Stability by linear approximation for time scale dynamical systems.

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Abstract

We study systems on time scales that are generalizations of classical differential or difference equations. In this paper we consider linear systems and their small nonlinear perturbations. In terms of the timescale and eigenvalues of a constant matrix we formulate conditions, sufficient for stability by linear approximation. We demonstrate that, like in classical cases, those conditions are close to necessary ones. For non-constant matrices and/or non-periodic time scales we use techniques of central upper Lyapunov exponents (a common tool of the theory of linear ODEs) to study stability of solutions. Also, time scale versions of the famous Chetaev theorem on conditional instability are proved.

Keywords: time-scale system, linearization, Lyapunov function, stability.

1 Introduction

We study dynamic equations on time scales e.g. on unbounded closed subsets of $\mathbb{R}$. The time-scale approach was quickly developing during last decades. The first advantage is that it gives a common language that fits both for flows and diffeomorphisms. On the other hand, there are many numerical methods that correspond to non-uniform steps. Especially, this is applicable for modeling non-smooth or strongly non-linear dynamical systems.

Consider a motion of a particle in two distinct media, e.g. water and air. Evidently, to model such system, it is not effective to use equidistant nodes. It is better to take more of them inside time periods, corresponding to motions in water. This is a natural way to obtain a non-trivial time scale in a real life problem.

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In this sense it seems to be useful to generalize some results on stability theory, well-known for ODEs for time-scale case. Mainly, we consider a general linear system (autonomous or non-autonomous) and its uniformly small non-linear perturbation. For the continuous dynamics, there exists a well-developed theory on stability by first approximation. For autonomous case there are classical stability criteria related to eigenvalues of a matrix A.

For non-autonomous systems and for uncertain cases there might be two approaches. The first one is based on the theory of Lyapunov functions. The second one involves integral inequalities where the most important is Grönwall–Bellman inequality. There is a very powerful tool that allows to find stability of solution via so-called central Lyapunov exponents.

Exponential stability of a time-varying dynamic equations on a time scale have been investigated by many authors. We mention recent papers by Bohner and Martynyuk [1] (this article is also a good introduction to theory of time-scale systems), Du and Tien [2], Hoffacker and Tisdell [3] and Martynyuk [4].

They have studied asymptotic properties of time-scale systems via Lyapunov functions and integral inequalities.

However, the following problems were open by now.

1. For constant matrices A, are there any criteria on stability by first approximation?

2. Is there any analog of Chetaev theorem on instability by first approximation for time-scale systems?

3. Are there any sufficient conditions on stability by first approximation, close to necessary ones?

In our paper we give positive answers to all these questions. The main idea of our paper is very simple: methods of classical theory of linear non-autonomous differential equations are applicable for time-scale systems.

More precisely, we have two principal aims. First, we would like to provide sufficient conditions on stability by first approximation. We demonstrate that the obtained conditions are close to necessary ones. In our proofs, we use the techniques of central upper Lyapunov exponents. This approach is novel for time scale analysis. Secondly, we prove two analogs of Chetaev theorem on instability by first approximation. Specifics of time scales demands a novel, non-classical approach to proof since, generally speaking, we cannot use tools of the theory of autonomous systems, any more.

The paper is organized as follows. In Section 2 we give a brief introduction to time-scale analysis mostly related to the concept of Δ – derivative. In Section 3, we give a review on existing results on stability of difference equations. Section 4 is very short, we study how properties of solutions linear autonomous systems are related with eigenvalues of the matrix of coefficients. In Section 5 we introduce one of the main characters of our paper: central upper exponent. Using this concept and Grönwall–Belmann lemma, we transfer the criterium of stable hyperbolicity for non-autonomous ODEs to the time-scale case. In Section 6, similarly to what happens in ODEs theory, we construct Lyapunov functions for time-scale systems as quadratic forms and thus relate stability by first approximation with certain estimates on eigenvalues of the matrix of coefficients. The rest of the paper concerns instability. In Section 7 we provide a time-scale generalization of the classical Millionschikov's result on accessibility of the upper center exponent. As a corollary, we deduce a time scale version of the criterium on instability by first approximation [5]. In Section 8 we give two time-scale version of Chetaev Theorem and a criterium on instability by first approximation. For reuslts of Section 7 we need some details of the original Millionschikov's proof so we reproduce it in appendix. Then Conclusion and Discussion section is given.

2 Time-scale analysis

Let a time scale $\mathbb{T}$ be an unbounded closed subset of $\mathbb{R}$ with the inherited metrics. We use following notions: $[a,b]_T = \{t \in \mathbb{T}; a \le t \le b\}$ provided $a, b \in \mathbb{T}$, $a < b$. We set $\mathbb{T}_a^+ = [a,\infty) \bigcap \mathbb{T}$. We introduce now some basic notions connected to the theory of time scales, which summarize the material from the recent book by Bohner and Peterson [10].

Introduce some other notions. Let $\mathbf{M}_{n,n}$ be the space of $n \times n$ complex matrices, $|\cdot|$ stand for a vector norm in $\mathbb{R}^n$ or $\mathbb{C}^n$ and the corresponding operator norm. We consider two spaces of matrix functions: $\mathcal{M}_R$ that is a space of continuous functions $A : \mathbb{R} \to \mathbf{M}_n$ and $\mathcal{M}_T$ that is the space of similarly defined functions $P : \mathbb{T} \to \mathbf{M}_n$.

Definition 1. Let $t \in \mathbb{T}$. We define the *forward jump* operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T}; s > t\}$ and backward jump operator $\rho(t) := \sup\{s \in \mathbb{T}; s < t\}$.

In this definition, we set $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $\sigma(t) = t$, then t is called right- dense, and if $t > \inf T$ and $\rho(t) = t$, then t is called left-dense. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$.

Definition 2. The function $f : \mathbb{T} \to \mathbb{R}$ is called Δ - *differentiable* at a point $t \in \mathbb{T}$ if there exist $\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a W–neighborhood of $t \in \mathbb{T}$ satisfying

$$|[f(\sigma(t)) - f(s)] - \gamma[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|$$

for all $s \in W$. In this case we write $f^\Delta(t) = \gamma$. When $\mathbb{T} = \mathbb{R}$, $x^\Delta(t) = \dot{x}(t)$. When $\mathbb{T} = \mathbb{Z}$,

$x(t)$ is the standard forward difference operator $x(n+1)-x(n)$.
Definition 3. A function $f:\mathbb{T}\to\mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and left-sided limits exists (finite) at left-dense points in $\mathbb{T}$. Denote the class of rd-continuous functions by $\mathcal{C}_{rd}=\mathcal{C}_{rd}(\mathbb{T})=\mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$.
Definition 4. If $F^{\Delta}(t)=f(t)$, $t\in\mathbb{T}$, then F is a Δ - *antiderivative* of f and the Cauchy Δ - integral is given by

$$\int_{\tau}^{s} f(t)\Delta t = F(s)-F(\tau), \qquad \text{for all} \quad s,\tau\in\mathbb{T}.$$

The following result has first been proved in [10].
Theorem A. *Let $t_0\in\mathbb{T}$ and $w:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ be continuous at (t_0,t), for all $t\in\mathbb{T}$ with $t>t_0$. Assume that $w^{\Delta}(t,.)$ is rd-continuous on $[t_0,\sigma(t)]$. Suppose that for each $\varepsilon>0$ there exists a neighborhood U of t, not depending on $\tau\in[t_0,\sigma(t)]$, such that*

$$|w(\sigma(t),\tau)-w(s,\tau)-w^{\Delta}(t,\tau)(\sigma(t)-s)|\le\varepsilon|\sigma(t)-s| \qquad \forall s\in U,$$

where w^{Δ} denotes the derivative of w with respect to the first variable. Then

$$g(t)=\int_{t_0}^{t} w(t,\tau)\,\Delta\tau$$

implies

$$g^{\Delta}(t)=\int_{t_0}^{t} w^{\Delta}(t,\tau)\Delta\tau + w(\sigma(t),t).$$

Definition 5. A function $p:\mathbb{T}\to\mathbb{R}$ is called *regressive* provided that $1+\mu(t)p(t)\neq 0$ for all $t\in\mathbb{T}$. The set of all regressive and rd-continuous functions is denoted by $\mathcal{R}=\mathcal{R}(\mathbb{T},\mathbb{R})$. The set $\mathcal{R}^{+}$ of all positively regressive function is

$$\mathcal{R}^{+}=\{p\in\mathcal{R}:1+\mu(t)p(t)>0 \qquad \forall t\in\mathbb{T}\}.$$

A matrix mapping $A:\mathbb{T}\to\mathbf{M}_n(\mathbb{R})$ is called *regressive* if for each $t\in\mathbb{T}$ the $n\times n$ matrix $E+\mu(t)A$ is invertible, where E the identity matrix. The matrix function $A(t):\mathbb{T}\to M_{n,n}$ is called *uniformly regressive* if the matrix function $(E+\mu(t)A(t))^{-1}$, is bounded.

