H^∞ and the Grothendieck approximation property

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Grothendieck's Approximation

Definition

$X \in AP$ *iff*

$$\forall\ Y, \forall\ \ compact\ \ K \subset X,\ \forall\ \varepsilon > 0,\ \forall\ T : X \to Y,$$

$$\exists\ R \in X^* \otimes Y :\ \sup_{x \in K} ||Rx - Tx|| \le \varepsilon.$$

Or, the same:

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$X \in AP$ *iff for every* $(x_n) \in c_0(X)$ *and for every* $\varepsilon > 0$ *there exists a finite rank operator* R *in* X *such that* $\sup_n ||Rx_n - x_n|| \le \varepsilon$.

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$X \in BAP$ *iff* $\exists\ C \geq 1$:

$$\forall \ \textit{compact} \ \ K \subset X, \ \forall\ \varepsilon > 0,$$

$$\exists\ R \in X^* \otimes X : \ \sup_{x \in K} ||Rx - x|| \leq \varepsilon, \ \ ||R|| \leq C.$$

We say also C-*MAP*. *If* $C = 1$, $X \in MAP$.

Easy reformulation:
X has the property C-MAP, if given $\varepsilon > 0$, a Banach space Y, an operator $T \in \mathrm{L}(X, Y)$ and any finite sequence $(x_k) \subset X$, there exists a finite rank operator R from X to Y such that
1) $||Tx_k - Rx_k|| < \varepsilon$ for all k,
2) $||R|| \leq C||T||$.

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Classical Spaces

Separable classical Banach spaces and L^∞ have the MAP. E.g.: $C(K), L_p(\mu), 1 \leq p < \infty \dots$ [Groth.]
Each of spaces $H^p, L^p/H_0^p$ $(1 \leq p < \infty), A, C/A_0$ have the MAP. Moreover, all these spaces have bases.

- R. P. Boas, Jr., Isomorphism between H^p and L^p, Amer. J. Math. 77 (1955), 655-656. + a result of Marcinkiewicz and Paley on $L^p, 1 < p < \infty$.
- P. Billard, Bases dans H et bases de sous espaces de dimension finie dans A, Proc. Conf.,Oberwolfach (August 14-22, 1971), ISNM Vol. 20, Birkhauser, Basel and Stuttgart, 1972.
- S. V. Bockarev, Existence of a basis in the space of functions in the disk, and some properties of the Franklin system, Mat. Sb. (N.S.) 95 (137) (1974), 3-18 == Math. USSR Sbornik 24 (1974), 1-16.

See, generally,

Pełczyński A. *Banach spaces of analytic functions and absolutely summing operators* – AMS Regional Conference Series in Mathematics 30, Providence, 1977.

- $B(H) \notin AP$.
- $L^\infty/H^\infty \in AP$.
- H^∞ — ???

Preliminaries by A. Pełczyński, $B(H)$ ...

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Absolutely summing operators

Definition

An operator T from a Banach space X into a Banach space Y is said to be p-absolutely summing, notation $T \in \Pi_p(X, Y)$, where $0 < p \leq \infty$, if there is a constant $C \in (0, \infty)$ such that for all finite families $(x_i)_{i=1}^n \subset X$

$$\sum_{i=1}^n ||Tx_i||^p \leq C^p \sup\{\sum_{i=1}^n |\langle x', x_i\rangle|^p : \ x' \in X, \ ||x'|| \leq 1\}.$$

The p-summing norm $\pi_p(T)$ is defined as $\inf C$.

Properties of order p

Definition

For $p > 1$, X has the property AP_p (repectively, the property $K - MAP_p$), if given $\varepsilon > 0$, a Banach space Y, an operator $T \in \Pi_{p'}(X, Y)$ and a weakty p'- summable sequence $(x_k) \subset X$, there exists a finite rank operator R from X to Y such that

$$\sum ||Tx_k - Rx_k||^{p'} < \varepsilon$$

(respectively,. and $\pi_{p'}(R) \leq K\pi_{p'}(T)$).

Easy to see:

X has the property $K - MAP_p$, if given $\varepsilon > 0$, a Banach space Y, an operator $T \in \Pi_{p'}(X, Y)$ and any finite sequence $(x_k) \subset X$, there exists a finite rank operator R from X to Y such that

1) $||Tx_k - Rx_k|| < \varepsilon$ for all k,

2) $\pi_{p'}(R) \leq K\pi_{p'}(T)$.

Grothendieck's Approximation

Recall:

Definition

$X \in AP$ iff for every $(x_n) \in c_0(X)$ and for every $\varepsilon > 0$ there exists a finite rank operator R in X such that $\sup_n ||Rx_n - x_n|| \leq \varepsilon$.

