation property

Definition

Let $T \in L(X, W)$, $C \geq 1$. We say that T has the C-BAP if for every compact subset K of X, for every $\varepsilon > 0$ there is a finite rank operator $R : X \to W$ such that $||R|| \leq C\,||T||$ and $\sup_{x \in K} ||Rx - Tx|| \leq \varepsilon$. T has the BAP, if it has the C-BAP for some $C \in [1, \infty)$.

Theorem

Let X be a separable Banach space, W be any Banach space and $T \in L(X, W)$. TFAE:

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Let X be a separable Banach space, W be any Banach space and $T \in L(X, W)$. TFAE:

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- **Comparing the usual Banach frames with O-frames.**
- Known: If X has an unconditional Banach frame, then:
  1. The frame is shrinking iff X does not contain l_1 iff X is almost reflexive.
  2. X is reflexive iff it does not contain both l_1 and c_0.

  For O-frames, the situation is different. We have

Example

There exists an operator $T : l_1 \to C[0,1]$ such that
1. T is conditionally weakly compact and, thus, does not contain l_1. T has no shrinking O-frame.
2. T does not contain also c_0, but is not weakly compact.

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Comparing Banach frames and O-frames

- Trivially, If X does not have the approximation property, then it can not have a Banach frame.

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Example

There exist two separable reflexive Banach spaces X, Y and and an operator $T : X \to Y$ so that:
Both X and Y do not have the approximation property, but T has an unconditional O-frame.

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Example

There exists a weakly compact operator, which has the approximation property, but has no operator frame.

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Thank you for your attention!