We use the cylinder transformation to define a generalized exponential function for an arbitrary time scale $\mathbb{T}$.
Definition 6. If $p\in\mathcal{R}(\mathbb{T},\mathbb{R})$, then define *the generalized exponential function* $e_p(t,s)$ by

$$e_p(t,s)=\exp\left(\int_{s}^{t}\xi_{\mu(\tau)}p(\tau)\right)\Delta\tau \tag{1}$$

where $\xi_h z$ is the cylinder transformation given by

$$\frac{\log(1+zh)}{h}$$

if $h \neq 0$ and z if $h = 0$.
Remark. Consider the regressive dynamic initial value problem

$$x^{\Delta}(t) = p(t)x(t), \qquad x(t_0) = x_0, \qquad t_0 \in T. \tag{2}$$

The exponential function $x(t) = e_p(t, t_0)x_0$ is the unique solution of (2). For what follows, we need several theorems given below.
Theorem B (Comparison Theorem). *Let $t_0 \in \mathbb{T}$, $x, f \in \mathcal{C}_{rd}$ and $p \in \mathcal{R}^+$. Then $x^{\Delta}(t) \leq p(t)x(t) + f(t)$, for all $t \in \mathbb{T}^+_{t_0}$ implies*

$$x(t) \leq x(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau, \qquad \forall t \in \mathbb{T}^+_{t_0}.$$

It is clear from the proofs of the last two results by Bohner and Peterson [10] that in each case, reversing the inequalities in the assumptions yields corresponding lower (instead of upper) estimates for the solution.
Remark. If p is an rd-continuous function, then

$$1 + \int_a^t p(u)\Delta u \leq e_p(t, a) \leq \exp\left(\int_a^t p(u)\Delta u\right) \qquad \forall t \in \mathbb{T}^+_a.$$

Later on we always assume that matrices of considered linear systems are regressive so that any Cauchy problem for the corresponding linear homogenous system has a unique forward solution.
Definition 7. Let $t_0 \in \mathbb{T}$. The unique matrix-valued solution of the initial value problem

$$X^{\Delta}(t) = A(t)X(t), \qquad X(t_0) = E_n, \tag{3}$$

where $A \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}, \mathbf{M}_n(\mathbb{R}))$ is called *the matrix exponential function* and it denoted by $\Phi_A(t, t_0)$. Accordingly, the matrix function $\Phi_A(t, t_0)$ possesses following two properties:

$$\Phi^{\Delta}_A(t, t_0) = A(t)\Phi_A(t, t_0), \qquad \Phi_A(t_0, t_0) = E_n.$$

This matrix function is referred to as the state transition matrix, and our assumption in the nature of $A(t)$ turns out that the state transition matrix exists and is unique.
Theorem C. *Suppose $A, B \in \mathcal{C}_{rd}(\mathbb{T}, \mathbf{M}_n(\mathbb{R}))$ are regressive matrix-valued functions on $\mathbb{T}$, then*

(i) $\Phi_A(t,r)\Phi_A(r,s) = \Phi_A(t,s)$ *for* $r,s,t \in \mathbb{T}$ *such that all terms of the expession above are well-defined;*

(ii) $\Phi_A(\sigma(t),s) = (E + \mu(t)A(t))\Phi_A(t,s)$;

(iii) If $\mathbb{T} = \mathbb{R}$ *and* A *is constant, then* $\Phi_A(t,s) = \exp(A(t-s))$*;*

(iv) If $\mathbb{T} = h\mathbb{Z}$*,with* $h > 0$*,* $t,s \in \mathbb{T}$ *and* A *is constant, then* $\Phi(t,s) = (E + hA)^{\frac{t-s}{h}}$.

3 Types of stability

Let us consider a linear system

$$x^{\Delta} = A(t)x \tag{4}$$

and its nonlinear perturbation

$$x^{\Delta} = A(t)x + f(t,x). \tag{5}$$

Definition 8.

a) System (4) is said to be *stable* if, for every $t_0 \in \mathbb{T}$ and for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $|x_0| = |x(t_0)| < \delta$ implies

$$|x(t,t_0,x_0)| < \varepsilon, \tag{6}$$

for all $t \in T^+_{t_0}$.

b) System (4) is said to be *uniformly stable* if it is stable and for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ independent on initial point t_0, such that inequality (6) is satisfied.

c) System (4) is said to be *uniformly asymptotically stable* if it is uniformly stable and it is uniformly attractive, i.e., there exists a positive constant c, independent of t_0, such that $|x_0| < c$ implies $x(t,t_0,x_0) \to 0$, $t \to +\infty$ uniformly w.r.t. t_0.

S.K.Choi and al. [6] proved that the stability of (4) is equivalent to the boundedness of all its solutions when $A \in \mathcal{R}(\mathbb{T}, \mathbf{M}_n(\mathbb{R}))$. Also, DaCunha [7] proved that the uniform stability of (3) is equivalent to the uniform boundedness of all its solutions with respect to the initial state (t_0, x_0), when $A \in \mathcal{R}(\mathbb{T}, \mathbf{M}_n(\mathbb{R}))$. He obtained the following characterization of uniform stability by means of the operator norm.

Theorem D (DaCunha, [7]). *Linear system* (4) *is uniformly stable if and only if there exists a positive constant* γ*, such that* $|\Phi_A(t, t_0)| \leq \gamma$*, for all* $t \in T^+_{t_0}$*. It is uniformly exponentially stable if and only if there exist a positive constants* λ *and* $\gamma \geq 1$ *independent on initial point* t_0*, such that* $|\Phi_A(t, t_0)| \leq \gamma \exp(-\lambda(t - t_0))$*, holds for all* $t \in \mathbb{T}^+_{t_0}$*.*

4 Uniformly regressive autonomous linear systems

Let $\mathbb{T}$ be a time scale. Let Δ and $\mu(t)$ be the corresponding graniness function, A be a constant $n \times n$ matrix.

Denote eigenvalues of the matrix A by λ_k.

Lemma 1. *The following two statements are equivalent.*

1.
$$|\lambda_k \mu(t) + 1| > 0. \tag{7}$$

2. *System*
$$x^\Delta = Ax, \qquad x \in \mathbb{R}^n \tag{8}$$
is uniformly regressive.

Particularly, solutions of Eq. (8) *are unique and have finite Lyapunov exponents.*

To prove this lemma, it suffices to reduce system (8) to the normal form and thus reduce it to a set of linear first order equations.

Later on, we always assume that inequality (7) is true unless the opposite statement is specified.

5 Stability. Grönwall-Bellmann approach.

Unlike cases diffeomorphisms and flows, it seems that autonomous systems do not play that important role in general timscale dynamics. The reason is trivial, generally speaking, time scale is not invariant with respect to shifts.

The principal aim of this section is to establish a criterium on stability of a solution of a time scale system by first approximation. We use a tool, well-known in the theory of linear systems that is central upper exponents.