A generalization:

Definition

Let $0 < q \leq \infty$ and $1/s = 1/q + 1$. We say that X has the approximation property of order s, $X \in AP_s$, if for every $(x_n) \in l_q(X)$ (where $l_q(X)$ means $c_0(X)$ for $q = \infty$) and for every $\varepsilon > 0$ there exists a finite rank operator R in X such that $\sup_n ||Rx_n - x_n|| \leq \varepsilon$.

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H^∞ – the simplest

- The space H^∞ has the property AP_p for any $p > 0, p \neq 1$. Moreover, if $p > 1$, then H^∞ and all its even duals have the property $1 - MAP_p$; if $p < 1$, then all the duals of H^∞ have the property AP_p.

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A generalization

Manuel D. Contreras and Santiago Diaz-Madrigal, Uniform Approximation Properties for Spaces of Analytic Functions, Math. Nachr. 210 (2000), 85 -91

X has the uniform $K-MAP_p$, if given $\varepsilon > 0$, there is a function $N \in \mathbf{N} \to m(N) \in \mathbf{N}$ such that for every Banach space Y, any operator $T \in \Pi_{p'}(X, Y)$ and any finite sequence $(x_k)_1^N \subset X$, there exists a finite rank operator R from X to Y such that
1) $||Tx_k - Rx_k|| < \varepsilon$ for all $k \leq N$,
2) $\pi_{p'}(R) \leq K\pi_{p'}(T)$,
3) dim $R(X) \leq m(N)$.

- *Theorem.* Let δ be a positive number. Then the space H^∞ has the uniform $(1+\delta)$-bounded approximation property of order p for every $1 < p < \infty$.

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J.M. Delgado, E. Oja, C. Pineiro, E. Serrano The p-approximation property in terms of density of finite rank operators, J. Math. Anal. Appl. 354 (2009) 159-164

General situation

Main theorem on AP for H^∞ : the space has the AP "up to logarithm".

- **Theorem.** Let $(x_n)_n$ be a sequence in H^∞ such that

$$||x_n|| \leq \frac{1}{\log(1+n)}.$$

Then for every $\varepsilon > 0$ there is a finite rank operator R in H^∞ with

$$\sup_n ||Rx_n - x_n|| \leq \varepsilon.$$

Moreover, we can control both the rank and the norm of R :

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Moreover, we can control both the rank and the norm of R :

- **Theorem.** There is a function $B(\varepsilon), \varepsilon > 0$, such that if $(x_n)_n$ is a sequence in H^∞, satisfying

$$||x_n|| \leq \frac{1}{\log(1+n)},$$

then there exists an operator $R : H^\infty \to H^\infty$ such that
1) $\sup_n ||Rx_n - x_n|| \leq \varepsilon$,
2) $rank T < B(\varepsilon)$,
3) $||T|| < B(\varepsilon.)$

Bourgain J., Reinov O.I. *On the approximation properties for the space H^∞* // Math. Nachr. - 122 (1985). - P. 19-27.

Nuclear operators

An operator $T : X \to Y$ is nuclear if it is of the form

$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle y_k$$

for all $x \in X$, where $(x'_k) \subset X^*, (y_k) \subset Y$, $\sum_k ||x'_k|| \, ||y_k|| < \infty$.
We use the notation $N(X, Y)$

If T is nuclear, then

$$T : X \to c_0 \to l_1 \to Y.$$

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Nuclear operators: a question

Question: Let T map X^{**} into X and π_X be the natural isometric injection from X to X^{**}. Suppose that

$$\pi_X T : X^{**} \to X \to X^{**}$$

is nuclear. Is it true that

$$T : X^{**} \to X$$

is nuclear too?

In general, the answer is NO.

Application

- **Theorem.** *Let a linear operator* $T : H^\infty \to A$ *be such that there are two sequences of functions* $\{g_n\} \subset L^1$ *and* $\{f_n\} \subset H^\infty$, *for which* $\sum_k \int |g_k|\, dm < \infty$, $\|f_n\| < 1/\log(n+1)$ *for each* n *and*

$$T(f) = \sum_{k=1}^{\infty} \int g_k(t)\, f(t)\, dm(t)\, f_k.$$

Then the operator T *is nuclear as an operator, acting from* H^∞ *into the disk-algebra* A.

n-dimensional case

For the case of $H^\infty(M)$ on complex manifolds M, see

Alexander Brudnyi, On the approximation property for Banach spaces predual to H^∞-spaces, Journal of Functional Analysis 263 (2012) 2863-2875

Thank you for your attention!