On nuclearity of operators with s-nuclear adjoints

Oleg Reinov

Nuclear operators

An operator $T : X \to Y$ is nuclear if it is of the form

$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle y_k$$

for all $x \in X$, where $(x'_k) \subset X^*, (y_k) \subset Y$, $\sum_k ||x'_k|| \, ||y_k|| < \infty$.
We use the notation $N(X, Y)$

If T is nuclear, then T^* is nuclear.

A. Grothendieck, Produits tensoriels topologiques et espases nucléaires, Mem. Amer. Math. Soc., Volume 16, 1955, 196 + 140.

Suppose T is a bounded linear operator acting between Banach spaces X and Y, Is it true that if T^* is nuclear then T is nuclear too?

As is well known, a negative answer was obtained already by T. Figiel and W.B. Johnson in:

T. Figiel, W.B. Johnson, The approximation property does not imply the bounded approximation property, Proc. Amer. Math. Soc., Volume 41 (1973), 197–200.

Grothendieck's Approximation

Definition

$X \in AP$ iff

$$\forall\ Y, \forall\ \ compact\ \ K \subset X,\ \forall\ \varepsilon > 0,\ \forall\ T : X \to Y,$$

$$\exists\ R \in X^* \otimes Y :\ \sup_{x \in K} ||Rx - Tx|| \leq \varepsilon.$$

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- Or, the same:

Definition

$X \in AP$ iff for every $(x_n) \in c_0(X)$ and for every $\varepsilon > 0$ there exists a finite rank operator R in X such that $\sup_n ||Rx_n - x_n|| \leq \varepsilon$.

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First part of "la proposition 15,2; chap. I, p. 86":

Case $\mathbf{X}^*$. Let $T \in L(X, Y)$ and assume that X^* has the AP. If $T^* \in N(Y^*, X^*)$, then $T \in N(X, Y)$.

A proof can be found in

J. Diestel and J. J. Uhl Jr., Vector measures, American Mathematical Society, Providence, RI, 1977.

Grothendieck's Approximation

Second part of "la proposition 15,2; chap. I, p. 86":
Case Y**. Let $T \in L(X, Y)$ and assume that $Y^{**} \in AP$. If $T^* \in N(Y^*, X^*)$, then $T \in N(X, Y)$.

Eve Oja

E. Oja, O.I. Reinov, Un contre-exemple à une affirmation de A.Grothendieck, C. R. Acad. Sc. Paris. — Serie I, Volume 305 (1987), 121–122.

- **I.** Let $T \in L(X, Y)$ and assume that $Y^{***} \in AP$. If $T^* \in N(Y^*, X^*)$, then $T \in N(X, Y)$.
- **II.** There exist Banach spaces X, Y and a non-nuclear operator $T : X \to Y$ so that X and Y^{**} have the metric approximation property and T^* is nuclear.

E. Oja, O.I. Reinov, Un contre-exemple à une affirmation de A.Grothendieck, C. R. Acad. Sc. Paris. — Serie I, Volume 305 (1987), 121–122.

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s-nuclear operators – Applications de puissance p.éme sommable

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for all $x \in X$, where
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- A. Hinrichs, A. Pietsch, p-nuclear operators in the sense of Grothendieck, Math. Nachr., Volume 283, No. 2 (2010), 232–261.

We are interested in the following question [Problem 10.1]: *Suppose T is a (bounded linear) operator acting between Banach spaces X and Y, and let $s \in (0, 1)$. Is it true that if T^* is s-nuclear then T is s-nuclear too?*

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It is not difficult to see that if T^* is s-nuclear, then T is p-nuclear with $1/s = 1/p + 1/2$.
This is the best possible general result one can obtain without imposing any conditions on the Banach spaces involved. The sharpness of the assertion $1/s = 1/p + 1/2$, for $s \in (2/3, 1]$, can be seen, for instance, in

O.I. Reinov, Approximation properties $\mathrm{AP_s}$ and p-nuclear operators (the case $0 < s \leq 1$), Journal of Mathematical Sciences, Volume 115, No. 2 (2003), 2243-2250. [Zapiski Nauchnykh Seminarov POMI, Vol. 270, 2000, pp. 277-291.]

So, we consider a slightly different question: *Under which conditions on the Banach spaces involved is it valid that* $(*)$ *an operator* $T \in L(X, Y)$ *is nuclear if its adjoint* T^* *is* s*-nuclear?*

To formulate the theorem, we need a definition:

- Let $0 < q \leq \infty$ and $1/s = 1/q + 1$. We say that X has the approximation property of order s, if for every $(x_n) \in l_q(X)$ (where $l_q(X)$ means $c_0(X)$ for $q = \infty$) and for every $\varepsilon > 0$ there exists a finite rank operator R in X such that $\sup_n ||Rx_n - x_n|| \leq \varepsilon$.
- **Theorem 1.** Let $s \in (0, 1]$, $T \in L(X, Y)$ and assume that either $X^* \in AP_s$ or $Y^{***} \in AP_s$. If $T \in N_s(X, Y^{**})$, then $T \in N_1(X, Y)$.

  In other words, under these conditions, from the s-nuclearity of the conjugate operator T^*, it follows that the operator T is nuclear.

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  In other words, under these conditions, from the s-nuclearity of the conjugate operator T^*, it follows that the operator T is nuclear.

Theorem

The examples in the following result show that the condition "X^* or Y^{***} has the approximation property of order s" is essential.

- **Theorem.** For each $s \in (2/3, 1]$ there exist a Banach space Z_s and a non-nuclear operator $T_s : Z_s^{**} \to Z_s$ so that Z_s^{**} has the metric approximation property, Z_s^{***} has the AP_r for every $r \in (0, s)$ and T_s^* is s-nuclear.
- *Remark*: The space Z_1^{***} is isomorphic to a space of type $Z_1^* \oplus E$, where E is an asymptotically Hilbertian space. This gives us one more example of an asymptotically Hilbertian space which fails the approximation property.

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Example we use

- Let $r \in (2/3, 1], q \in [2, \infty), 1/r = 3/2 - 1/q$. There exist a separable reflexive Banach space Y_0 and a tensor element $w \in Y_0^* \widehat{\otimes}_r Y_0$ so that $w \neq 0, \tilde{w} = 0$, the space Y_0 (as well as Y_0^*) has the AP_s for every $s < r$ (but, evidently, does not have the AP_r). Moreover, Y_0 is of type 2 and of cotype q_0 for any $q_0 > q$.
- For $q = 2$ (that is, $r = 1$), the space Y_0 is a subspace of a space of the type $\left(\sum_j l_{p_j}^{k_j}\right)_{l_2}$ with $p_j \searrow 2$ and $k_j \nearrow \infty$. Every such space is an asymptotically Hilbertian space (for definitions and some discussion, see

P. G. Casazza, C. L. García, W. B. Johnson, An example of an asymptotically Hilbertian space which fails the approximation property, Proc. Amer. Math. Soc,, Volume 129, No. 10 (2001), 3017-3024.

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Reference

O. I. Reinov, On linear operators with s-nuclear adjoints, $0 < s \leq 1$, J. Math. Anal. Appl., Volume 415 (2014) 816-824.

Thank you for your attention!