Consider systems (4) and (5) where

$$|f(t, x)| \leq \varepsilon |x| \tag{9}$$

Let $\Phi(t,s)$ be the fundamental matrix of (4). We say that $u(t) : \mathbb{T} \to \mathbb{R}$ is *upper function* for system (4) if there exists a $C > 0$ such that for all $t \geq s$, $t, s \in \mathbb{T}$

$$|\Phi(t,s)| \leq C e_u(s,t)$$

(see (1) for definition of generalized exponent). Let U be the set of all upper functions of (4). We call the value

$$\chi(A) = \inf_{u \in U} \limsup_{t \to \infty} \frac{1}{t} \int_s^t u(\tau) \Delta\tau.$$

Observe that this central exponent is not less than the greatest Lyapunov exponent for solutions of system (4). On the other hand, it is not greater than

$$\limsup \frac{1}{t} \int_0^t |A(s)| \, \Delta s.$$

Remark. Center exponents may be greater than senior Lyapunov exponents. This can be illustrated by the following example given by Perron [8] for linear systems of ordinary differential equations. Take $\alpha \in (0.5, 0.5 + \exp(-\pi)/4)$ and consider the system

$$\begin{aligned} \dot{x} &= -\alpha y; \\ \dot{y} &= (\sin \log t + \cos \log t - 2\alpha) y \end{aligned} \tag{10}$$

and the perturbed one

$$\begin{aligned} \dot{x} &= -\alpha y; \\ \dot{y} &= (\sin \log t + \cos \log t - 2\alpha) y + \mu \exp(-\alpha t) x \end{aligned} \tag{11}$$

where μ is a small parameter. Both considered systems are integrable. Lyapunov exponents for solutions of Eq. (10) equal to $-\alpha$ and $1 - 2\alpha$, so, they are negative. The senior exponent of Eq. (11) equals to $1 - 2\alpha + \exp(-\pi)/2 > 0$. In fact, this is the central upper Lyapunov exponent for system (10).

Theorem 1. *If $\chi(A) < 0$, there exists $\varepsilon > 0$ such that for any f satisfying* (9)*, the zero solution of Eq.* (5) *is asymptotically stable.*

This statement is very close to Lemma 3.1 of [1].

To prove it we use the time-scale version of Grönwall-Bellmann lemma for time-scale systems first given in [10]. Denote $\mathbb{T}^+_{t_0} = \{t \in \mathbb{T} : t \geq t_0\}$.

Lemma (Grönwall-Bellmann Inequality). *Let $t_0 \in \mathbb{T}$, $x, g \in \mathcal{C}_{rd}$ and $p \in \mathcal{R}^+$, $p \geq 0$. Then*

$$x(t) \leq g(t) + \int_{t_0}^{t} x(s) p(s) \, \Delta s \qquad \textit{for all} \quad t \in \mathbb{T}^+_{t_0}$$

implies

$$x(t) \le g(t) + \int_{t_0}^{t} e_p(t, \sigma(s)) g(s) p(s)\, \Delta s \qquad \textit{for all} \quad t \in \mathbb{T}^+_{t_0}.$$

Proof of Theorem 1. Any solution $x(t)$ of Eq. (5) satisfies the non-homogenous linear system

$$x^{\Delta} = A(t)x + f(t, x(t)).$$

So,

$$x(t) = x(t_0)\Phi(t) + \int_{t_0}^{t} \Phi(t, s) f(s, x(s))\, \Delta s$$

and we can write down the inequality

$$|x(t)| \le |\Phi(t)||x(t_0)| + \int_{t_0}^{t} |\Phi(t, s)|\varepsilon|x(s)|\, \Delta s \tag{12}$$

that is true for $t \in \mathbb{T}^+_{t_0}$. Fix an upper function u with the upper mean value $\chi \in (\chi(A), 0)$. It follows from (12) that

$$|x(t)| \le |x(t_0)| e_u(t_0, t) + C\varepsilon \int_{t_0}^{t} e_u(t, s)|x(s)|\, \Delta s.$$

Let

$$v(t) = |x(t)| e_u(t, t_0).$$

Then

$$v(t) \le v(t_0) + C\varepsilon \int_{t_0}^{t} v(s) \Delta s$$

and, consequently, due to Grönwall – Bellmann Lemma

$$v(t) \le v(t_0) e_{C\varepsilon}(t, t_0).$$

This finishes the proof □.

Now we proceed to the case of the constant matrix A and, respectively, to system (8). Let λ_k be eigenvalues of the matrix A ($k = 1, \dots, n$). Define

$$\nu_k = \limsup_{t \to \infty} \mathrm{Re} \frac{1}{t} \int_{t_0}^{t} \xi_{\mu(t)} \lambda_k \Delta t$$

(see (1) for definition of the transformation ξ_μ).

It is evident that system (8) is exponentially asymptotically stable if all $\nu_k < 0$. Reducing system (8) to the normal form, we can easily see that

$$\chi(A) \geq \max \nu_k. \tag{13}$$

There is at least one important case, when inequality (13) becomes equality.

Definition 9. We call the time scale $\mathbb{T}$ periodic if there exists $t_0 > 0$ such that $\mathbb{T} = \mathbb{T} + t_0$.

Similarly to Floquet theory, we can reduce an autonomous system on a periodic time-scale to a discrete system on the time scale $\mathbb{Z}$. Namely, there exist a $\kappa \in [0, t_0)$ such that the time scale $\mathbb{T}_0 = t_0\mathbb{Z} + \kappa$ is such that

1. $\mathbb{T}_0 \subset \mathbb{T}$;

2. There exists a constant matrix B and a time scale system

$$x^\Delta = Bx \tag{14}$$

   on the time scale $\mathbb{T}_0$ such that the reduction of the fundamental matrix $\Phi(t, 0)$ of system (8) is a fundamental matrix of system (14).

Then central upper exponents of systems (8) and (14) coincide and $\chi(A) = \max \nu_k$.

For some non-periodic time scales center upper exponents of constant matrices may still be calculated explicitly. We use a concept, similar to one, well-known in Combinatorics.

Definition 11. The time scale $\mathbb{T}$ is *syndetic* if $\sup_{t \in \mathbb{T}} \mu(t) < +\infty$.

Using technique of normal forms, we can write down a precise formula for center upper exponents of linear autonomous systems on syndetic time scales. Given a syndetic time scale $\mathbb{T}$, for any $t \in \mathbb{R}$ we introduce $[t]_\mathbb{T} = \sup\{s \in \mathbb{T}, s \leq t\}$ (observe that this is not $\rho(t)$). Then, similarly to see [9, Page. 116, Eq. (8.7)] , we can prove that

$$\chi(A) = \lim_{T \to \infty} \limsup_{k \to \infty} \frac{1}{kT} \sum_{i=0}^{k-1} \int_{[iT]_\mathbb{T}}^{[(i+1)T]_\mathbb{T}} \max_j \operatorname{Re} \xi_{\mu(t)} \lambda_j \, \Delta t.$$

For non-syndetic time scales we only can say that

$$\chi(A) \leq \limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \max \operatorname{Re} \xi_{\mu(t)} \lambda_k \Delta t.$$

6 Stability by first approximation

First of all, we recall, the "time-scale" version of Lyapunov theorem on asymptotic stability by first approximation, proved in [11].

Definition 11. *Let $r > 0$. We say that $V(t, x) : \mathbb{T} \times B(0, r)$ is a strict Lyapunov function for a time scale system*

$$x^{\Delta} = F(t, x), \qquad t \in \mathbb{T}, \quad x \in \mathbb{R}^n, \quad F(t, 0) \equiv 0. \tag{15}$$

if the following conditions are fulfilled.

1. *For all t, x we have $V(t, x) \geq w_+(x)$ where $w_+(x) : B(0, r) \to \mathbb{R}$ is a positive definite function.*

2. $$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) F(t, x) < -w_-(x)$$
   for all $t \in \mathbb{D}$.

3. *$V(\sigma(t), x + \mu(t)F(t, x)) < -\mu(t)w(x)$ for all $t \in \mathbb{S}$, $x \in \Omega_t \bigcap B(0, r)$. Here $w_-(x) : B(0, r) \to \mathbb{R}$ is a positive definite function.*

Lyapunov Theorem [10]. *If there is a strict Lyapunov function for system (15), then the zero solution of the considered system is asymptotically stable.*

Theorem 2. *If the matrix A is stable hyperbolic then there exists $\varepsilon > 0$ such that for for any $a > 0$ and any $f : \mathbb{T} \times B(0, a)$, satisfying condition (9) the solution $x = 0$ of the system*

$$x^{\Delta} = Ax + f(t, x) \tag{16}$$

is asymptotically stable.

First of all, we prove the following lemma.

Lemma 2. *If the matrix A is stable hyperbolic, system (16) has a Lyapunov function that is a quadratic form $V(x) = x^T Bx$ where B is a positively defined matrix.*

Proof. Making a non-degenerate transformation $x = Sy$, we can reduce system (8) to the form

$$y^{\Delta} = Jy \tag{17}$$

where $J = \mathrm{diag}(J_1, \ldots, J_k)$ where for any $i = 1, \ldots, k$ either

$$J_m = \begin{pmatrix} \lambda & 0 & 0 & \ldots & 0 \\ \delta & \lambda & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \delta & \lambda \end{pmatrix} = \lambda E + \delta I$$

for a real eigenvalue λ or

$$J_m = \begin{pmatrix} \Lambda & 0 & 0 & \dots & 0 \\ \delta E_2 & \Lambda & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \delta E_2 & \Lambda \end{pmatrix}$$

where

$$\Lambda = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for a pair of eigenvalues $\alpha \pm i\beta$. A parameter $\delta > 0$ may be selected arbitrarily small.

Observe first of all that it suffices to prove existence of the desired quadratic form for Eq. (16) and, consequently, for every system

$$z^\Delta = J_m z. \tag{18}$$

We select $V(z) = z^2$ for Eq. (18). Direct calculations demonstrate that

$$(z^2)^\Delta = 2(z, z^\Delta) = 2\lambda z^2 + 2\delta z^T I z \le \lambda z^2 < 0$$

for small δ and $z \neq 0$ provided λ is real. Similarly, we can demonstrate that

$$(z^2)^\Delta \le \alpha z^2$$

for a block, corresponding to a pair of eigenvalues with a negative real part α. This proves the lemma. □

Now the statement of Theorem 2 follows from Lyapunov Theorem.

7 Instability via Millionschikov's rotations

Now we prove a statement that is in a certain sense converse to Theorem 1.

Theorem 3. *Let the matrix $A(t)$, corresponding to system (4) be bounded, uniformly regressive and*

$$\chi(A) > 0. \tag{19}$$

Let the time scale $\mathbb{T}$ be such that

$$\liminf_{\mathbb{T} \ni t \to \infty} \frac{\chi(A)t - \mu(t) \sup_s |A(s)|}{\sigma(t)} > 0. \tag{20}$$

Then for any $\varepsilon > 0$ *there exists a matrix* $B_\varepsilon(t)$ *such that*

$$\sup |B_\varepsilon(t)| \leq \varepsilon$$

such that the system

$$x^\Delta = (A(t) + B_\varepsilon(t))x \tag{21}$$

is unstable. For syndetic time scales the similar statement is true if $\chi(A) = 0$.
Remark. Evidently, (19) implies (20) for syndetic time scales.
Corollary. *In conditions of Theorem 3 there exists a continuous map* $f : \mathbb{T}_0^+ \times B(0,1) \to \mathbb{R}^m$ *such that*

1. $f(t, 0) = 0$ *for any* $t \geq 0$;
2. $\dfrac{\partial f}{\partial x}(t, 0) = 0$ *for any* $t \geq 0$;
3. *the Jacobi matrix*
$$\frac{\partial f}{\partial x}(t, x)$$
is uniformly continuous at $[0, \infty) \times B(0, 1)$ *(particularly, it is bounded);*
4. *the solution* $x(t) = 0$ *of the corresponding system* (5) *is unstable.*

Proving Theorem 3, we essentually base on the proof of the main result of [12], originally proved for ordinary differential equations. We have to reproduce it literally (see Appendix). We observe that the similar statement is true for linear systems over $\mathbb{C}^n$. The matrix $A(t)$ can also be taken imaginary.
Proof of Theorem 3. First of all, observe that a linear uniformly regressive system on a time scale can be embedded to a linear system of ordinary differential equations with a bounded matrix.

Let $t \notin \mathbb{T}$. We define

$$t_- = \sup(\mathbb{T} \bigcap (-\infty, t]) =: [t]_\mathbb{T}.$$

Given a time scale system (4), we introduce a system of ordinary differential equations

$$\dot{x} = \widetilde{A}(t)x \tag{22}$$

The matrix $\widetilde{A}(t)$ that can be imaginary is defined by formulae: $\widetilde{A}(t) = A(t)$ if $t \in \mathbb{T}$;

$$\widetilde{A}(t) = \frac{1}{\mu(t_-)} \mathrm{Log}\,(E + \mu(t_-)A(t_-)) \qquad \text{if} \quad t \notin \mathbb{T}.$$

Logarithm of a matrix is never unique. Sometimes it cannot be selected real. For a given value t_- we take a same branch for $t \in (t_-, \sigma(t_-))$ so that $\widetilde{A}$ was continuous and bounded provided A is.

Any solution of Eq. (22), restricted to the time scale $\mathbb{T}$, is a solution of time scale system (4). Moreover, if the solution is unbounded, it is also unbounded on the time scale $\mathbb{T}$ (matrix $\widetilde{A}(t)$ is constant on every connected component of $\mathbb{R} \setminus \mathbb{T}$).

Given a bounded continuous and uniformly regressive matrix $A(t)$, a small value $\sigma > 0$, a matrix norm $|\cdot|$ and a time scale $\mathbb{T}$, we introduce the set

$$\Xi = \{P = A + B \in \mathcal{M}_R \, : \, \sup |B(t)| \leq \sigma\}.$$

Given an extension $\widetilde{A}$ can select a continuous nonlinear operator

$$\mathcal{L} : \mathcal{M}_T \to \mathcal{M}_R$$

so that if $B \in \Xi$, $\widetilde{B} = \mathcal{L}B$, the reduction of the Cauchy matrix $\Phi(t, s)$ of the system of ordinary differential equations

$$\dot{x} = (\widetilde{A}(t) + \widetilde{B}(t))x \tag{23}$$

to $\mathbb{T} \times \mathbb{T}$ is the Cauchy matrix of the system

$$x^{\Delta} = (A(t) + B(t))x. \tag{24}$$

Additionally, we assume that the matrix $\widetilde{A}(t) + \widetilde{B}(t)$ must be constant out of $\mathbb{T}$.

Given a matrix $\widetilde{B}(t))$ we may construct a fundamental matrix of Eq. (23). Reducing this fundamental matrix to $\mathbb{T}$, we obtain a fundamental matrix of (24) that reconstructs a matrix $B(t)$. In this case we set $B = \widetilde{\mathcal{L}}\widetilde{B}$. This operator $\widetilde{\mathcal{L}}$ is continuous and left inverse to $\mathcal{L}$.

Remark. For non-syndetic time scales operator $\mathcal{L}$ is still continuous. Operator $\widetilde{\mathcal{L}}$ is not continuous. More precisely, there might exist a small matrix $\widetilde{B}$ such that $\widetilde{\mathcal{L}}\widetilde{B}$ is continuous and unbounded with respect to t.

We start with the case when $\chi(A) > 0$.

Let $\widetilde{P} = \widetilde{A} + \widetilde{B}$ be an ε – small perturbation of the matrix A that makes the senior exponent of system (24) positive. Consequently, the system itself becomes unstable.

So, any fundamental matrix of this system is unbounded with the Lyapunov exponent close to $\chi(A)$. Due to (20), the reduction of the fundamental matrix to the time scale does also have a positive Lyapunov exponent.

It follows from the proof of Millionschikov's theorem (see Appendix) that for every t_0 such that $\mu(t_0) \neq 0$ there exist at most two subsegments of $[t_0, \sigma(t_0))$ where $\widetilde{B}(t) \neq 0$.

Moreover, length of all such segments is uniformly bounded and does not depend on parameters of the time scale. The reason is very simple: for any t_0 the reduction of $\widetilde{A}(t)$ is a constant matrix, so all initial conditions x_i,corresponding to solutions of maximal growth (see Appendix, Eq. (43)), coincide. Following the lines of the proof of Millionschikov's Theorem we see that for any splitting of $[t_0, \sigma(t_0))$ to segments of length T, only the first and the last of corresponding rotations (and, consequently, perturbations) may be non-zero.

This means that there exists a $C > 0$ such that $\sup |\widetilde{B}(t)| < \varepsilon$ implies $\sup |B(t)| < C\varepsilon$ for small values of ε. To finish the proof it suffices to take $\varepsilon \leq \chi(A)/(2C + 1)$.

Now we study the case when $\chi(A) = 0$. First of all, observe that the transformation $y = \exp(\varepsilon t)x$ transferes a system

$$\dot{x} = P(t)x$$

to $\dot{y} = (P(t) + \varepsilon E)y$.

So, if a family of perturbations $B_n(t)$ makes senior exponents systems

$$\dot{x} = (\widetilde{A}(t) + B_n(t))x$$

tending to zero, there exists a sequence $\varepsilon_n \to 0$ such that senior exponent of the system

$$\dot{x} = (\widetilde{A}(t) + B_n(t) + \varepsilon_n E)x$$

is positive. If the time scale $\mathbb{T}$ is syndetic, the corresponding perturbations of the time scale system do also tend to zero that finishes the proof. □

Proof of Corollary. Let $\widetilde{B}_n(t)$ is the perturbation that exists due to Theorem 3 and such that

1) $B_n(t) \leq 2^{-n}$;

2) systems

$$x^{\Delta} = (A(t) + B_n(t))x \tag{25}$$

are all unstable.

Fix an unbounded solution $x_1(t)$ of system (25) ($n = 1$) such that $|x_1(0)| = 1$. Select T_1 so that $|x_1(T_1)| = 2$, $|x_1(t)| < 2$ while $0 \leq t < T_1$. Then we construct an unbounded solution $x_2(t)$ of system (25) ($n = 2$) such that $|x_2(t)| < |x_1(t)|/2$ for $0 \leq t \leq T_1$. Given $x_2(t)$ we select the first time instant T_2 such that $|x_2(T_2)| = 2$. Then we construct $x_3(t)$ and T_3 and so on.

Evidently, $T_{n+1} > T_n$ for any n and $x_n(0) \to 0$.

Now we are ready to construct a map $f : [0, \infty) \times B(0,1) \to \mathbb{R}^N$, that satisfies conditions of the theorem. Set $\epsilon_n(t) = |x_n(t)|/4$ for $t < T_n$. Evidently, all $\epsilon_n(t)$ are continuous; $\epsilon_n(t) > 0$ for all n $t < T_n$ and

$$|x_n(t) - x_k(t)| > |\epsilon_n(t)| + |\epsilon_k(t)|,$$

if $t < \min(T_k, T_n)$. This follows from inequalities

$$|x_n(t)| < |x_{n-1}(t)|/2 < \cdots < |x_k(t)|/2^{n-k},$$

that are true if $n > k$.

Consider a function $\phi : [0, \infty) \to [0, \infty)$ with following properties:

1) $\phi \in C^\infty([0, \infty))$;

2) $\phi(t) = 1$ for $t \le 1/10$; $\quad \phi(t) = 0$ if $t \ge 1$;

3) $-2 < \phi'(t) < 0$ if $1/10 < t < 1$.

For any pair $(t, x) \in [0, \infty) \times B(0,1)$ we set

$$\Psi_n(t,x) = \begin{cases} \phi(|x - x_n(t)|^2/\epsilon_n^2(t))B_n(t), & \text{if} \quad |x - x_n(t)| \le \epsilon_n(t), t < T_n; \\ 0 \quad \text{otherwise.} \end{cases}$$

All Ψ_n vanish in left neighborhoods of T_n for all $x \in B(0,1)$. Since $0 \le \phi(\tau) \le 1$ for all τ and $|B_n(t)| < 1/2^n$, we obtain $\Psi_n(t,x) < 1/2^n$ for all t and x. Define

$$F(t,x) = \begin{cases} \Psi_n(t,x), & \text{if exists } n\text{:} \quad \Psi_n(t,x) \ne 0; \\ 0, & \text{if such} \quad n \quad \text{does not exist.} \end{cases} \tag{26}$$

For any fixed t and x there exists at most one such number n.

It follows from (26) that

$$F(t,x) = \sum_{n=1}^{\infty} \Psi_n(t,x). \tag{27}$$

Since $|\Psi_n(t,x)| < 1/2^n$, (27) implies that F is uniformly continuous w.r.t. x. Since $\varepsilon_n(t) = |x_n(t)|/4$, we have $F(t,0) = 0$. Introduce $f(t,x) = F(t,x)x$. Let us prove that $f \in C^1_x$.

Since for any $x_n(t) \rightrightarrows 0$ on compact subsets of $[0, +\infty)$, for any $\varepsilon > 0$ the reduction of sum (27) to $[0,T] \times \mathbb{R}^m \setminus B(0, \varepsilon)$ may contain a finite number of non-zero terms only. All these terms are C^1 smooth with respect to x. Consequently, for any (t,x) such that $x \ne 0$

the Jacobi matrix $\partial f/\partial x(t,x)$ is correctly defined and locally continuous. Moreover, for any $i = 1\dots m$:

$$\frac{\partial f}{\partial x_i}(t,x) = F_i(t,x) + \frac{\partial F}{\partial x_i}(t,x)\cdot x, \tag{28}$$

where $x = \mathrm{col}(x_1,\dots,x_n)$, and F_i is the i-th column of the matrix F.

Let

$$\Theta_{ni}(t,x) = \frac{\partial \Psi_n}{\partial x_i}(t,x) = 2\frac{(x_i - x_{ni}(t))\phi'(|x-x_n(t)|^2/\varepsilon_n^2(t))B_n(t)}{\varepsilon_n^2(t)}.$$

Here $x_{ni}(t)$ is the i-th component of the vector $x_n(t)$. We have demonstrated that the first term of the right hand side of (28) is uniformly continuous with respect to x and equals 0 if $x = 0$. We prove that the similar is true for the second term if we set it equal to zero if $x = 0$.

Indeed,

$$\frac{\partial F}{\partial x_i}(t,x)\cdot x = \begin{cases} \Theta_{ni}(t,x)\cdot x, \\ \text{if there exists} \quad n \quad \text{such that} \quad |x - x_n(t)| \le \varepsilon_n(t), t < T_n; \\ 0, \quad \text{if such} \quad n \quad \text{does not exist.} \end{cases} \tag{29}$$

Each solution $x_n(t)$ is bounded away from zero while $t \le T_n$. So, for any $N \in \mathbb{N}$ there exists a $\delta > 0$ such that if given (t,x) : $|x| < \delta$ there exists n, that corresponds to this pair in the sense of (26) and (29), it cannot be less than N. So,

$$|x| < |x_n(t)| + |x - x_n(t)| \le |x_n(t)| + \varepsilon_n(t) = 5\varepsilon_n(t).$$

This implies that

$$\begin{aligned} |\Theta_{ni}(t,x)\cdot x| &= 2\frac{|x_i - x_{ni}(t)|\,|\phi'(|x-x_n(t)|^2/\varepsilon_n^2(t))|\,|B_n(t)|\,|x|}{\varepsilon_n^2(t)} \le \\ &\le 4\frac{|x_i - x_{ni}(t)|}{\varepsilon_n(t)}\cdot\frac{|x|}{\varepsilon_n(t)}\cdot|B_n(t)| \le 4\frac{|B_n(t)|5\varepsilon_n(t)}{\varepsilon_n(t)} \le 20/2^n. \end{aligned} \tag{30}$$

It follows from (27) that:

$$\frac{\partial F}{\partial x_i}(t,x)\cdot x = \sum_{n=1}^{\infty}\Theta_{ni}(t,x)\cdot x. \tag{31}$$

All terms of (31) are uniformly continuous, which follows from (30). Moreover, series (31) can be estimated by $\sum_{n=1}^{\infty} 20/2^n$. So, due to Weierstrass theorem, the second term of (28) is continuous. Then the constructed mapping f satisfies conditions 1. — 4. It follows from (26) that $A(t)x_n(t) + f(t, x_n(t)) = A_n(t)x_n(t)$ for all n and $t \in [0, T_n]$. Then all $x_n(t)$ are solutions of (5). This demonstrates that the zero solution of (5) is unstable. □

8 Instability via Chetaev theorem

Now we prove an analogue of famous Chetaev theorem on instability by first approximation for time-scale systems.

Consider a time-scale system (15). Recall that it has the zero solution. Represent $\mathbb{T} = \mathbb{S} \bigcup \mathbb{D}$ where $\mathbb{S}$ is the set of right-scattering points and $\mathbb{D}$ is the set of right-dense points. Evidently, this is a disjoint union.

Let $B(0, r)$ be the r – ball in $\mathbb{R}^n$, centered at zero.

Definition 12. We say that a continuous function $V(t, x) : \mathbb{T} \times \mathbb{R}^n$ is a *Chetaev function* for Eq. (15) if there exists a family of domains $\Omega_t \subset \mathbb{R}^n$ such that the folowing conditions are satisfied.

1. $0 \in \partial\Omega_t$ for all $t \in \mathbb{T}$ (here the symbol ∂ denotes boundary);
2. $V(t, x) > 0 \qquad \text{for all} \quad t \in \mathbb{T}, x \in \Omega_t \bigcap B(0, r)$;
3. $V(t, x) = 0 \qquad \text{for all} \quad t \in \mathbb{T}, x \in \partial\Omega_t \bigcap B(0, r)$;
4. for any $\varepsilon > 0$ there a is $\delta > 0$ such that the inequality $V(t, x) > \delta$ implies $|x| > \varepsilon$ (this can be called uniform continuity in a neighborhood of zero);
5. there exists a function $w(x)$ such that $w(0) = 0$, $w(x) > 0$ for all $x \neq 0$ and

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)F(t, x) \geq w(x)$$

   for all $t \in \mathbb{D}$, $x \in \Omega_t \bigcap B(0, r)$;

$$V(\sigma(t), x + \mu(t)F(t, x)) - V(t, x) \geq \mu(t)w(x)$$

   for all $t \in \mathbb{S}$, $x \in \Omega_t \bigcap B(0, r)$.

Here we recall that $\sigma(t) = t + \mu(t)$.

Theorem 4. *If there is a Chetaev function for system* (15)*, then the trivial solution of this system is unstable.*

Proof. Suppose that the zero solution is stable despite the existence of the Chetaev function. Fix a $t_0 \in \mathbb{T}$. Consider the domain $A = (\mathbb{T} \times B(0, r)) \bigcap \Omega_{t_0}$. Fix $x_0 \in A$ sufficiently close to the origin. Define $A_{x_0} = \{(t, x) : x \in \Omega_t \bigcap B(0, r), V(t, x) \geq V(t_0, x_0)\}$. This set is non-empty, moreover, due to stability of zero solution, we can select x_0 so that $(t, x(t, t_0, x_0)) \in A_{x_0}$ for all t. Here $x(t, t_0, x_0)$ is the solution of (4) corresponding to initial conditions $x(t_0) = x_0$.

Indeed, this solution (provided $|x_0|$ is small) cannot leave the ball $B(0,r)$ due to stability of zero solution. Observe that the function $V(t, x(t, t_0, x_0))$ increases as t increases. So, starting with positive values of this function, the solution cannot leave the domain A_{x_0} via the part of the boundary, given by the condition $V(t,x) = V(t_0, x_0)$ while $t \in \mathbb{D}$. Neither, the solution cannot intersect the boundary $\partial\Omega_t$, $t \in \mathbb{D}$.

Due to similar reasons, it is impossible that $V(t, x(t, t_0, x_0)) < V(t_0, x_0)$ (that also includes the case $V(t, x(t, t_0, x_0))) \le 0$ for a $t \in \mathbb{S}$).

On the other hand

$$V(t, x(t, t_0, x_0)) - V(s, x(s, t_0, x_0)) \ge \gamma(t - s)$$

where $t > s$, $t, s \in \mathbb{T}$ while $(t, x(t, t_0, x_0)) \in A_{x_0}$. Here

$$\gamma = \min\{w(x) : (t_0, x) \in A_{x_0}\}.$$

This gives a contradiction since the the function $V(t, x(t, t_0, x_0))$ must be bounded. □

Now we consider another type of Chetaev functions.

Definition 13. We say that a continuous function $V(t,x) : \mathbb{T} \times \mathbb{R}^n$ is a *second type Chetaev function* for Eq. (15) if there exists a family of domains $\Omega_t \subset \mathbb{R}^n$ and a function $\gamma : \mathbb{T} \to \mathbb{R}$ such that

1. items 1–4 of Definition 12 are satisfied;
2. $\limsup\limits_{T\to\infty} \dfrac{1}{T} \int\limits_0^T \xi_{\mu(t)}\gamma(t)\,\Delta t > 0$;
3. $$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)F(t,x) \ge \gamma(t)V(t,x) \tag{32}$$
   for all $t \in \mathbb{D}$, $x \in \Omega_t \bigcap B(0,r)$;
   $$V(\sigma(t), x + \mu(t)F(t,x)) - V(t,x) \ge \mu(t)\gamma(t)V(t,x) \tag{33}$$
   for all $t \in \mathbb{S}$, $x \in \Omega_t \bigcap B(0,r)$.

Theorem 5. *If there is a second type Chetaev function for system* (15)*, then the trivial solution of this system is unstable.*

Proof. Similarly to proof of Theorem 4 and due to Grönwall – Bellmann lemma, we demonstrate that (32) and (33) imply that

$$V(t, x(t, t_0, x_0)) \ge e_\gamma(t, t_0)V(t_0, x_0)$$

while $(t, x(t, t_0, x_0)) \in \Omega_t$. This means that the left hand side of this inequality cannot be bounded. □

Now we are ready to prove an analogue of the classical theorem on instability by first approximation. Let $a > 0$. Consider the class $\mathcal{F}_a$ of functions $f : \mathbb{T}_0^+ \times B(0, a) \to \mathbb{R}^m$ satisfying conditions

1. $f(t, 0) = 0$ for any $t \geq 0$;
2. $\dfrac{\partial f}{\partial x}(t, 0) = 0$ for any $t \geq 0$;
3. the Jacobi matrix $\dfrac{\partial f}{\partial x}(t, x)$ is uniformly continuous at $[0, \infty) \times B(0, 1)$.

Theorem 6. *Let $\mathbb{T}$ be a syndetic time scale. If one of values ν_k corresponding to a constant matrix A is positive, then there exists $\varepsilon > 0$ such that for for any $a > 0$ and any $f : \mathbb{T} \times B(0, a)$, $f \in \mathcal{F}_a$ the solution $x = 0$ of system (16) is unstable.*

Proof. Fix a $\delta > 0$ First of all, applying an invertible linear transformation $x = S_\delta y$, we may reduce the time scale system (16) to the form

$$y^\Delta = J_\delta y + g(t, y) \tag{34}$$

where $g(t, y) = S_\delta^{-1} f(t, S_\delta y) \in \mathcal{F}_a$, J_δ is the real normal form of the matrix A defined by (17). This transformation does not hurt stability or instability of the trivial solution.

Without loss of generality we assume that $\nu_1 \geq \nu_2 \geq \ldots \nu_n$. For Eq. (34), we select $m \in \{1, \ldots, n\}$ so that $\nu_m > 0$, $\nu_{m+1} \leq 0$ or set $m = n$ if $\nu_m > 0$. Let

$$V = \sum_{i=1}^{m} y_i^2 - \sum_{i=m+1}^{n} y_i^2.$$

Let $\Omega = \Omega_t$ be one of connected components (cones) that correspond to the inequality $V > 0$. Direct calculations demonstrate that for small value of δ the function V is a second type Chetaev function for Eq. (34) in a vicinity of zero. Here we may take $\gamma = \mathrm{Re}\,\lambda_m - \delta$. □

Appendix. Proof of Millionschikov's theorem.

Theorem (Millionschikov). *Let the matrix $A(t)$, corresponding to system*

$$\dot{x} = A(t)x \tag{35}$$

be bounded, and continuous. Let the central upper Lyapunov exponent of system (35) *be equal to* Ω. *Then for any* $\varepsilon > 0$ *there exists a matrix* $B_\varepsilon(t)$ *such that*

$$\sup |B_\varepsilon(t)| \le \varepsilon \tag{36}$$

and the seniour exponent of the system

$$\dot{x} = (A(t) + B_\varepsilon(t))x \tag{37}$$

is greater than $\Omega - \varepsilon$.
Proof. We start with the main idea of the proof. Consider the Cauchy matrix $\Phi(t,s)$ and a value $T > 0$ so that the value

$$\limsup_{k\to\infty} \frac{1}{kT} \sum_{i=0}^{k-1} \ln |\Phi((i+1)T, iT)| \tag{38}$$

is close to $\chi(A) = \Omega$ (see [9]). Let x_i $(i = 0, 1, 2, \dots)$ be the unit vector such that

$$|\Phi((i+1)T, iT)x_i| = |\Phi((i+1)T, iT)|.$$

Let $x_i(t) = \Phi(t, iT)x_i$.

It is $x_0(t)$ that has the quickest growth among solutions of (35). Without loss of generality, we may say that on $[T, 2T]$, the solution $x_1(t)$ increases faster than $x_0(t)$.

We perturb system (35) in the following way. First of all, we rotate the solution $x_0(t)$ in the plain $\langle x_0(t), x_1(t)\rangle$ by angle ε. This can be done on a time segment of length $\ll T$. Then, for greater values of t, we set perturbation equal to zero. Since $x_1(t)$ increases faster than $x_0(t)$, the angle between solutions $y_0(t)$ and $x_1(t)$ becomes $\le \varepsilon$. This happens on a time period of length $\ll T$. Then we perturb system (35) so that $y_0(t)$ becomes parallel to $x_1(t)$. Then we set perturbation equal to zero up to $t = 2T$.

Similarly, we consider segment $[2T, 3T]$ and later ones. Finally, we obtain a solution $y_0(t)$ of the perturbed system that has Lyapunov exponent, close to Ω.

Now we proceed to the detailed proof. Fix a $\varepsilon > 0$.
1) Given a triangle ABC, due to Sine Theorem

$$\frac{BC}{AC} = \frac{\sin\sphericalangle A}{\sin\sphericalangle B} \ge \sin\sphericalangle A. \tag{39}$$

Fix a T_0 so that

$$\exp(\varepsilon T_0/2) \cdot \sin^2 \varepsilon \ge 1. \tag{40}$$

Let triangles $\triangle ABC\ \ \triangle A_1B_1C_1$ be such that

$$\frac{B_1C_1}{A_1C_1} : \frac{BC}{AC} \geq \exp(\varepsilon T_0/2); \quad \sphericalangle A = \varepsilon. \tag{41}$$

Then (39), (40) and (41) imply that

$$\sin \sphericalangle B_1 \leq \frac{A_1C_1}{B_1C_1} \leq \exp(-\varepsilon T_0/2) \cdot \frac{1}{\sin \varepsilon} \leq \sin \varepsilon.$$

Since $A_1C_1/B_1C_1 \leq 1$ we have $\sphericalangle B_1 \leq \sphericalangle A_1$, and, consequently, $\sphericalangle B_1 \leq \pi/2$). Therefore, $\sphericalangle B_1 \leq \varepsilon$.
2) Fix a fundamental matrix $\Phi(t)$ of system (35) and set $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$. Fix a $T > 0$ so that $4(2a+\varepsilon)T_0/T < \varepsilon/4$ (here $a = \sup |A(t)|$),

$$T/T_0 = s \ - \text{ is an integer,}$$

$$\limsup_{k\to\infty} \frac{1}{kT} \sum_{i=0}^{k-1} \ln |\Phi((i+1)T, iT)| > \Omega - \frac{\varepsilon}{4}. \tag{42}$$

3) Take a unit vector x_i $(i = 0, 1, 2, \dots)$ such that

$$|\Phi((i+1)T, iT)x_i| = |\Phi((i+1)T, iT)|. \tag{43}$$

Let

$$x_i(t) = \Phi(t, iT)x_i \tag{44}$$

be solutions of Eq. (35).

Set $B_\varepsilon(t) = 0$ for $0 \leq t \leq T$. Suppose that

$$\frac{|x_1(2T)|}{|x_1(T)|} : \frac{|x_0(2T)|}{|x_0(T)|} \geq \exp\left(\frac{\varepsilon T}{2}\right).$$

If this is wrong, we set $B_\varepsilon(t) = 0$ for $T < t \leq 2T$.

Divide the segment $[T, 2T]$ to s segments of length T_0: $Q_1, Q_2, \dots, Q_s$. Let Q_{j_0} be the first of segments $Q_i = [a_i, b_i]$ $\quad (i = 2, 3, \dots, s-1)$, where

$$\frac{|x_1(b_i)|}{|x_1(T)|} : \frac{|x_0(2T)|}{|x_0(T)|} \geq \exp(\varepsilon T_0/2). \tag{45}$$

If such segments do not $B_\varepsilon(t) = 0$ for $T < t \leq 2T$.) Let $\tau_1 < \tau_2 < \tau_3$ be ends of segments Q_{j_0-1} and Q_{j_0}.

Define the perturbation $B_\varepsilon(t)$ for $T < t \leq 2T$ in the following way.
A. If $T < t \leq 2T, \quad t \notin Q_{j_0-1} \cup Q_{j_0} = [\tau_1, \tau_3]$ we set $B_\varepsilon(t) = 0$.
B. If $t \in Q_{j_0-1} = [\tau_1, \tau_2]$ we set

$$B_\varepsilon(t) = U_\varepsilon^{-1}(t)A(t)U_\varepsilon(t) - U_\varepsilon^{-1}(t)\dot{U}_\varepsilon(t) - A(t), \tag{46}$$

where $U_\varepsilon(t)$ is an orthogonal matrix such that:

i)

$$U_\varepsilon(\tau_1) = E, \tag{47}$$

ii)

$$|\dot{U}_\varepsilon(t)| \leq \varepsilon/T_0, \tag{48}$$

Observe that since $x_0(t)$ is a solution of Eq. (35)), the function

$$y_0(t) = \begin{cases} U_\varepsilon^{-1}(t)x_0(t) & \text{if} \quad t \in [\tau_1, \tau_2], \\ x_0(t) & \text{if} \quad t < \tau_1 \end{cases}$$

is a solution of system (37) where matrix $B_\varepsilon(t)$ satisfies (36).

C. There exist $\alpha_1 \geq 0, \alpha_2 > 0$ such that

$$y_0(\tau_2) = U_\varepsilon^{-1}(\tau_2)x_0(\tau_2) = \alpha_1 x_0(\tau_2) + \alpha_2 x_1(\tau_2) \tag{49}$$

and

$$\sphericalangle(x_0(\tau_2), y_0(\tau_2)) = \varepsilon. \tag{50}$$

If $\sphericalangle(x_0(\tau_2), y_0(\tau_2)) < \varepsilon$, we set $B_\varepsilon(t) = 0$ for $t \in [\tau_1, \tau_2]$. It follows from (47) and (48) that

$$|B_\varepsilon(t)| \leq (2a+1)\varepsilon.$$

The matrix $U_\varepsilon(t)$ can be constructed as follows. Let E^{n-2} be the orthogonal completion to the plain $E^2 = \langle x_0(\tau_2), x_0(\tau_2)\rangle$, we set $U_\varepsilon(t)|_{E^{n-2}} = \text{id}$ and define $U_\varepsilon(t)|_{E^2}$ as a rotation by angle $-\varepsilon(t-\tau_1)/T_0$, in the direction from $x_0(\tau_2)$ to $x_0(\tau_2)$ in the plain E^2.
D. Due to statements of item 1) and inequalities (40), (45), (49) and (50),

$$\sphericalangle(z_0(\tau_3), x_1(\tau_3)) \leq \varepsilon,$$

where $t \in [\tau_2, \tau_3]$ and

$$z_0(t) = \alpha_1 x_0(t) + \alpha_2 x_1(t),$$

α_1 and α_2 be defined by formula (49).

For $t \in [\tau_2, \tau_3]$ we take $B_\varepsilon(t)$, that satisfies (46) — (48)), replacing τ_1 with τ_2 in (47). Instead of inequalities (49)) and (50) we demand that the vector $U_\varepsilon^{-1}(\tau_3)(\alpha_1 x_0(\tau_3) + \alpha_2 x_1(\tau_3))$ was parallel to $x_1(\tau_3)$.
4) We construct the perturbation $B_\varepsilon(t)$ on segment $[2T, 3T]$ basing on solution $x_1(t)$ similarly to what we have done on the segment $[T, 2T]$ basing on solution $x_0(t)$ and so on . . .

Function $y_0(t)$ is a solution of the constructed system (37) with the Lyapunov exponent, greater than $\Omega - \varepsilon$. Indeed, due to (42), (43) and (44) it suffices to prove that for any $i = 0, 1, 2, \ldots$

$$\frac{|y_0((i+1)T)|}{|y_0(iT)|} \geq \frac{|x_i((i+1)T)|}{|x_i(iT)|} \exp\left(-\frac{3\varepsilon T}{4}\right).$$

It follows from construction of $y_0(t)$ that for any fixed i the number of k such that inequality

$$\frac{|y_0(iT + (k+1)T_0)|}{|y_0(iT + kT_0)|} \geq \frac{|x_i(iT + (k+1)T_0)|}{|x_i(iT + kT_0)|} \exp\left(-\frac{\varepsilon T_0}{2}\right) \tag{51}$$

is not fulfilled, does not exceed 4. If (51) is not satisfied, we use the inequality

$$\frac{|y_0(iT + (k+1)T_0)|}{|y_0(iT + kT_0)|} \geq \frac{|x_i(iT + (k+1)T_0)|}{|x_i(iT + kT_0)|} \exp(-(2a + \varepsilon)T_0).$$

Multiplying all inequalities (50) and (51) corresponding to $k = 1, 2, \ldots, s-1$, we obtain

$$\frac{|y_0((i+1)T)|}{|y_0(iT)|} > \frac{|x_i((i+1)T)|}{|x_i(iT)|} \exp\left(\left(-\frac{\varepsilon}{2} + \frac{4(2a+\varepsilon)T_0}{T}\right) T\right) \geq \frac{|x_i((i+1)T)|}{|x_i(iT)|} \exp\left(-\frac{3\varepsilon T}{4}\right).$$

This finishes the proof $\Box$.
Corollary. *The perturbation $B_\varepsilon(t)$ may be taken continuous.*
Proof. It follows from the proof that $B_\varepsilon(t)$ is piecewise continuous e.g. has finitely many discontinuity points on bounded subsets of $\mathbb{R}$. So, for any $\delta > 0$, we may construct a continuous matrix $C_\varepsilon(t)$, such that $|C_\varepsilon(t)| < \varepsilon$ and the Lebesgue measure of the set

$$M = \{t : B_\varepsilon(t) \neq C_\varepsilon(t)\}$$

is not greater than δ.

Consider the system

$$\dot{x} = (A(t) + C_\varepsilon(t))x. \tag{52}$$

Let $\Psi(t)$ and $\Xi(t)$ be fundamental matrices of (37) and (52) respectively, so that $\Phi(0) = \Xi(0) = E$. Then

$$\Xi(t) = \Psi(t) + \Psi(t) \int\limits_0^t \Psi^{-1}(\tau)(B_\varepsilon(\tau) - C_\varepsilon(\tau))\Xi(\tau)\, d\tau,$$

and, respectively,

$$|\Xi(t)| \leq |\Psi(t)| + |\Psi(t)| \int\limits_0^t |\Psi^{-1}(\tau)||B_\varepsilon(\tau) - C_\varepsilon(\tau)||\Xi(\tau)|\, d\tau. \tag{53}$$

Denote $u(t) = |\Xi(t)|/|\Psi(t)|$, $v(t) = |\Psi^{-1}(\tau)||\Psi(\tau)||B_\varepsilon(\tau) - C_\varepsilon(\tau)|$. Dividing both parts of (53) by $|\Psi(t)|$, we obtain

$$u(t) \leq 1 + \int\limits_0^t u(\tau)v(\tau)\, d\tau,$$

which implies by Grönwall - Bellmann lemma

$$\frac{|\Xi(t)|}{|\Psi(t)|} = u(t) \leq \exp\left(\int\limits_0^t v(\tau)\, d\tau\right).$$

This implies that the greatest Lyapunov exponents of systems (37) and (52) coincide. □

Conclusion and Discussion

Dealing with non-periodic time scales, one can hardly apply methods suitable for autonomous systems of ordinary differential equations. The main reason is that, generally speaking, due to structure of time scales the set of solutions of a dynamic system cannot be invariant with respect to shifts.

However, methods of non-autonomous stability theory can still be applied. One should look at central exponents to obtain a criterium of stability that seems to be generally more effective than constructing a Lyapunov function.

The problem of instability needs a similar approach. The standard method of Chetaev functions needs to be modified to be applicable for the case of a generic time scale. However, the principal idea of the approach still works.

Let us briefly repeat main results of our paper.

1. We have proved the stability criterium for time scale system, based on the so-called center Lyyapunov exponent. If this exponent is negative, the linear system is stable as well as the trivial solution of any small perturbation. If it is positive, the system is unstable and, moreover, the zero solution cannot be stabilized by a small perturbation.

2. We give estimates on eigenvalues of a constant matrix that are sufficient for stability by first approximation. For this we prove two versions of generalization of the classical Chetaev theorem. Of course, both of them are already known for ordinary differential equations. However, we have to modify techniques of proofs to deal with arbitrary time scales.

Observe that the structure of the time scale was important for qualitative estimates only. Actually, we studied some properties of integral operators on Banach spaces, just varying measures.

This gives many opportunities for farther research.

Many results, well-known in the theory of differential equations and local diffeomorphisms e.g. Perron theorem on existence of stable and unstable manifolds may also be proved for systems on time scales. We think, this is the most obvious development of our results.

There is a lot of work that can be done in the field of linear time scale systems. For example, in this paper we did not say anything about regular systems, irregularity coefficients, exponentially small perturbations of linear systems, stability of all Lyapunov exponents, famous Oseledets' results on regularity of almost all linear approximation of a conservative system etc. The modern theory of linear differential equations is too large to be covered in one paper.

Another possible way would be considering another class of time scale dynamical systems, namely those that correspond to the ∇ -derivative, defined as follows.

Definition 14. The function $f : \mathbb{T} \to \mathbb{R}$ is called ∇ - *differentiable* at a point $t \in \mathbb{T}$ if there exist $\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a W–neighborhood of $t \in \mathbb{T}$ satisfying

$$|[f(t) - f(\rho(s))] - \gamma[t - \rho(s)]| \leq \varepsilon|t - \rho(s)|$$

for all s. Let us recall that $\rho(t) = \sup\{s \in \mathbb{T} : s > t\}$. In this case we write $f^{\Delta}(t) = \gamma$. When $\mathbb{T} = \mathbb{R}$, $x^{\Delta}(t) = \dot{x}(t)$. When $\mathbb{T} = \mathbb{Z}$, $x(t)$ is the standard forward difference operator $x(n) - x(n-1)$.

Despite some technical differences (Cauchy problems for time scale systems with ∇ – derivtives may be non-unique in *forward* direction) it seems that results of this paper can be spread to that class of time scale systems.

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